MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 11 January 2016 2 – 6 pm

(With two figures.)

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

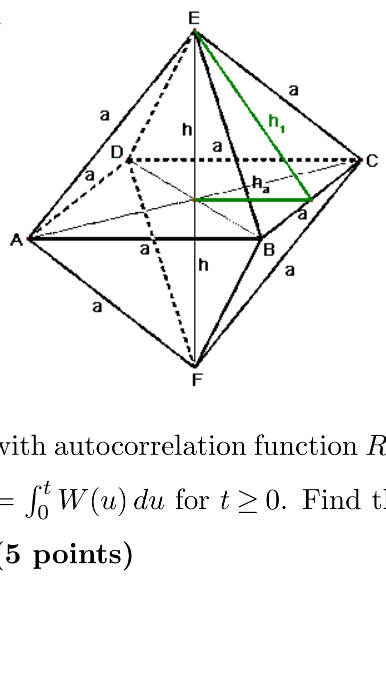
Motivations: All answers/solutions must be motivated. Good Luck!

Task 1. Calculate the probability $P(X(1)=1,\ X(4)=4\,|\,X(2)=2,\ X(3)=3,\ X(5)=5)$ for a Poisson process $X(t)$ with arrival rate $\lambda>0$. **(5 points)**

Task 2. A discrete time martingale M_n, $n\in\mathbb{N}$, is defined by the requirement that $E[M_{n+1}|F_n]=M_n$ for $n\in\mathbb{N}$, where $F_n=\sigma(M_0,\ldots,M_n)$ is the information obtained by observing the values of $M_0,\ldots,M_n$. Prove that this definition implies the seemingly more demanding requirement $E[M_{m+n}|F_n]=M_n$ for $m=1,2,3,\ldots$. **(5 points)**

Task 3. Is it possible for a two state discrete time Markov chain not to have a stationary distribution? **(5 points)**

Task 4. A continuous time random walk on the corners $\{A,B,C,D,E,F\}$ of an octaeder spends a unit mean exponentially distributed random time at each corner after which it selects one of the four neighbour corners as its next position with equal probabilities 1/4. Find the characteristic function for the random time it takes the random walk to move from A to C (see figure below).



(5 points)

Task 5. Let $\{W(t)\}_{t\geq 0}$ be a random process with autocorrelation function $R_{WW}(s,t)=\min\{s,t\}$. Form a new process $Y(t)$ as $Y(t)=\int_0^t W(u)\,du$ for $t\geq 0$. Find the autocorrelation function $R_{YY}(s,t)$ for $0\leq s\leq t$. **(5 points)**

Task 6. The passport issuing service in former East Germany (=the German Democratic Republic) opened at a certain unpredictable time in the morning each day after which it was open exactly six hours after which it closed down for the day. It was forbidden for passport applicants to queue outside the passport issuing service before the opening time. When the passport issuing service opened each morning passport applicants started to arrive according to a Poisson process with arrival rate 6 applicants per hour. The passport issuing service had just one passport issuer who needed an exponential distributed time with mean 1/2 hour to issue a passport. A passport applicant that was in progress with her/his passport issuing at the closing time was abandoned (didn't get a passport). Write a computer programme that by means of stochastic simulation find an approximation of the expected value of the number of passport applicants that entered the passport issuing service but did not get a passport each day. (Or in other words, find the mean number value of the number of customers that is not being finished served during the first six time units for an M/M/1 queueing system with $\lambda = 6$ and $\mu = 2$ when it is started empty.)

(5 points)

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Solutions to written exam 11 January 2016

Task 1. $P(X(1)=1,\, X(4)=4 \,|\, X(2)=2,\, X(3)=3,\, X(5)=5)$

$$= \frac{P(X(1)=1,\, X(4)=4,\, X(2)=2,\, X(3)=3,\, X(5)=5)}{P(X(2)=2,\, X(3)=3,\, X(5)=5)}$$

$$= \frac{P(X(1)=1,\, X(2)-X(1)=1,\, X(3)-X(2)=1,\, X(4)-X(3)=1,\, X(5)-X(4)=1)}{P(X(2)=2,\, X(3)-X(2)=1,\, X(5)-X(3)=2)}$$

$$= \frac{[P(X(1)=1)]^5}{P(X(2)=2)\, P(X(2)=1)\, P(X(2)=2)} = \frac{[\lambda^1/((1!)\cdot \mathrm{e}^{\lambda})]^4}{[(2\lambda)^2/((2!)\cdot \mathrm{e}^{2\lambda})]^2} = \frac{1}{4}.$$

Task 2. This is solved Exercise 5.66 in Hsu's book: See his solution.

