MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 17 December 2012 2 pm - 6 pm

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Good Luck!

Task 1. Consider a Markov chain $\{X_n : n \geq 0\}$ with state space E, initial state probabilities $\mathbf{p}(0)$ and transition probability matrix P given by

$$E = \{0, 1\}, \quad \mathbf{p}(0) = \hat{\mathbf{p}} \quad \text{and} \quad P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix},$$

respectively, where $\hat{\mathbf{p}}$ is the stationary distribution of the Markov chain and $\alpha, \beta \in (0,1]$ are constants. Under what additional conditions on α and β do we have $E[X_n] = 1/3$ for all $n \geq 0$? **(5 points)**

Task 2. Let $\{X(t) : t \geq 0\}$ be Wiener process. Show that the process $\{Y(t) : t \geq 0\}$ given by $Y(t) = t\,X(1/t)$ for $t > 0$ and $Y(0) = 0$ is also a Wiener process. [Hint: As $Y(t)$ is a normal process it is sufficient to show that $Y(t)$ has the same mean and autocorrelation function as has a Wiener process.] **(5 points)**

Task 3. Let $\{X(t) : t \geq 0\}$ be a Poisson process with rate (/intensity) $\lambda > 0$. Show that the process $\{M(t) : t \geq 0\}$ given by $M(t) = (X(t) - \lambda t)^2 - \lambda t$ for $t \geq 0$ is a martingale with respect to the knowledge (of the σ-field) F_t of all historic process values of the Poisson process. **(5 points)**

Task 4. Which continuous-time LTI-system has output signal $Y(t)$ that is a WSS random process with autocorrelation function $R_Y(\tau) = 1/(1+\tau^2)$ when the input signal is continuous-time white noise? **(5 points)**

Task 5. For an $M/M/1$-queueing system with traffic intensity ρ and service times that are exponentially distributed with parameter μ the average amount of time a customer spends waiting in the queue (begfore being served) is given by $W_q = \rho/(\mu\,(1-\rho))$: Derive this formula! **(5 points)**

Task 6. The function $f(x)$ defined for $x \in [0,1]$ by $f(x) = 1$ for rational $x \in [0,1] \cap \mathbb{Q}$ and $f(x) = 0$ for irrational $x \in [0,1] \cap (\mathbb{R} - \mathbb{Q})$ is obviously not Riemann integrable $\int_0^1 f(x)\,dx$ as all upper integrals are at least 1 and all lower integrals are at most 0.

However, the Lebesgue integral $\int_0^1 f(x)\,dx$ is well-defined with value 0: The reason for this is that if we enumerate $x \in [0,1] \cap \mathbb{Q}$ as $\{q_n\}_{n=1}^{\infty}$, then the region of all non-zero function values of $f(x)$ is contained in the set $\cup_{n=1}^{\infty}[q_n - 2^{-n}\varepsilon, q_n + 2^{-n}\varepsilon]$ for each $\varepsilon > 0$, the "length" of which is at most $\sum_{n=1}^{\infty} 2\,2^{-n}\varepsilon = 2\,\varepsilon$. As the length of that region is at most $2\,\varepsilon$ for each $\varepsilon > 0$ the length must in fact be 0, and so the integral $\int_0^1 f(x)\,dx = 0$.

It is impossible to calculate $\int_0^1 f(x)\,dx$ numerically by means of ordinary deterministic numerical mathematical methods as they require regularity properties (smoothness) of the function $f(x)$ which our function $f(x)$ in turn completely lacks.

Say something about how it in principle would be possible to calculate $\int_0^1 f(x)\,dx$ numerically by means of the so called Monte-Carlo method (remember the computational problem of Exercise Session 1) – that is, to make appropriate use of a long sequence of uniformly distributed random variables over the unit interval (or unit square). Also, say something about why it will be problematic (impossible?) to make a working implementation of this numerical calculation to find that $\int_0^1 f(x)\,dx = 0$. (Pretending that you didn't know about the above cited fact from Lebesgue integration theory about the value of the integral, that is ... .) **(5 points)**

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Solutions to written exam Monday 17 December

Task 1. It is easy to see that the equation $\hat{\mathbf{p}}\,P = \hat{\mathbf{p}}$ for the stationary distribution has uniquie solution $\hat{\mathbf{p}} = [\beta/(\alpha{+}\beta)\ \ \alpha/(\alpha{+}\beta)]$, so that $E[X_n] = \alpha/(\alpha{+}\beta)$. This in turn equals 1/3 if and only if $\beta = 2\alpha$ and $\alpha \leq 1/2$.

