

# Fourier-transform/continuous time

(1)

1-dim Fourier transform of  $f: \mathbb{R} \rightarrow \mathbb{C}$

$$\hat{f}(\omega) = (Ff)(\omega) = \int_{-\infty}^{+\infty} e^{-j\omega x} f(x) dx \quad (j^2 = -1)$$

1-dim inverse Fourier transform of  $\hat{f}: \mathbb{R} \rightarrow \mathbb{C}$

$$f(x) = (F^{-1}\hat{f})(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega x} \hat{f}(\omega) d\omega$$

n-dim Fourier transform of  $f: \mathbb{R}^n \rightarrow \mathbb{C}$

$$\hat{f}(\omega_1, \dots, \omega_n) = (Ff)(\vec{\omega}) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} e^{-j(\omega_1 x_1 + \dots + \omega_n x_n)} f(x_1, \dots, x_n) dx_1 \dots dx_n$$

n-dim inverse Fourier transform of  $\hat{f}: \mathbb{R}^n \rightarrow \mathbb{C}$

$$f(x_1, \dots, x_n) = (F^{-1}\hat{f})(x_1, \dots, x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} e^{j(\omega_1 x_1 + \dots + \omega_n x_n)} \hat{f}(\omega_1, \dots, \omega_n) d\omega_1 \dots d\omega_n$$

example for  $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$  we have

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu+j\omega\sigma^2)/2\sigma^2} dx = e^{-j\omega\mu + \frac{1}{2}\omega^2\sigma^2} = e^{-j\omega\mu - \frac{1}{2}\omega^2\sigma^2} \quad \#$$

integral of  $N(\dots)$ -PDF which is 1

example for  $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$  we have

$$\hat{f}(\omega) = \int_0^{+\infty} \lambda e^{-(\lambda+j\omega)x} dx = \frac{\lambda}{\lambda+j\omega} \quad \#$$

1-dim properties

$$F(f(x-x_0))(\omega) = e^{-j\omega x_0} (Ff)(\omega)$$

$$F(e^{j\omega_0 x} f(x))(\omega) = (Ff)(\omega - \omega_0)$$

$$F(f(-x))(\omega) = (Ff)(-\omega)$$

$$F(f'(x))(\omega) = j\omega (Ff)(\omega)$$

## Fourier-transform/discrete time

(2)

1-dim Fourier transform of  $f: \mathbb{Z} \rightarrow \mathbb{C}$

$$\hat{f}(\omega) = (Ff)(\omega) = \sum_{k=-\infty}^{+\infty} e^{-j\omega k} f(k)$$

1-dim inverse Fourier transform of  $\hat{f}: \mathbb{R} \rightarrow \mathbb{C}$

$$f(k) = (F^{-1}\hat{f})(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega k} \hat{f}(\omega) d\omega$$

properties  $F(f(k-k_0))(\omega) = e^{-j\omega k_0} (Ff)(\omega)$

$$F(e^{j\omega_0 k} f(k))(\omega) = (Ff)(\omega - \omega_0)$$

$$F(f(-k))(\omega) = (Ff)(-\omega)$$

example for  $f(k) = \frac{\lambda^k}{k!} e^{-\lambda}$  for  $k=0,1,2,\dots$  (0 otherwise)

$$(\hat{f}(\omega)) = \sum_{k=0}^{+\infty} \underbrace{(\lambda e^{-j\omega})^k}_{\text{Taylor-exp of exp}} \frac{1}{k!} e^{-\lambda} = e^{-\lambda} (e^{j\omega} - 1) \quad \#$$

$\delta$ -function

\* discrete <sup>time</sup> Kronecker  $\delta$ -function  $\delta: \mathbb{Z} \rightarrow \{0,1\}$   $\delta(k) = \begin{cases} 0 & k \neq 0 \\ 1 & k=0 \end{cases}$

\* continuous <sup>time</sup> Dirac  $\delta$ -function  $\delta: \mathbb{R} \rightarrow \{0,\infty\}$  given by

$$\int_{-\infty}^{+\infty} f(x) \delta(x) dx = f(0) \text{ for } f: \mathbb{R} \rightarrow \mathbb{R} \text{ "smooth" and vanishing at infinity (with compact support)}$$

$\delta(x)$  is the distribution derivative of the Heaviside step-function  $\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$  as

$$\int_{-\infty}^{+\infty} f(x) \theta'(x) dx = \underbrace{[f(x)\theta(x)]_{-\infty}^{+\infty}}_{=0 \text{ since compact support}} - \int_{-\infty}^{+\infty} f'(x) dx = f(0)$$

$(F\delta)(\omega) = 1$  (both in discrete and continuous time)

## Convolution

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continuous-time convolution between  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  is given by

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-y)g(y) dy = \int_{-\infty}^{+\infty} g(x-y)f(y) dy = (g * f)(x)$$

discrete time convolution between  $f, g: \mathbb{Z} \rightarrow \mathbb{R}$  is given by

$$(f * g)(k) = \sum_{l=-\infty}^{+\infty} f(k-l)g(l) = \sum_{l=-\infty}^{+\infty} g(k-l)f(l) = (g * f)(k)$$

