MSG800/MVE170 Basic Stochastic Processes

Written exam Wednesday 6 April 2016 2 – 6 pm

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Task 1. Find the crosscorrelation function $R_{XY}(s,t)$ for the continuous time processes $X(t) = 2\,\sin(\omega t + \Theta)$ and $Y(t) = 2\,\cos(\omega t + B\Theta)$ for $t \in \mathbb{R}$, where $\omega \in \mathbb{R}$ is a constant and Θ and B are independent random variables with Θ continuous and uniformly distributed over the interval $[0,\pi]$ and B discrete with possible values -1 and 1 that have equal probability 1/2. [Hint: $2\sin(x)\cos(y) = \sin(x+y) + \sin(x-y)$.] **(5 points)**

Task 2. Prove that for a discrete time homogeneous Markov chain X_n, $n \in \mathbb{N}$, with transition matrix P it holds that $P(X_{m+n} = j \,|\, X_m = i) = (P^n)_{ij}$ for $n \geq 1$ and all (i,j), that is, element (i,j) of the n'th power P^n of P. **(5 points)**

Task 3. A continuous time random walk on the corners of a cube spends a unit mean exponentially distributed random time at each corner after which it selects one of the three neighbour corners as its next position with equal probabilities 1/3. Find the characteristic function for the random time it takes the random walk to move between two diametrically opposite corners. **(5 points)**

Task 4. Prove that for a discrete time martingale M_n, $n \in \mathbb{N}$, it holds that $E[M_n] = E[M_{n-1}] = \ldots = E[M_0]$. **(5 points)**

Task 5. A continuous time random process $X(t)$, $t \in \mathbb{R}$, with mean function $E[X(t)] = \mathrm{e}^t$ is input to an LTI system with impulse response $h(t) = 2\,\mathrm{e}^{-t}$ for $t \geq 0$ and $h(t) = 0$ for $t < 0$. Find the mean function $E[Y(t)]$ of the output process $Y(t)$. **(5 points)**

Task 6. Consider an M/M/1/3 queueing system with $\lambda = \mu = 1$ that is started empty at time zero. Show how one can find an approximation of the probability that the queueing system gets full some time during the first three time units of service by means of computer simulations. **(5 points)**

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Solutions to written exam 6 April 2016

Task 1. $R_{XY}(s,t) = E[X(s)Y(t)] = 4\,E[\sin(\omega s+\Theta)\,\cos(\omega t+B\Theta)] = 2\,E[\sin(\omega s+\Theta)\,\cos(\omega t+\Theta)] + 2\,E[\sin(\omega s+\Theta)\,\cos(\omega t-\Theta)] = E[\sin(\omega(s+t)+2\Theta)] + E[\sin(\omega(s-t))] + E[\sin(\omega(s+t))] + E[\sin(\omega(s-t)+2\Theta)] = \sin(\omega(s-t)) + \sin(\omega(s+t)) = 2\sin(\omega s)\cos(\omega t)$.

Task 2. The claim of the task is true by definition for $n=1$. Further, if the claim is true for $n-1$ then it is also true for n as $P(X_{m+n}=j\,|X_m=i) = \sum_k P(X_{m+n}=j,\, X_{m+1}=k\,|X_m=i) = \sum_k P(X_{m+n}=j\,|X_{m+1}=k)\,P(X_{m+1}=k\,|X_m=i) = \sum_k (P^{n-1})_{kj}P_{ik} = (P^n)_{ij}$. Hence the claim of the task in general follows by means of induction.

Task 3. We are looking for $\Psi(\omega)$ in the system of equations

$$\begin{aligned}
\Psi(\omega) &= E[\mathrm{e}^{j\omega\,\exp(1)}]\,\Psi_1(\omega)\\
\Psi_1(\omega) &= E[\mathrm{e}^{j\omega\,\exp(1)}]\,[(1/3)\cdot\Psi(\omega) + (2/3)\cdot\Psi_2(\omega)]\\
\Psi_2(\omega) &= E[\mathrm{e}^{j\omega\,\exp(1)}]\,[1/3 + (2/3)\cdot\Psi_1(\omega)],
\end{aligned}$$

with solution $\Psi(\omega) = 2\,(E[\mathrm{e}^{j\omega\,\exp(1)}])^3/(9-7\,(E[\mathrm{e}^{j\omega\,\exp(1)}])^2)$.

Task 4. This is solved Exercise 5.67 in Hsu's book: See his solution.

Task 5. $E[Y(t)] = E[\int_{-\infty}^{\infty} X(t-s)\,h(s)\,ds] = \int_0^{\infty} \mathrm{e}^{t-s}\,2\,\mathrm{e}^{-s}\,ds = \int_0^{\infty} 2\,\mathrm{e}^{t-2s}\,ds = \mathrm{e}^t$.

Task 6.

```
In[1]:= For[i=1; Reps=10000; SumSucc=0, i<=Reps, i++, T=0; X=0; Succ=0;
          While[T<=3,
            If[X==0, T=T+Random[ExponentialDistribution[1]]; X=1,
            If[X==1, T=T+Random[ExponentialDistribution[2]];
              If[Random[]<=1/2, X=0, X=2],
            If[X==2, T=T+Random[ExponentialDistribution[2]];
              If[Random[]<=1/2, X=1, X=3],
            If[X==3, Succ=1; T=T+Random[ExponentialDistribution[1]]; X=2]]]]];
          SumSucc=SumSucc+Succ]; N[SumSucc/Reps]
Out[1]= 0.3317
```