MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 9 January 2017 2 – 6 pm

→→ I-STUDENTER: skriv kurskod MVE171 istf. MVE170 på era tentalösningar. ←←

TEACHER AND JOUR: Patrik Albin, telephone 070 6945709.

AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Consider an M(1)/M(1)/1/2 queueing system (that is, unit mean exponentially distributed times between arrivals of new customers, unit mean exponentially distributed service times, one server and one quequeing slot giving a total of two slots in the whole queueing system). The queueing system starts up empty $X(0) = 0$ at time $t = 0$ and is then run for 5 time units after which it is closes down. Write programme code for a computer programme that by means of stochastic simulation finds an approximative value for the probability that some time during the 5 units the queueing system is operational it happens that the value of $X(t)$ changes in one sequence from 0 to 1 to 2, that is, there are two straight arrivals to the queueing system without any customer being finished served in between. **(5 points)**

Task 2. A pair of WSS random processes $X(t)$ and $Y(t)$ are called jointly WSS if their cross-correlation function $R_{XY}(t, t+\tau) = \mathbf{E}\{X(t)Y(t+\tau)\}$ depends on τ only (but not on t). Show by an example that a pair of WSS processes $X(t)$ and $Y(t)$ [that are not equal, i.e., $X(t) = Y(t)$ is not permitted] can be jointly WSS and by another example that they can also fail to be so. [**Hint:** You may want to use $Y(t) = X(f(t))$ with some suitable (non-random) function $f(t)$ for the second example.] **(5 points)**

Task 3. The time homogeneous Markov property for a discrete time stochastic process X_n, $n = 0, 1, 2, \ldots$, is the requirement that

$$\mathbf{P}\{X_{n+1}=j|X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0\} = \mathbf{P}\{X_{n+1}=j|X_n=i\} = p_{ij}$$

do not depend on neither $i_0, \ldots, i_{n-1}$ or n. Show that this property implies that also

$$\mathbf{P}\{X_{n+2}=j|X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0\} = \mathbf{P}\{X_{n+2}=j|X_n=i\} = p_{ij}^{(2)}$$

do not depend on neither $i_0, \ldots, i_{n-1}$ or n. **(5 points)**

Task 4. Let $N(t)$, $t \geq 0$, be a Poisson process with intensity $\lambda = 1$ and $W(t)$, $t \geq 0$, a Wiener process with $\mathbf{E}\{W(1)^2\} = 1$ that is independent of the aformentioned Poisson process. Show that the process $X(t) = (N(t) - t)\,W(t)$, $t \geq 0$, is a martingale with respect to the information F_s given by (observing) all the process values of $N(r)$ and $W(r)$ for $r \in [0, s]$. **(5 points)**

Task 5. Consider a WSS continuous time random process $X(t)$ with autocorrelation function $R_X(\tau) = 1/(1+\tau^2)$ as input to an LTI system with one of the following three types of frequency responses $H_1(\omega)$, $H_2(\omega)$ and $H_3(\omega)$ given by

$$H_1(\omega) = \begin{cases} 1 & \text{for } |\omega| \leq \underline{\omega} \\ 0 & \text{for } |\omega| > \underline{\omega} \end{cases}, \quad H_2(\omega) = \begin{cases} 1 & \text{for } |\omega| \in (\underline{\omega}, \overline{\omega}) \\ 0 & \text{for } |\omega| \notin (\underline{\omega}, \overline{\omega}) \end{cases} \quad \text{and} \quad H_3(\omega) = \begin{cases} 1 & \text{for } |\omega| \geq \overline{\omega} \\ 0 & \text{for } |\omega| < \overline{\omega} \end{cases}.$$

Find the two frequencies $0 < \underline{\omega} < \overline{\omega} < \infty$ which are such that the corresponding outputs from the LTI system $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ have the same power $\mathbf{E}\{Y_1(t)^2\} = \mathbf{E}\{Y_2(t)^2\} = \mathbf{E}\{Y_3(t)^2\}$. **(5 points)**

Task 6. A continuous time Markov chain has state space $\{0, 1, 2\}$ and generator

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Show that the time it takes the chain to move from state 0 to state 1 is exponentially distributed with parameter 1. **(5 points)**

MSG800/MVE170 Basic Stochastic Processes

Solutions to written exam 9 January 2017

Task 1.

```
No = 10000; Prob = 0;
For[i=1, i<=No, i++,
  TT = Random[ExponentialDistribution[1]]
          + Random[ExponentialDistribution[2]];
  OK = False;
  While[TT < 5, Unif = Random[];
    If[Unif <= 1/2, OK = True];
    TT = TT + Random[ExponentialDistribution[1]]
          + Random[ExponentialDistribution[2]]];
  If[OK, Prob = Prob + 1/No]];
Prob
```

Task 2. If $X(t)$ and $Y(t)$ are any independent WSS processes, then $\mathbf{E}\{X(t)Y(t+\tau)\} = \mathbf{E}\{X(t)\}\,\mathbf{E}\{Y(t+\tau)\} = \mu_X\,\mu_Y$ depends on neither τ or t, so $X(t)$ and $Y(t)$ are jointly WSS. If $X(t)$ is any WSS process and $Y(t) = X(-t)$, then $\mathbf{E}\{X(t)Y(t+\tau)\} = \mathbf{E}\{X(t)\}\,\mathbf{E}\{X(-(t+\tau))\} = R_X(-2t-\tau) = R_X(2t+\tau)$ which depends on t [unless R_X is a constant, meaning that $X(t)$ equals the one and same random variable for all t].

