MSG800/MVE170 Basic Stochastic Processes

MVE171 Grundläggande stokastiska processer och finansiella tillämpningar

Written exam Thursday 24 August 2017 2 – 6 pm

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $X(t)$ and $Y(t)$ be independent Poisson processes with unit arrival rate. Calculate the probability $P[X(1)=1\,|\,X(2)+Y(2)=4]$. **(5 points)**

Task 2. Let $X(t)$ be a discrete time Markov chain with state space $\{0,1,2\}$ and transition matrix

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Find the expected value of the time it takes the chain to move from state 0 to state 2. **(5 points)**

Task 3. Write a computer programme that by means of stochastic simulation determines an approximative value for the probability $\mathbf{P}\{X(t) > t \text{ for some } t \in [0,10]\}$ for a unit rate Poisson process $\{X(t)\}_{t\geq 0}$. **(5 points)**

Task 4. The input to a continuous time LTI system is a WSS continuous time random process $X(t)$ with autocorrelation function

$$R_X(\tau) = \frac{A\omega_0}{\pi}\,\frac{\sin(\omega_0\tau)}{\omega_0\tau} \quad \text{for } \tau \in \mathbb{R},$$

where $A, \omega_0 > 0$ are constants, whilst the LTI system has impulse response of the same form

$$h(t) = \frac{\omega_1}{\pi}\,\frac{\sin(\omega_1 t)}{\omega_1 t} \quad \text{for } t \in \mathbb{R},$$

where $\omega_1 > 0$ is a constant. Find the autocorrelation function of the output $Y(t)$ from the system. **(5 points)**

Task 5. Let $X(t)$ be a continuous time Markov chain with state space $\{0,1\}$ and transition matrix (cf. Grimmett & Stirzaker Exercise 6.9.1)

$$P_t = \frac{1}{\lambda+\mu}\begin{bmatrix} \lambda+\mu\,\mathrm{e}^{-(\lambda+\mu)t} & \mu-\mu\,\mathrm{e}^{-(\lambda+\mu)t} \\ \lambda-\lambda\,\mathrm{e}^{-(\lambda+\mu)t} & \mu+\lambda\,\mathrm{e}^{-(\lambda+\mu)t} \end{bmatrix},$$

where $\lambda, \mu > 0$ are constants. Find $\mathbf{Cov}\{X(s), X(t)\}$ under the assumption that the Markov chain is started according to its stationary distribution. **(5 points)**

Task 6. Let $\{N(t)\}_{t\geq 0}$ be a unit rate Poisson process. Find a deterministic function $g(t)$ such that $M(t) = g(t)\,\mathrm{e}^{N(t)}$ is a martingale with respect to the filtration F_s containing information of all values $\{N(r),\ r \in [0,s]\}$ of the Poisson process up to time s. **(5 points)**

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Solutions to written exam 24 August 2017

Task 1. $P[X(1)=1\,|\,X(2)+Y(2)=4] = P[X(1)=1,\, X(2)+Y(2)=4]/P[X(2)+Y(2)=4] = P[X(1)=1,\, (X(2)-X(1))+X(1)+Y(2)=4]/P[\mathrm{Po}(4)=4] = P[\mathrm{Po}(1)=1]\,P[\mathrm{Po}(3)=3]/P[\mathrm{Po}(4)=4] = (\mathrm{e}^{-1}\,1^1/(1!))\,(\mathrm{e}^{-3}\,3^3/(3!))/(\mathrm{e}^{-4}\,4^4/(4!)) = 27/64$.

Task 2. By symmetry and with obvious notation we have $E_{0\to 2} = 1+(1/3)\cdot 0+(1/3)\cdot E_{0\to 2} + (1/3)\cdot E_{1\to 2} = 1 + (2/3)\cdot E_{0\to 2} = 3$.

Task 3.

```
rep=100000
    For[i=1; count=0, i<=rep, i++, t=0; X=0; succ=0;
        While[t<10, t=t+Random[ExponentialDistribution[1]]; X=X+1;
            If[(X>t)&&(t<=10), succ=1]];
        count=count+succ];
    N[count/rep]
```

Task 4. Writing $p_a(\omega) = 1$ for $|\omega| < a$ and $p_a(\omega) = 0$ for $|\omega| > a$ we have $S_X(\omega) = A\,p_{\omega_0}(\omega)$ and $H(\omega) = p_{\omega_1}(\omega)$ so that $S_Y(f) = |H(f)|^2 S_{XX}(f) = A\,p_{\omega_0}(\omega)\,p_{\omega_1}(\omega) = A\,p_{\min\{\omega_0,\omega_1\}}(\omega)$ giving

$$R_Y(\tau) = \frac{A\min\{\omega_0,\omega_1\}}{\pi}\,\frac{\sin(\min\{\omega_0,\omega_1\}\tau)}{\min\{\omega_0,\omega_1\}\tau} \quad \text{for } \tau\in\mathbb{R}.$$

Task 5. The stationary distribution is $(\pi_0\ \ \pi_1) = \lim_{t\to\infty}((P_t)_{00}\ \ (P_t)_{11}) = (\frac{\lambda}{\lambda+\mu}\ \ \frac{\mu}{\lambda+\mu})$ giving $\mathbf{E}\{X(t)\} = \mathbf{P}\{X(t)=1\} = \pi_1 = \frac{\mu}{\lambda+\mu}$ so that $\mathbf{Cov}\{X(s),X(t)\} = \mathbf{E}\{X(s)X(t)\} - \mathbf{E}\{X(s)\}\,\mathbf{E}\{X(t)\} = \mathbf{P}\{X(s)=1,\, X(t)=1\} - (\frac{\mu}{\lambda+\mu})^2 = (P_{t-s})_{11}\pi_1 - (\frac{\mu}{\lambda+\mu})^2 = \big(\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\,\mathrm{e}^{-(\lambda+\mu)\,(t-s)}\big)\,\frac{\mu}{\lambda+\mu} - (\frac{\mu}{\lambda+\mu})^2 = \frac{\lambda\mu}{(\lambda+\mu)^2}\,\mathrm{e}^{-(\lambda+\mu)\,(t-s)}$ for $t\geq s\geq 0$ so that $\mathbf{Cov}\{X(s),X(t)\} = \frac{\lambda\mu}{(\lambda+\mu)^2}\,\mathrm{e}^{-(\lambda+\mu)\,|t-s|}$ in general.

Task 6. $\mathbf{E}\{M(t)\,|\,F_s\} = g(t)\,\mathbf{E}\{\mathrm{e}^{N(t)-N(s)}|F_s\}\,\mathrm{e}^{N(s)} = g(t)\,\mathbf{E}\{\mathrm{e}^{N(t)-N(s)}\}\,\mathrm{e}^{N(s)} = g(t) \times \mathrm{e}^{(\mathrm{e}-1)\,(t-s)}\,\mathrm{e}^{N(s)}$ so we may take $g(t) = \mathrm{e}^{-(\mathrm{e}-1)\,t}$.