$F(f * g)(\omega) = (Ff)(\omega) (Fg)(\omega)$  (both in discrete and continuous time)

example for  $f(x) = g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$  we have

$$(f * g)(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\sqrt{2}y - x/\sqrt{2})^2}{2\sigma^2}} e^{-\frac{x^2}{4\sigma^2}} dy = \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} e^{-\frac{x^2}{2(\sqrt{2}\sigma)^2}}$$

integrates to  $\sqrt{2}$

when  $\mu=0$  and in a similar fashion

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} e^{-\frac{(x-\mu)^2}{2(\sqrt{2}\sigma)^2}} \text{ for } \mu \text{ in general} \quad \#$$

## Axioms of Probability Theory

$S$  is the sample space of possible outcomes  $\subseteq S$  of a random experiment, subsets of  $S$  (=collection of outcomes) are called events, a probability measure  $P(A)$  is defined for events  $A \subseteq S$  according to the axioms

$$P(\emptyset) = 0, P(S) = 1, \text{ and } P(A) \geq 0 \text{ and } P(A \cup B) = P(A) + P(B) \text{ for } A \cap B = \emptyset$$

From these axioms several additional rules follow:

$$P(A^c) = P(\bar{A}) = 1 - P(A)$$

$$P(A) \leq P(B) \text{ for } A \subseteq B$$

$$P(A) \leq 1$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i) \text{ when } A_i \cap A_j = \emptyset \text{ for } i \neq j$$

Independence events A and B are independent if  $P(A \cap B) = P(A)P(B)$

Conditional probability  $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Law of total probability  $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$   
whenever  $A_i \cap A_j = \emptyset$  for  $i \neq j$  and  $\cup_{i=1}^n A_i = S$

Random variable function  $X: S \rightarrow \mathbb{R}$   $X(s)$   
function  $(X, Y): S \rightarrow \mathbb{R}^2$   $(X(s), Y(s))$

Cumulative Distribution Function (CDF)  ~~$F_X(x) = P(X$~~

$$F_X(x) = P(X \leq x) = P(\{s \in S : X(s) \leq x\})$$

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

Properties  $0 \leq F_X(x) \leq 1$ ,  $F_X(x_1) \leq F_X(x_2)$  for  $x_1 \leq x_2$

$$F_X(\infty) = 1, F_X(-\infty) = 0, P(a < X \leq b) = F_X(b) - F_X(a)$$

$0 \leq F_{X,Y}(x,y) \leq 1$ ,  $F_{X,Y}(x,y)$  is increasing in each argument

$$F_{X,Y}(x, \infty) = F_X(x), F_{X,Y}(\infty, y) = F_Y(y)$$

$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1)$$

Independence X and Y independent if  $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$  for all  $A, B \subseteq \mathbb{R}$

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$X$  and  $Y$  independent  $\Leftrightarrow F_{X,Y}(x,y) = F_X(x)F_Y(y)$

\* Continuous random variable  $X$  or  $(X,Y)$  has uncountably infinitely many possible values, probability mass density function PDF

$$f_X(x) = \frac{d}{dx} F_X(x) \quad f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

Properties  $f_X(x) \geq 0$ ,  $\int_{-\infty}^{+\infty} f_X(x) dx = 1$ ,  $P(a < X \leq b) = \int_a^b f_X(x) dx$   
 $P(X \in A) = \int_A f_X(x) dx$

$$f_{X,Y}(x,y) \geq 0, \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dx dy = 1, P((X,Y) \in A) = \iint_A f_{X,Y}(x,y) dx dy$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dy, f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dx$$

$X$  and  $Y$  independent  $\Leftrightarrow f_{X,Y}(x,y) = f_X(x) f_Y(y)$

example Gaussian  $N(\mu, \sigma^2)$  r.v.  $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

example exponential r.v. with parameter  $\lambda$   $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

example uniform distribution over  $[a,b]$   $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$

Expectation  $E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx$   $E(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$   
 $E(g(X,Y)) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y) dx dy$

Conditional PDF of  $X$  given that  $Y=y$

$$f_{X|Y}(x|y) = f_{X,Y}(x,y) / f_Y(y)$$

$$P(X \in A | Y=y) = \int_A f_{X|Y}(x|y) dx \quad P(X \in A) = \int_{-\infty}^{+\infty} P(X \in A | Y=y) f_Y(y) dy$$

$$E(X | Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y) dx \quad E(X) = \int_{-\infty}^{+\infty} E(X | Y=y) f_Y(y) dy$$

\* Discrete random variable  $X$  or  $(X, Y)$  has finitely or countably infinitely many possible values, probability mass function PMF

$$P_X(x) = P(X=x) \quad P_{X,Y}(x,y) = P(X=x, Y=y)$$

Properties  $P_X(x) \in [0, 1]$ ,  $\sum_{\text{all } x} P_X(x) = 1$ ,  $P(X \in A) = \sum_{x \in A} P_X(x)$

