MSG800/MVE170 Basic Stochastic Processes

Exercise Session 4

Chapter 6 in Hsu's book

Solved problems. Problems 6.14, 6.16, 6.20, 6.26, 6.27, 6.29 and 6.32 in Hsu's book.

Problems for own work. Problems 6.53, 6.57, 6.59, 6.62, 6.64 and 6.65 in Hsu's book.

Computer problem for own work. If $\{W(t)\}_{t\geq 0}$ is a Wiener process, then a so called Ornstein-Uhlenbeck (OU) process $\{X(t)\}_{t\in\mathbb{R}}$ is given by $X(t) = \mathrm{e}^{-t}\,W(\mathrm{e}^{2t})$. Using that the Wiener process is zero-mean with autocorrelation function $R_{WW}(s,t) = \min\{s,t\}$ it is not hard to establish that the OU-process is WSS zero-mean with autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}$ for $\tau \in \mathbb{R}$.

Let $\{X(t)\}_{t\in\mathbb{R}}$ and $\{O(t)\}_{t\in\mathbb{R}}$ be independent OU-process. Consider a noise process $\{N(t)\}_{t\in\mathbb{R}}$ given by $N(t) = \sqrt{2}\,\cos(\Theta+10t)\,O(t)$, where Θ is a random variable that is independent of X and O and uniformly distributed over $[-\pi,\pi]$. Then X has PSD $S_{XX}(\omega) = 2/(1+\omega^2)$ (see Problem 6.26) while N has PSD $S_{NN}(\omega) = (S_{XX}(\omega-10) + S_{XX}(\omega+10))/2$ (see Problem 6.53): Make plots of these PSD's.

The signal $X(t)$ is sent on a noisy channel where it is disturbed by the additive noise $N(t)$ so that the recived observed signal is $Y(t) = X(t) + N(t)$. In order to reconstruct the sent signal $X(t)$ form the observed recived signal $Y(t)$ as well as possible $Y(t)$ is filtered through a so called Wiener filter with frequencey response $H(\omega) = S_{XX}(\omega)/(S_{XX}(\omega)+S_{NN}(\omega))$. The corresponding impulse response is

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{j\omega t} H(\omega)\,d\omega = \frac{1}{\pi}\int_0^{\infty} \cos(\omega t)\,H(\omega)\,d\omega \approx \frac{1}{\pi}\int_0^{\omega_0} \cos(\omega t)\,H(\omega)\,d\omega$$

for a suitable sufficiently large $\omega_0 > 0$, say $\omega_0 = 10$ or $\omega_0 = 25$. The outsignal (attempted reconstruction of X) is $Z(t) = (h\star Y)(t)$.

Simulate $\{X(t)\}_{t\in[0,10]}$, $\{Y(t)\}_{t\in[0,10]}$ and $\{Z(t)\}_{t\in[0,10]}$ (with the latter computed numerically as $h\star Y$). Show by means of plots that despite the processes X and Y are very unlike each other, the processes X and Z are very alike.

In order to simulate an OU-process it is convenient to use that $X(t{+}\Delta) = \mathrm{e}^{-(t+\Delta)} \times (W(\mathrm{e}^{2(t+\Delta)}) - W(\mathrm{e}^{2t})) + \mathrm{e}^{-(t+\Delta)}W(\mathrm{e}^{2t}) = \mathrm{e}^{-(t+\Delta)}(W(\mathrm{e}^{2(t+\Delta)}) - W(\mathrm{e}^{2t})) + \mathrm{e}^{-\Delta}X(t)$ for $\Delta > 0$, where $\mathrm{e}^{-(t+\Delta)}(W(\mathrm{e}^{2(t+\Delta)}) - W(\mathrm{e}^{2t}))$ is $\mathrm{N}(0, 1-\mathrm{e}^{-2\Delta})$-distributed and independent of $\{X(s)\}_{s\in(-\infty,t]}$ (by defining properties of the Wiener process). Hence we may simulate, for example, a discrete sample $\{X(-15+\frac{i}{1000})\}_{i=0}^{40000}$ with step length $\Delta = \frac{1}{1000}$ of the process values $\{X(t)\}_{t\in[-15,25]}$ by taking $X(-15)$ $\mathrm{N}(0,1)$-distributed and then calculating $X(-15+\frac{i}{1000}) = \varepsilon_i + \mathrm{e}^{-1/1000} X(-15+\frac{i-1}{1000})$ recursively for $i = 1, \ldots, 40000$, where $\{\varepsilon_i\}_{i=1}^{40000}$ are IID $\mathrm{N}(0, 1-\mathrm{e}^{-1/2000})$-distributed and independent of $X(-15)$.

Note that it is common that software give you an $\mathrm{N}(0,\sigma)$-distributed random variable instead of an $\mathrm{N}(0,\sigma^2)$-distributed one, as is e.g., the case with both Mathematica and Matlab: This is argubly the most common reason for seemingly unexplainable erroneous results from simulations. In the same manner software may give you an $\exp(1/\lambda)$-distributed random variable when you ask for an $\exp(\lambda)$-distributed one, and so on ...
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