MSG800/MVE170 Basic Stochastic Processes

Written exam 4 April 2018 8.30–12.30

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) <u>or</u> Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Consider a time homogeneous Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space E, initial distribution $\mathbf{p}(0)$ and transition probability matrix P given by

$$E = \{0,1,2\}, \quad \mathbf{p}(0) = [1/3 \;\; 1/3 \;\; 1/3] \quad \text{and} \quad P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \end{bmatrix},$$

respectively. Write a computer program that by means of simulation finds a numerical approximation of the expected vaule of the number of time units it takes until the chain for the first time has spent two consequtive time units (two time units in row, that is) at state 0. **(5 points)**

Task 2. Give an example of a discrete-parameter Markov chain $\{X(n),\, n \geq 0\}$ that has both periodic and aperiodic states. **(5 points)**

Task 3. Let $X_1, X_2, \ldots$ be independent identically distributed random variables, where each X_i can take only two values $1/2$ and 2 with the probabilities p and $1-p$, respectively. For which value of $p \in (0,1)$ is the process $\{M_n,\, n \geq 0\}$ given by $M_0 = 1$ and $M_n = \prod_{i=1}^{n} X_i = X_1\, X_2 \ldots X_n$ for $n \geq 1$ a martingale? **(5 points)**

Task 4. Find the autocorrelation function $R_X(s,t)$ of the process $X(t) = \sqrt{2}\, A \cos(U t + \Theta)$ for $t \in \mathbb{R}$, where A, U and Θ are independent random variables with A standard normal distributed, U uniformly distributed over the inteval $[0,1]$ and Θ uniformly distributed over the inteval $[0,\pi]$. [Hint: The fact that $2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$ can be useful.] **(5 points)**

Task 5. A WSS random signal $\{X(t)\}_{t\in\mathbb{R}}$ with PSD $S_X(\omega)$ is transmitted on a noisy channel where it is disturbed by an additive zero-mean WSS random noise $\{N(t)\}_{t\in\mathbb{R}}$ that is independent of the signal X and has PSD $S_N(\omega)$. The recived signal $Y(t) =$

$X(t)+N(t)$ is input to a linear system (/filter) with output signal $Z(t)$ that has frequency response $H(\omega) = S_X(\omega)/(S_X(\omega)+S_N(\omega))$. Express the mean-square deviation $\mathbf{E}\{(Z(t)-X(t))^2\}$ in terms of S_X and S_N. **(5 points)**

Task 6. Consider a M/M/1 queueing system with traffic intensity $\rho = \lambda/\mu < 1$. Show that the total time T a customer spends in the queueing system is exponentially distributed with mean $1/(\mu-\lambda)$. [Hint: Show that T has the moment generating function $\mathbf{E}\{e^{sT}\} = (\mu-\lambda)/(\mu-\lambda-s)$ of an exponential distribution with mean $1/(\mu-\lambda)$.]

(5 points)

MSG800/MVE170 Solutions to written exam 4 April 2018

Task 1. Here is a Mathematica program that solves the task

```
For[i=1; Result={}, i<=100000, i++,
    X = Floor[Random[UniformDistribution[{0,3}]]];
    AtZero = 0; Wait = 0;
    While[AtZero<1, Wait=Wait+1;
        Move = Random[UniformDistribution[{0,1}]];
        If[X==0, If[Move<=1/2, Y=0, If[Move<=5/6, Y=1, Y=2]]];
        If[X==1, If[Move<=2/3, Y=1, Y=2]];
        If[X==2, If[Move<=1/2, Y=0, Y=2]];
        If[X==0&&Y==0, AtZero=AtZero+1; AppendTo[Result,Wait], X=Y]]
N[Mean[Result]]
```

Task 2. For example the chain with state space E and transition probability matrix P given by

$$E = \{0,1,2,3\} \qquad \text{and} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix},$$

respectively, where the states $\{0,1\}$ have period 2 while the states $\{2,3\}$ are aperiodic.

Task 3. We have $E(M_{n+1}|F_n) = E(M_{n+1}|M_1,\ldots,M_n) = E(X_{n+1}M_n|M_n) = E(X_{n+1}) \times M_n = M_n$ when $E(X_{n+1}) = (1/2)\,p + 2\,(1-p) = 2 - 3\,p/2 = 1$, that is, when $p = 2/3$.

Task 4. We have $R_X(s,t) = \mathbf{E}\{X(s)X(t)\} = \mathbf{E}\{A^2\cos(U(s+t)+2\Theta)\} + \mathbf{E}\{A^2\cos(U(s-t))\} = \mathbf{E}\{\cos(U(s+t)+2\Theta)\} + \mathbf{E}\{\cos(U(s-t))\} = \mathbf{E}\{\cos(U(s-t))\} = \sin(s-t)/(s-t)$.

Task 5. Writing h for the impulse response of the filter and $\star$ for convolution the fact that $h\star N$ is zero-mean and independent of X (as N is) readily gives that $\mathbf{E}\{(Z(t)-X(t))^2\} = \mathbf{E}\{((h\star X)(t) + (h\star N)(t) - X(t))^2\} = \mathbf{E}\{(((h-\delta)\star X)(t) + (h\star N)(t))^2\} = \mathbf{E}\{((h-\delta)\star X)(t)^2 + (h\star N)(t)^2\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[|H(\omega)-1|^2 S_X(\omega) + |H(\omega)|^2 S_N(\omega)]\,d\omega = \ldots = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)S_N(\omega)/(S_X(\omega)+S_N(\omega))\,d\omega$.

Task 6. As T is the sum of the service time of the customer under consideration plus the service times of the $X(t)$ customers before that customer queueing for service, it follows that T is the sum of $X(t)+1$ independent exponentially distributed random variables with mean $1/\mu$. As $X(t)$ has the stationary distribution $P[X(t)=n] = (1-\lambda/\mu)\,(\lambda/\mu)^n$ for $n \geq 0$ it follows that $E[\mathrm{e}^{sT}] = \sum_{n=0}^{\infty} E[\mathrm{e}^{sT}|X(t)=n]\,P[X(t) =$

$n] = \sum_{n=0}^{\infty} E[\mathrm{e}^{s(T_1+\ldots+T_{n+1})}]\,(1-\lambda/\mu)\,(\lambda/\mu)^n = \sum_{n=0}^{\infty}(E[\mathrm{e}^{sT_1}])^{n+1}\,(1-\lambda/\mu)\,(\lambda/\mu)^n = \sum_{n=0}^{\infty}(\mu/(\mu-s))^{n+1}\,(1-\lambda/\mu)\,(\lambda/\mu)^n = \ldots = (\mu-\lambda)/(\mu-\lambda-s)$, were $T_1, T_2, \ldots$ are independent exponentially distributed random variables with mean $1/\mu$.