$P_{X,Y}(x,y) \in [0, 1]$ ,  $\sum_{\text{all } x,y} P_{X,Y}(x,y) = 1$ ,  $P((X,Y) \in A) = \sum_{(x,y) \in A} P_{X,Y}(x,y)$

$P_X(x) = \sum_{\text{all } y} P_{X,Y}(x,y)$ ,  $P_Y(y) = \sum_{\text{all } x} P_{X,Y}(x,y)$

$X$  and  $Y$  independent  $\Leftrightarrow P_{X,Y}(x,y) = P_X(x)P_Y(y)$

example Bernoulli r.v.  $X$ , possible values  $\{0, 1\}$ ,  $P_X(0) = 1-p$ ,  $P_X(1) = p$

example Binomial r.v.  $X$ , possible values  $\{0, \dots, n\}$ ,  $P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$  for  $k=0, \dots, n$ , same as the sum of  $n$  independent Bernoulli r.v.'s

example Poisson r.v.  $X$ , possible values  $\{0, 1, \dots\} = \mathbb{N}$ ,  $P_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$  for  $k=0, 1, 2, \dots$ ,  $P_0(\lambda_1) + P_0(\lambda_2) = P_0(\lambda_1 + \lambda_2)$ , that is sum of two independent Po is Po

example geometric (waiting time) r.v.  $X$ , possible values  $\{1, 2, 3, \dots\}$ ,  $P_X(k) = (1-p)^{k-1} p$  for  $k=1, 2, \dots$ , number of Bernoulli trials that have to be made until first 1 occur...

Expectation  $E(X) = \sum_{\text{all } x} x P_X(x)$   $E(g(x)) = \sum_{\text{all } x} g(x) P_X(x)$   
 $E(g(X, Y)) = \sum_{\text{all } x,y} g(x,y) P_{X,Y}(x,y)$

Conditional PMF of X given that Y=y

$$P_{X|Y}(x|y) = P_{X,Y}(x,y) / P_Y(y)$$

$$P(X \in A | Y=y) = \sum_{x \in A} P_{X|Y}(x|y) \quad P(X \in A) = \sum_{\text{all } y} P(X \in A | Y=y) P_Y(y)$$

$$E(X | Y=y) = \sum_{\text{all } x} x P_{X|Y}(x|y) \quad E(X) = \sum_{\text{all } y} E(X | Y=y) P_Y(y)$$

Linearity of expectation  $E(\sum_{i=1}^n X_i) = \sum_{i=1}^n E(X_i)$

Variance  $Var(X) = \sigma_X^2 = E((X - E(X))^2) = E(X^2) - (E(X))^2$

Covariance  $Cov(X, Y) = E((X - E(X))(Y - E(Y))) = E(XY) - E(X)E(Y)$

$$Var(X) = Cov(X, X)$$

X and Y are called uncorrelated if  $Cov(X, Y) = 0$

X and Y independent  $\Rightarrow$  X and Y uncorrelated

Bilinearity of covariance  $Cov(\sum_{i=1}^m a_i X_i, \sum_{j=1}^n b_j Y_j) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j Cov(X_i, Y_j)$

in particular  $Var(\sum_{i=1}^n a_i X_i) = Cov(\sum_{i=1}^n a_i X_i, \sum_{j=1}^n a_j X_j) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j Cov(X_i, X_j)$

Bivariate normal distribution

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{\frac{(x-\mu_x)^2}{\sigma_x^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}}{2(1-\rho^2)}\right)$$

n-variate normal distribution see page 111 in Hsu  
(page 87 in first edition)

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2} \sqrt{|\det K|}} \exp\left(-\frac{1}{2}(x-\mu)^T K^{-1}(x-\mu)\right) \text{ where}$$

$\mu_i = E(X_i)$  and  $K_{ij} = Cov(X_i, X_j)$  where  $K$  is an  $n \times n$ -matrix and  $x$  is a column matrix

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Functions of random variables  $Y = g(X)$ ,  $g: \mathbb{R} \rightarrow \mathbb{R}$  increasing  
 $\Rightarrow f_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{d}{dy} P(X \leq g^{-1}(y)) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$

$$Z = g(X, Y) \quad f_Z(z) = \frac{d}{dz} P(g(X, Y) \leq z) = \frac{d}{dz} \iint_{g(x, y) \leq z} f_{X, Y}(x, y) dx dy$$

Characterizatz function (CF) of random variable(s) <sup>continuous</sup>

$$\varphi_X(\omega) = E(e^{j\omega X}) = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x) dx \quad | f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-j\omega x} \varphi_X(\omega) d\omega$$

$$\varphi_{X, Y}(\omega_1, \omega_2) = E(e^{j(\omega_1 X + \omega_2 Y)}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{j(\omega_1 x + \omega_2 y)} f_{X, Y}(x, y) dx dy$$

$$f_{X, Y}(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^2} e^{-j(\omega_1 x + \omega_2 y)} \varphi_{X, Y}(\omega_1, \omega_2) d\omega_1 d\omega_2$$