Task 3. By the Markov property we have

$$\begin{aligned}
&\mathbf{P}\{X_{n+2}=j|X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}\\
&\quad= \frac{\mathbf{P}\{X_{n+2}=j, X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}}{\mathbf{P}\{X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}}\\
&\quad= \sum_k \frac{\mathbf{P}\{X_{n+2}=j, X_{n+1}=k, X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}}{\mathbf{P}\{X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}}\\
&\quad= \sum_k \frac{\mathbf{P}\{X_{n+2}=j, X_{n+1}=k, X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}}{\mathbf{P}\{X_{n+1}=k, X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}}\\
&\qquad\qquad\times \frac{\mathbf{P}\{X_{n+1}=k, X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}}{\mathbf{P}\{X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}}\\
&\quad= \sum_k \mathbf{P}\{X_{n+2}=j|X_{n+1}=k, X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}\\
&\qquad\qquad\times \mathbf{P}\{X_{n+1}=k|X_n=i, X_{n-1}=i_{n-1},\ldots,X_0=i_0\}\\
&\quad= \sum_k \mathbf{P}\{X_{n+2}=j|X_{n+1}=k\}\,\mathbf{P}\{X_{n+1}=k|X_n=i\}\\
&\quad= \sum_k p_{kj}\,p_{ik},
\end{aligned}$$

which obviously do not depend on neither $i_0,\ldots,i_{n-1}$ or n.

Task 4. $\mathbf{E}\{(N(t)-t)\,W(t)|F_s\} = \mathbf{E}\{[(N(t)-N(s))-(t-s)]\,(W(t)-W(s))|F_s\}+\mathbf{E}\{[(N(t)-N(s))-(t-s)]\,W(s)|F_s\}+\mathbf{E}\{(N(s)-s)\,(W(t)-W(s))|F_s\}+\mathbf{E}\{(N(s)-s)\,W(s)|F_s\} = \mathbf{E}\{[(N(t)-N(s))-(t-s)]\,(W(t)-W(s))\}+\mathbf{E}\{(N(t)-N(s))-(t-s)|F_s\}\,W(s)+(N(s)-s)\,\mathbf{E}\{W(t)-W(s)|F_s\}+(N(s)-s)\,W(s) = \mathbf{E}\{N(t)-N(s))-(t-s)\}\,\mathbf{E}\{W(t)-W(s)\}+\mathbf{E}\{(N(t)-N(s))-(t-s)\}\,W(s)+(N(s)-s)\,\mathbf{E}\{W(t)-W(s)\}+(N(s)-s)\,W(s) = 0\cdot 0+0\cdot W(s)+(N(s)-s)\cdot 0+(N(s)-s)\,W(s) = (N(s)-s)\,W(s)$ for $0\leq s\leq t$.

Task 5. Clearly, $1=\mathbf{E}\{X(t)^2\}=\mathbf{E}\{Y_1(t)^2\}+\mathbf{E}\{Y_2(t)^2\}+\mathbf{E}\{Y_3(t)^2\}$ so when these three are equal we have $\mathbf{E}\{Y_1(t)^2\}=\mathbf{E}\{Y_2(t)^2\}=\mathbf{E}\{Y_3(t)^2\}=1/3$. As $S_X(\omega)=\pi\,\mathrm{e}^{-|\omega|}$ we get $\mathbf{E}\{Y_1(t)^2\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_1(\omega)|^2 S_X(\omega)\,d\omega=\int_0^{\underline{\omega}}\mathrm{e}^{-\omega}\,d\omega=1-\mathrm{e}^{-\underline{\omega}}$, $\mathbf{E}\{Y_2(t)^2\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_2(\omega)|^2 S_X(\omega)\,d\omega=\int_{\underline{\omega}}^{\overline{\omega}}\mathrm{e}^{-\omega}\,d\omega=\mathrm{e}^{-\underline{\omega}}-\mathrm{e}^{-\overline{\omega}}$ and $\mathbf{E}\{Y_3(t)^2\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_3(\omega)|^2 S_X(\omega)\,d\omega=\int_{\overline{\omega}}^{\infty}\mathrm{e}^{-\omega}\,d\omega=\mathrm{e}^{-\overline{\omega}}$ so that $\underline{\omega}=\ln(3/2)$ and $\overline{\omega}=\ln(3)$.

Task 6. As the time spent in each state is exponential distributed with parameter 2 the characteristic function of the sought after time T_{01} satisfies (with obvious notation)

$$\Psi_{T_{01}}(\omega)=\Psi_{\exp(2)}(\omega)\,\big((1/2)+(1/2)\,\Psi_{\exp(2)}(\omega)\,[(1/2)+(1/2)\,\Psi_{T_{01}}(\omega)]\big)$$

or simpler

$$\Psi_{T_{01}}(\omega)=\Psi_{\exp(2)}(\omega)\,[(1/2)+(1/2)\,\Psi_{T_{21}}(\omega)]=\Psi_{\exp(2)}(\omega)\,[(1/2)+(1/2)\,\Psi_{T_{01}}(\omega)],$$

which both give

$$\Psi_{T_{01}}(\omega)=\frac{(1/2)\,\Psi_{\exp(2)}(\omega)}{1-(1/2)\,\Psi_{\exp(2)}(\omega)}=\ldots=\frac{1}{1-j\omega}=\Psi_{\exp(1)}(\omega).$$