Task 2. We have $E[Y(t)] = 0$ and $R_Y(s,t) = E[Y(s)Y(t)] = E[s\,X(1/s)\,t\,X(1/t)] = st\,R_X(1/s,1/t) = st\,\sigma^2 \min(1/s,1/t) = \sigma^2 \min(s,t)$, as is required for a Wiener process.

Task 3. We have $E[M(t)|F_s] = E[(X(t)-\lambda t)^2 - \lambda t|F_s] = E[(X(t)-X(s)+X(s)-\lambda t)^2 - \lambda t|F_s] = E[(X(t)-X(s))^2|F_s] + 2\,E[(X(s)-\lambda t)\,(X(t)-X(s))|F_s] + E[(X(s)-\lambda t)^2|F_s] - \lambda t = E[(X(t){-}X(s))^2] + 2\,(X(s){-}\lambda t)\,E[X(t){-}X(s)|F_s] + (X(s){-}\lambda t)^2 - \lambda t = \lambda\,(t{-}s) + \lambda^2(t{-}s)^2 + 2\,(X(s){-}\lambda t)\,E[X(t){-}X(s)] + (X(s){-}\lambda t)^2 - \lambda t = \lambda\,(t{-}s) + \lambda^2(t-s)^2 + 2\,(X(s)-\lambda t)\,\lambda\,(t-s) + (X(s)-\lambda t)^2 - \lambda t = (X(s)-\lambda s)^2 - \lambda s = M(s)$ for $t \geq s$.

Task 4. As the power spectral density of $Y(t)$ is $S_Y(\omega) = \pi\,\mathrm{e}^{-|\omega|}$ and satisfies $S_Y(\omega) = |H(\omega)|^2 S_W(\omega) = |H(\omega)|^2\sigma^2$, where $S_W(\omega) = \sigma^2$ is the power spectral density of white noise, we must have $|H(\omega)| = \sqrt{\pi}\,\mathrm{e}^{-|\omega|/2}/\sigma$ so that we can take the LTI-system with impulse response $h(t) = 1/(2\sigma\sqrt{\pi}\,(1/4{+}t^2))$.

Task 5. An arriving customer has n customers ahead of it in the queuing system with probability $p_n = (1-\rho)\,\rho^n$ for $n \geq 0$ giving rise to an average waiting time of $\sum_{n=0}^{\infty} n\,(1-\rho)\,\rho^n\,E[\exp(\mu)] = \sum_{n=1}^{\infty} n\,(1-\rho)\,\rho^n/\mu = (1-\rho)\rho/\mu\,(d/d\rho)\sum_{n=1}^{\infty}\rho^n = (1-\rho)\rho/\mu\,(d/d\rho)[\rho/(1-\rho)] = \rho/(\mu\,(1-\rho))$.

Task 6. The Monte-Carlo algorithm for numerical calculation of $\int_0^1 f(x)\,dx$ is to generate a very great number n of independent bivariate random variables $\{(X_i,Y_i)\}_{i=1}^n$ that all have a common unifom distribution over the unit square with PDF $f_{X,Y}(x,y) = 1$ for $0 \leq x,y \leq 1$ and $f_{X,Y}(x,y) = 0$ elsewhere and check how great a fraction of these random numbers that satisfy $f(X_i) \geq Y_i$. Or alternativey to consider the sample mean of the random variables $\{f(X_i)\}_{n=1}^{\infty}$ where $\{X_i\}_{i=1}^n$ are independent random variables that have a common unifom distribution over the unit interval. As $n \to \infty$ both these numerical approximations will in principle converge to $\int_0^1 f(x)\,dx$ by the law of large numbers - not only for our choice of the function $f(x)$ but for any function $f(x)$. In fact, in our case, as it is zero probability that X_i will take any of the values $\{q_n\}_{n=1}^{\infty}$ the mentioned approximating fractions and sample means will be identically zero all the time, giving the exact value of the integral rather than just a numerical approximation. However, due to the finite precision in representing numbers in a computer, the random numbers $\{X_i\}_{i=1}^n$ we get from the computer will always be rational so what we will end up with in practice trying the Monte-Carlo algorithm is the faulty result $\int_0^1 f(x)\,dx = 1$, despite the fact that the method in principle is correct (even exact).