MSG800/MVE170 Basic Stochastic Processes

MVE171 Grundläggande stokastiska processer och finansiella tillämpningar

Written exam Wednesday 12 April 2017 $2-6$ pm

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Task 1. Consider a M/M/2/4 queueing system with mean arrival rate $\lambda = 1$ and mean service rate $\mu = 1$. Write a computer programme that by means of stochastic simulation finds an approximative value of the average length of a busy period of the queueing system (that is, a period during which at least one of the servers are busy).

(5 points)

Task 2. A hot dog vendor operates a hot dog stand where the number of hot dogs he sells each day is modeled as a Poisson random variable with expected value α. Let $X[k]$ represent the number of hot dogs the vendor has in stock at the beginning of each day. At the end of the day, if his stock of hot dogs has fallen below some minimal value β, then the vendor immediately purchases enough new hot dogs to bring up his total stock to γ for the next day. On the other hand, if at the end of the day the stock of hot dogs is at least β, then the stock is not increased for the next days sails. Fine the transition matrix for the Markov chain $X[k]$. **(5 points)**

Task 3. Let $X(t)$, $t \geq 0$, be the standard Wiener process with $\mathbf{E}\{X(t)^2\} = t$. Which (if any) deterministic function (or functions) $f(t)$ makes the stochastic process $M(t) = f(t)\,\mathrm{e}^{X(t)}$, $t \geq 0$, a martingale? (The answer must be motivated!) **(5 points)**

Task 4. Let $X(t)$, $t \in \mathbb{R}$, be a continuous time zero-mean stationary Gaussian process with autocorrelation function $R_X(t) = \mathrm{e}^{-|t|}$. Find the $t > 0$ which is such that $\mathbf{P}\{X(0) + X(t) > 1\} = 1/4$. **(5 points)**

Task 5. The derivative $\delta'(t)$ of the Dirac δ-function $\delta(t)$ can be defined as the function with the property that $\int_{-\infty}^{\infty} \delta'(t) g(t)\, dt = -g'(0)$ for (sufficiently nice) functions $g(t)$, or equivalently, as having Fourier transform $\int_{-\infty}^{\infty} \mathrm{e}^{-j\omega t}\delta'(t)\,dt = j\omega$. Show that $\delta'(t)$ is the impulse response of the mathematical operation of differentiation. **(5 points)**

Task 6. A continuous time Markov chain $X(t)$ with state space (possible values) $\{0, 1, 2, 3\}$ has generator

$$G = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}.$$

Find the expected value of the time it takes for the chain to move from state 0 to state 1. **(5 points)**

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Solutions to written exam Wednesday 12 April

Task 1.

```
Clear[aver,reps,que]; reps=100000;
For[i=1; aver=0, i<=reps, i++, que=1; While[que>=1,
  If[que==4, aver=aver+Random[ExponentialDistribution[2]]; que=3];
  If[que>=2&&que<=3, aver=aver+Random[ExponentialDistribution[3]];
    If[Random[UniformDistribution[0,1]]<=2/3, que=que-1, que=que+1];
  If[que==1, aver=aver+Random[ExponentialDistribution[2]];
    If[Random[UniformDistribution[0,1]]<=1/2, que=0, que=2]]]];
aver/reps
```

Task 2. The states (possible values) for $X[k]$ are $\{\beta,\ldots,\gamma\}$ and lettning Y denote a Poisson random variable with expected value α we have $p_{i,j} = \Pr(i-Y=j) = \alpha^{i-j}\,\mathrm{e}^{-\alpha}/((i-j)!)$ for $j=\beta,\ldots,i$ and $p_{i,\gamma} = \Pr(i-Y<\beta) = \sum_{k=i-\beta+1}^{\infty}\alpha^k\,\mathrm{e}^{-\alpha}/(k!)$ while $p_{i,j}=0$ for $j=i+1,\ldots,\gamma-1$ unless $i=j=\gamma$ in which case we instead have $p_{\gamma,\gamma} = \Pr(Y=0)+\Pr(Y>\gamma-\beta) = \mathrm{e}^{-\alpha}+\sum_{k=\gamma-\beta+1}^{\infty}\alpha^k\,\mathrm{e}^{-\alpha}/(k!)$.

Task 3. One possibility is $f(t)=0$. If on the other hand $f(t)\neq 0$, then the easily established fact that $\mathbf{E}\{\mathrm{e}^{X(t)-X(s)}\} = \mathbf{E}\{\mathrm{e}^{X(t-s)}\} = \mathrm{e}^{(t-s)/2}$ together with independent increments of $X(t)$ shows that $\mathbf{E}\{f(t)\,\mathrm{e}^{X(t)}|F_s\} = f(t)\,\mathbf{E}\{\mathrm{e}^{X(t)-X(s)}\mathrm{e}^{X(s)}|F_s\} = f(t)\,\mathbf{E}\{\mathrm{e}^{X(t)-X(s)}\}\,\mathrm{e}^{X(s)} = f(t)\,\mathrm{e}^{(t-s)/2}\mathrm{e}^{X(s)}$ for $0\leq s\leq t$, which in turn is equal to $f(s)\,\mathrm{e}^{X(s)}$ [the requirement for $M(t)$ to be a martingale] if and only if $f(t)=\mathrm{e}^{-t/2}$.

Task 4. $\mathbf{P}\{X(0)+X(t)>1\} = \mathbf{P}\{\mathrm{N}(0,2+2\,\mathrm{e}^{-t})>1\} = 1-\Phi(1/\sqrt{2+2\,\mathrm{e}^{-t}}\,) = 1/4$ means that $\Phi(1/\sqrt{2+2\,\mathrm{e}^{-t}}\,)=3/4$ so that $1/\sqrt{2+2\,\mathrm{e}^{-t}} = \Phi^{-1}(3/4)$.

Task 5. $\int_{-\infty}^{\infty}X(t-s)\delta'(s)\,ds = -\frac{d}{ds}X(t-s)|_{s=0} = X'(t)$, or alternatively $\mathcal{F}(\delta'\star X)(\omega) = \mathcal{F}(\delta')(\omega)\,\mathcal{F}(X)(\omega) = j\omega\,\mathcal{F}(X)(\omega) = \mathcal{F}(X')(\omega)$.

Task 6. Writing $E_{0\to 1}$, $E_{2\to 1}$ and $E_{3\to 1}$ for the expected value of the time it takes the chain to move from 0 to 1, from 2 to 1 and from 3 to 1, respectively, we see that

$$E_{0\to 1} = \mathbf{E}\{\exp(3)\}+(1/3)\cdot 0+(1/3)\cdot E_{2\to 1}+(1/3)\cdot E_{3\to 1} = 1/3+(2/3)\cdot E_{0\to 1},$$

giving $E_{0\to 1}=1$.