Fourier-transform / continuous time

1-dim Fourier-transform of $f: \mathbb{R} \to \mathbb{C}$

$$\hat{f}(\omega) = (\mathcal{F}f)(\omega) = \int_{-\infty}^{+\infty} e^{-j\omega x} f(x)\, dx \qquad (j^2 = -1)$$

1-dim inverse Fourier transform of $\hat{f}: \mathbb{R} \to \mathbb{C}$

$$f(x) = (\mathcal{F}^{-1}\hat{f})(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega x} \hat{f}(\omega)\, d\omega$$

n-dim Fourier transform of $f: \mathbb{R}^n \to \mathbb{C}$

$$\hat{f}(\omega_1, \ldots, \omega_n) = (\mathcal{F}f)(\vec{\omega}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-j(\omega_1 x_1 + \ldots + \omega_n x_n)} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

n-dim inverse Fourier transform of $\hat{f}: \mathbb{R}^n \to \mathbb{C}$

$$f(x_1, \ldots, x_n) = (\mathcal{F}^{-1}\hat{f})(x_1, \ldots, x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{j(\omega_1 x_1 + \ldots + \omega_n x_n)} \hat{f}(\omega_1, \ldots, \omega_n)\, d\omega_1 \ldots d\omega_n$$

example for $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$ we have

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu+j\omega\sigma^2)^2/2\sigma^2}\, dx}_{\text{integral of } \mathcal{N}(\ldots,\ldots)\text{-PDF which is } 1} e^{-j\omega\mu + \frac{1}{2}\omega^2\sigma^2} = e^{-j\omega\mu - \frac{1}{2}\omega^2\sigma^2}$$ ✱

example for $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$ we have

$$\hat{f}(\omega) = \int_0^{+\infty} \lambda e^{-(\lambda + j\omega)x}\, dx = \frac{\lambda}{\lambda + j\omega}$$ ✱✱

1-dim properties

$$\mathcal{F}(f(x - x_0))(\omega) = e^{-j\omega x_0} (\mathcal{F}f)(\omega)$$
$$\mathcal{F}(e^{j\omega_0 x} f(x))(\omega) = (\mathcal{F}f)(\omega - \omega_0)$$
$$\mathcal{F}(f(-x))(\omega) = (\mathcal{F}f)(-\omega)$$
$$\mathcal{F}(f'(x))(\omega) = j\omega (\mathcal{F}f)(\omega)$$

Fourier-transform/discrete time

1-dim Fourier transform of $f: \mathbb{Z} \to \mathbb{C}$

$$\hat{f}(\omega) = (\mathcal{F}f)(\omega) = \sum_{k=-\infty}^{+\infty} e^{-j\omega k} f(k)$$

1-dim inverse Fourier transform of $\hat{f}: \mathbb{R} \to \mathbb{C}$

$$f(k) = (\mathcal{F}^{-1}\hat{f})(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega k} \hat{f}(\omega)\, d\omega$$

properties $\mathcal{F}(f(k-k_0))(\omega) = e^{-j\omega k_0} (\mathcal{F}f)(\omega)$

$\mathcal{F}(e^{j\omega_0 k} f(k))(\omega) = (\mathcal{F}f)(\omega - \omega_0)$

$\mathcal{F}(f(-k))(\omega) = (\mathcal{F}f)(-\omega)$

example for $f(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 0, 1, 2, \ldots$ (0 otherwise)

$$\hat{f}(\omega) = \sum_{k=0}^{+\infty} \underbrace{(\lambda e^{-j\omega})^k \frac{1}{k!}}_{\text{Taylor-exp of exp}} e^{-\lambda} = e^{\lambda(e^{-j\omega} - 1)} \quad \#$$

δ-function

* discrete (time) Kronecker δ-function $\delta: \mathbb{Z} \to \{0, 1\}$ $\delta(k) = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$

* continuous (time) Dirac δ-function $\delta: \mathbb{R} \to \{0, \infty\}$ given by

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\, dx = f(0)$$ for $f: \mathbb{R} \to \mathbb{R}$ "smooth" and vanishing at infinity (with compact support)

$\delta(x)$ is the distribution derivative of the Heavyside step-function $\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$ as

$$\int_{-\infty}^{+\infty} f(x)\theta'(x)dx = \underbrace{[f(x)\theta(x)]_{-\infty}^{+\infty}}_{=0 \text{ since compact support}} - \int_{-\infty}^{+\infty} f'(x)\, dx = f(0)$$

$(\mathcal{F}\delta)(\omega) = 1$ (both in discrete and continuous time)

Convolution

continuous-time convolution between $f, g : \mathbb{R} \to \mathbb{R}$ is given by