Task 3. A stationary distribution π always exists as we can always solve

$$\begin{cases} [\pi_1 \;\; \pi_2] = [\pi_1 \;\; \pi_2] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \\ \pi_1 + \pi_2 = 1 \end{cases} \Leftrightarrow \begin{cases} a\pi_1 = b\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases} \Leftrightarrow \begin{cases} \begin{cases} \pi_1 = b/(a+b) \\ \pi_2 = a/(a+b) \end{cases} & \text{if } a+b \neq 0 \\ \pi_1 + \pi_2 = 1 & \text{if } a+b=0 \end{cases}.$$

Task 4. There are two natural ways to solve this task:

Solution 1. We can clump together corners $\{B,D,E,F\}$ to one state resulting in a three state continuous time Markov chain with states $\{A, BDEF, C\}$. The times T_A and T_C the chain spends in states A and C are both unit mean exponentially distributed while the time T_{BDEF} the chain spends in state $BDEF$ by a calculation we have done many time on lectures has characteristic function

$$\sum_{k=1}^{\infty} \left(\mathbf{E}\{\mathrm{e}^{j\omega \exp(1)}\}\right)^k (1/2)^{k-1}(1/2) = \sum_{k=1}^{\infty} (1-j\omega)^{-k} (1/2)^{k-1}(1/2) = \ldots = \frac{(1/2)}{(1/2)-j\omega}$$

showing that T_{BDEF} is exponentially distributed with mean 2. The time $T_{A\to C}$ it takes to move from A to C thus has a characteristic function $\Psi(\omega) = \mathbf{E}\{\mathrm{e}^{j\omega T_{A\to C}}\}$ that satisfies

$$\begin{aligned} \Psi(\omega) &= \mathbf{E}\{\mathrm{e}^{j\omega T_A}\}\, \mathbf{E}\{\mathrm{e}^{j\omega T_{BDEF}}\} \left(1/2\cdot 1 + (1/2)\cdot \Psi(\omega)\right) \\ &= \frac{1}{1-j\omega} \frac{(1/2)}{(1/2)-j\omega} \left(1/2\cdot 1 + (1/2)\cdot \Psi(\omega)\right), \end{aligned}$$

so that by elementary calculations $\Psi(\omega) = (1/4)/\big((1/4)-(3/2)\, j\omega - \omega^2\big)$.

Solution 2. If we do not clump together corners $\{B,D,E,F\}$ we can instead in addition to the characteristic function $\Psi(\omega)$ defined as in Solution 1 consider also the common characteristic function $\Psi_1(\omega)$ for the times $T_{B\to C}$, $T_{D\to C}$, $T_{E\to C}$ and $T_{F\to C}$ it takes to move from B to C, from D to C, from E to C and from F to C, respectively. The we have

$$\Psi(\omega) = \mathbf{E}\{\mathrm{e}^{j\omega T_A}\}\, \Psi_1(\omega) = \frac{1}{1-j\omega}\, \Psi_1(\omega)$$

and

$$\Psi_1(\omega) = \frac{1}{1-j\omega}\big((1/4)\cdot\Psi(\omega) + (1/2)\cdot\Psi_1(\omega) + (1/4)\cdot 1\big),$$

so that (from the second equation)

$$\Psi_1(\omega)\left(1 - \frac{(1/2)}{1-j\omega}\right) = \frac{(1/4)}{1-j\omega}\,(\Psi(\omega)+1).$$

Inserting the first equation obtained above in this we get

$$\Psi(\omega)\,(1-j\omega)\left(1 - \frac{(1/2)}{1-j\omega}\right) = \frac{(1/4)}{1-j\omega}\,(\Psi(\omega)+1),$$

which of course can be easily solved and which (also of course) replicates the result we have obtained in Solution 1 already.

Task 5. $R_{YY}(s,t) = E\big[(\int_0^s W(u)\,du)\,(\int_0^t W(v)\,dv)\big] = \int_{u=0}^{u=s}\int_{v=0}^{v=t} E[W(u)W(v)]\,dudv = \int_{u=0}^{u=s}\int_{v=0}^{v=s} \min\{u,v\}\,dudv + \int_{u=0}^{u=s}\int_{v=s}^{v=t} \min\{u,v\}\,dudv = 2\int_{u=0}^{u=s}\int_{v=0}^{v=u} v\,dvdu + \int_{u=0}^{u=s}\int_{v=s}^{v=t} u\,dvdu = 2\int_0^s u^2/2\,du + \int_0^s (t-s)\,u\,du = \ldots = ts^2/2 - s^3/6.$

Task 6. The sought after average is the expected number of arrivals $6\cdot 6 = 36$ minus the average number of customers that is being served and can be approximated as

```
wiederholungen = 10000000000000000000000000000; reisepass = 0;
        for durchlauf = 1:wiederholungen
          ankunft = exprnd(1/6); betrieb = ankunft + exprnd(1/2);
          while betrieb < 6
            reisepass = reisepass + 1; ankunft = ankunft + exprnd(1/6);
            if betrieb < ankunft
              betrieb = ankunft + exprnd(1/2);
            else
              betrieb = betrieb + exprnd(1/2);
            end
          end
        end
        36-reisepass/wiederholungen
        >>24.496952604276987154761795215
```