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-y) g(y)\, dy = \int_{-\infty}^{+\infty} g(x-y) f(y)\, dy = (g * f)(x)$$

discrete time convolution between $f, g : \mathbb{Z} \to \mathbb{R}$ is given by

$$(f * g)(k) = \sum_{\ell=-\infty}^{+\infty} f(k-\ell) g(\ell) = \sum_{\ell=-\infty}^{+\infty} g(k-\ell) f(\ell) = (g * f)(k)$$

$$\mathcal{F}(f * g)(\omega) = (\mathcal{F} f)(\omega)\, (\mathcal{F} g)(\omega) \quad \text{(both in discrete and continuous time)}$$

example for $f(x) = g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$ we have

$$(f * g)(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \underbrace{\frac{1}{\sqrt{2\pi}\sigma} e^{-(\sqrt{2}y - x/\sqrt{2})^2/2\sigma^2}}_{\text{integrates to } 1/\sqrt{2}} e^{-\frac{x^2}{4\sigma^2}}\, dy = \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} e^{-\frac{x^2}{2(\sqrt{2}\sigma)^2}}$$

when $\mu = 0$ and in a similar fashion

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} e^{-\frac{(x-\mu)^2}{2(\sqrt{2}\sigma)^2}} \quad \text{for } \mu \text{ in general} \quad \#$$

Axioms of Probability Theory

S is the sample space of possible outcomes $s \in S$ of a random experiment, subsets of S (= collection of outcomes) are called events, a probability measure $P(A)$ is defined for events $A \subseteq S$ according to the axioms

$P(\emptyset) = 0$, $P(S) = 1$, ~~and~~ $P(A) \geq 0$ and $P(A \cup B) = P(A) + P(B)$ for $A \cap B = \emptyset$

From these axioms several additional rules follow:

$P(A^c) = P(\bar{A}) = 1 - P(A)$

$P(A) \leq P(B)$ for $A \subseteq B$

$P(A) \leq 1$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ when $A_i \cap A_j = \phi$ for $i \neq j$

Independence events A and B are independent if $P(A \cap B) = P(A)\,P(B)$

Conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Law of total probability $P(B) = \sum_{i=1}^{\infty} P(B|A_i)P(A_i)$
whenever $A_i \cap A_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} A_i = S$

Random variable function $X: S \to \mathbb{R}$ $X(S)$
function $(X, Y): S \to \mathbb{R}^2$ $(X(S), Y(S))$

Cumulative Distribution Function (CDF) ~~$F_X(x) = P(X$~~
$F_X(x) = P(X \leq x) = P(S \in S : X(S) \leq x)$
$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

Properties $0 \leq F_X(x) \leq 1$, $F_X(x_1) \leq F_X(x_2)$ for $x_1 \leq x_2$
$F_X(\infty) = 1$, $F_X(-\infty) = 0$, $P(a < X \leq b) = F_X(b) - F_X(a)$

$0 \leq F_{X,Y}(x,y) \leq 1$, $F_{X,Y}(x,y)$ is increasing in each argument
$F_{X,Y}(x, \infty) = F_X(x)$, $F_{X,Y}(\infty, y) = F_Y(y)$
$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1)$

Independence X and Y independent if $P(X \in A, Y \in B)$
$= P(X \in A)\,P(Y \in B)$ for all $A, B \subseteq \mathbb{R}$

X and Y independent $\Leftrightarrow F_{XY}(x,y) = F_X(x) F_Y(y)$

* **Continuous random variable** X or (X,Y) has uncountably infinitely many possible values, probability density function PDF

$$f_X(x) = \frac{d}{dx} F_X(x) \qquad f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

Properties $f_X(x) \geq 0$, $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$, $P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$

$P(X \in A) = \int_A f_X(x)\,dx$

$f_{X,Y}(x,y) \geq 0$, $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$, $P((X,Y) \in A) = \iint_A f_{XY}(x,y)\,dx\,dy$

$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$, $f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$

X and Y independent $\Leftrightarrow f_{XY}(x,y) = f_X(x) f_Y(y)$

example Gaussian $N(\mu, \sigma^2)$ r.v. $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$

example exponential r.v. with parameter λ $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

example uniform distribution over $[a,b]$ $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$

Expectation $E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx$ $E(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$

$E(g(X,Y)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$

Conditional PDF of X given that $Y = y$

$f_{X|Y}(x|y) = f_{X,Y}(x,y) / f_Y(y)$

$P(X \in A | Y = y) = \int_A f_{X|Y}(x|y)\,dx$ $P(X \in A) = \int_{-\infty}^{+\infty} P(X \in A | Y = y) f_Y(y)\,dy$

$E(X | Y = y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y)\,dx$ $E(X) = \int_{-\infty}^{+\infty} E(X | Y = y) f_Y(y)\,dy$

* **Discrete random variable** X or (X,Y) has finitely or countably infinitely many possible values, probability mass function PMF

$$P_X(x) = P(X=x) \qquad P_{X,Y}(x,y) = P(X=x, Y=y)$$

Properties $P_X(x) \in [0,1]$, $\sum_{\text{all } x} P_X(x) = 1$, $P(X \in A) = \sum_{x \in A} P_X(x)$

$P_{X,Y}(x,y) \in [0,1]$, $\sum_{\text{all } x,y} P_{X,Y}(x,y) = 1$, $P((X,Y) \in A) = \sum_{(x,y) \in A} P_{X,Y}(x,y)$

$P_X(x) = \sum_{\text{all } y} P_{X,Y}(x,y)$, $P_Y(y) = \sum_{\text{all } x} P_{X,Y}(x,y)$

X and Y independent $\iff P_{X,Y}(x,y) = P_X(x) P_Y(y)$

example Bernoulli r.v. X, possible values $\{0,1\}$, $P_X(0) = 1-p$, $P_X(1) = p$

example Binomial r.v. X, possible values $\{0, \dots, n\}$, $P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, \dots, n$, same as the sum of n independent Bernoulli r.v.'s

example Poisson r.v. X, possible values $\{0,1,\dots\} = \mathbb{N}$, $P_X(k) = \lambda^k e^{-\lambda}/k!$ for $k = 0,1,2,\dots$, $Po(\lambda_1) + Po(\lambda_2) = Po(\lambda_1 + \lambda_2)$, that is sum of two independent Po is Po

example geometric (waiting time) X, possible values $\{1,2,3,\dots\}$, $P_X(k) = (1-p)^{k-1} p$ for $k = 1,2,\dots$, number of Bernoulli trials that have to be made until first 1 occur...

Expectation $E(X) = \sum_{\text{all } x} x P_X(x)$ $\qquad E(g(X)) = \sum_{\text{all } x} g(x) P_X(x)$

$E(g(X,Y)) = \sum_{\text{all } x} \sum_{y} g(x,y) P_{X,Y}(x,y)$

Conditional PMF of X given that $Y=y$

$P_{X|Y}(x|y) = P_{X,Y}(x,y) / P_Y(y)$

$P(X \in A | Y=y) = \sum_{x \in A} P_{X|Y}(x|y)$ $\quad P(X \in A) = \sum_{\text{all } y} P(X \in A | Y=y) P_Y(y)$

$E(X | Y=y) = \sum_{\text{all } x} x P_{X|Y}(x|y)$ $\quad E(X) = \sum_{\text{all } y} E(X|Y=y) P_Y(y)$

Linearity of expectation $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$

Variance $Var(X) = \sigma_X^2 = E\left((X - E(X))^2\right) = E(X^2) - (E(X))^2$

Covariance $Cov(X,Y) = E\left((X-E(X))(Y-E(Y))\right) = E(XY) - E(X)E(Y)$

$Var(X) = Cov(X,X)$

X and Y are called uncorrelated if $Cov(X,Y)=0$

X and Y independent $\Rightarrow$ X and Y uncorrelated

Bilinearity of covariance $Cov\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j Cov(X_i, Y_j)$

in particular $Var\left(\sum_{i=1}^{n} a_i X_i\right) = Cov\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{n} a_j X_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j Cov(X_i, X_j)$

Bivariate normal distribution

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-p^2}} \exp\left(-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2p\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2}{2(1-p^2)}\right)$$

n-variate normal distribution see page 111 in Hsu
(page 89 in first edition)

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det K}} \exp\left(-\frac{1}{2}(x-\mu)^T K^{-1}(x-\mu)\right)$$ where

n/n-matrix

column matrices

$\mu_i = E(X_i)$ and $K_{ij} = Cov(X_i, X_j)$

Functions of random variables

$Y = g(X)$, $g: \mathbb{R} \to \mathbb{R}$ increasing

$$\Rightarrow f_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(X \le g^{-1}(y)) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

$$Z = g(X, Y) \quad f_Z(z) = \frac{d}{dz} P(g(X,Y) \le z) = \frac{d}{dz} \iint_{g(x,y) \le z} f_{X,Y}(x,y)\,dx\,dy$$

Characteristic function: CHF of continuous random variable(s)

$$\Psi_X(\omega) = E(e^{j\omega X}) = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x)\,dx \quad | \quad f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-j\omega x} \Psi_X(\omega)\,d\omega$$

$$\Psi_{X,Y}(\omega_1, \omega_2) = E(e^{j(\omega_1 X + \omega_2 Y)}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{j(\omega_1 x + \omega_2 y)} f_{X,Y}(x,y)\,dx\,dy$$

$$f_{X,Y}(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^2} e^{-j(\omega_1 x + \omega_2 y)} \Psi_{X,Y}(\omega_1, \omega_2)\,d\omega_1\,d\omega_2$$