MSG800/MVE170 Basic Stochastic Processes

Written exam Saturday 26 April 2014 8.30–12.30 am

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated.

THE GRADING of this exam will most likely be done on (but not before) Tuesday 7 May.

GOOD LUCK!

Task 1. Consider a time homogeneous Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space (/set of possible values) E, initial distribution $\mathbf{p}(0)$ and transition probability matrix P given by

$$E = \{0, 1, 2\}, \quad \mathbf{p}(0) = [1/3 \;\; 1/3 \;\; 1/3] \quad \text{and} \quad P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \end{bmatrix},$$

respectively. Write a computer program that by means of simulation finds (an approximation of) the expected vaule of the number of time units it takes until the chain for the first time has spent two consequtive time units (two time units in row, that is) at state 0. **(5 points)**

Task 2. Find the probability $\mathbf{P}\{X(5)-X(3) > 1\}$ [expressed in terms of the standard normal distribution function $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-x^2/2}\, dx$] for a WSS normal continuous time process $X(t)$ with autocorrelation function $R_X(t, t+\tau) = \mathrm{e}^{-|\tau|}$. **(5 points)**

Task 3. Let $\{W(t),\, t \geq 0\}$ be a Wiener process with $\mathbf{E}\{W(1)^2\} = 1$. For what selection of constants/coefficients $a, b \in \mathbb{R}$ is the process $X(t) = W(t)^2 + a\, W(t) + b\, t$ a martingale with respect to the filtration $\mathcal{F}_s$ containing information of all values $\{W(r),\, r \in [0, s]\}$ of the Wiener process up to time s? **(5 points)**

Task 4. A WSS continuous time process $X(t)$ with power spectral density $S_X(\omega)$ is input to an LTI system with frequency response $H(\omega)$ and output $Y(t)$. Find an expression for the cross power spectral density $S_{Y,Y'}(\omega)$ between $Y(t)$ and its derivative process $Y'(t)$ expressed in terms of $S_X(\omega)$ and $H(\omega)$. **(5 points)**

Task 5. Students shall do (solve) only one of the following to versions of Task 5. It is not permitted to hand in solutions to both versions - students who do so anyway will earn the score from the poorest of their two solutions, not from the best (: ... .

Task 5 - version 1. Consider a queueing system which basically is M/M/2/2 with $\lambda = \mu = 1$ except that the second server is half-ill so that its/her/his service time has changed from being exponentially distributed with parameter 1 (/mean 1) to being exponentially distributed with parameter 1/2 (/mean 2). The first (healthy) server is always employed when a customer arrives to an empty queueing system so that the second server is employed only for customers arriving when the first server is busy. Whenever a customer is finished being served after both servers has been busy/occupied, then it is the quickest (first) server that continues the service of the remaining customer in the system. Find the stationary distribution (steady state probabilities) $(p_0\ p_1\ p_2)$ of this modified (for a server being half-ill) queueing system. **(5 points)**

Task 5 - version 2. Let $\{M(t),\, t \geq 0\}$ and $\{N(t),\, t \geq 0\}$ be independent unit rate (/unit intensity) Poisson processes. Find the generator G of the continuous time Markov chain $X(t) = M(t) - N(t)$. **(5 points)**

Task 6. A busy period is the time it takes for a queueing system to become empty when started with one customer in the system. (Or in other words, the length of a time period during which the system is non-empty.) Find the expected value of a busy period of an M/M/2/2 queueing system with $\lambda = \mu = 1$. **(5 points)**

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Solutions to written exam Saturday 26 April 2014

Task 1. Here is a Mathematica program that solves the task

```
For[i=1; Result={}, i<=100000, i++,
    X = Floor[Random[UniformDistribution[{0,3}]]];
    AtZero = 0; Wait = 0;
    While[AtZero<1, Wait=Wait+1;
        Y = {Move = Random[UniformDistribution[{0,1}]],
        If[X==0, If[Move<=1/2, Z=0, If[Move<=5/6, Z=1, Z=2]],
        If[X==1, If[Move<=2/3, Z=1, Z=2],
        If[Move<=1/2, Z=0, Z=2]]], Z}[[3]];
        If[X==0&&Y==0, AtZero=AtZero+1]]; AppendTo[Result,Wait]]
N[Mean[Result]]
```

Task 2. As $X(5)-X(3)$ is zero-mean normal distributed with variance $\mathbf{E}\{(X(5)-X(3))^2\} = R_X(5,5)-2\,R_X(3,5)+R_X(3,3) = 2\,(1-\mathrm{e}^{-2})$ we have $\mathbf{P}\{X(5)-X(3)>1\} = 1-\Phi(1/\sqrt{2\,(1-\mathrm{e}^{-2})}\,)$.

Task 3. As $\mathbf{E}\{X(t)|\mathcal{F}_s\} = \mathbf{E}\{(W(t)-W(s))^2|\mathcal{F}_s\} + 2\,\mathbf{E}\{(W(t)-W(s))\,W(s)|\mathcal{F}_s\} + \mathbf{E}\{W(s)^2|\mathcal{F}_s\} + a\,\mathbf{E}\{W(t)-W(s)|\mathcal{F}_s\} + a\,\mathbf{E}\{W(s)|\mathcal{F}_s\} + b\,t = \mathbf{E}\{(W(t)-W(s))^2\} + 2\,\mathbf{E}\{(W(t)-W(s))\}\,W(s) + W(s)^2 + a\,\mathbf{E}\{W(t)-W(s)\} + a\,W(s) + b\,t = t-s+0+W(s)^2+0+a\,W(s)+b\,t = W(s)^2+a\,W(s)-s+(1+b)\,t$ for $s\leq t$, we see that a can be any real number while we must have $b=-1$.

Task 4. According to Equation 6.63 in Hsu's book we have $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$. And from the solution to Exercise 6.32 (a) in Hsu's book we then see that $S_{Y,Y'}(\omega) = j\omega\,S_Y(\omega) = j\omega\,|H(\omega)|^2 S_X(\omega)$.

Task 5 - version 1. By results for the birth-death process in Section 6.11 in the book by G&S or by the discussion of the same topic in Section 9.3 in the book by Hsu we see (with the notation of Hsu) that

$$p_1 = \frac{a_0}{d_1}\,p_0 \quad \text{and} \quad p_2 = \frac{a_0 a_1}{d_1 d_2}\,p_0 \quad \text{with} \quad a_0 = a_1 = d_1 = 1 \quad \text{and} \quad d_2 = 3/2,$$

so that (by adding the equation $p_0+p_1+p_1 = 1$ and solving the resulting system of equations) $(p_0\ \ p_1\ \ p_2) = (3/8\ \ 3/8\ \ 2/8)$.

Task 5 - version 2. What we have here is a birth-death process (see Section 6.11 in the book by G&S) modified to have both positive and negative integer values and

$\lambda_n = \mu_n = 1$ for all such $n \in \mathbb{Z}$. Therefore the generator is given by

$$G = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & -2 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & -2 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & -2 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & -2 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & -2 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Task 6. Letting $E(1)$ denote the searched for expectation and $E(2)$ the expected value of the time it takes for the queueing system to become empty when started with two customer in the system, we see that

$$E(1) = 1/2 + 1/2 \times E(2) \quad \text{and} \quad E(2) = 1/2 + E(1),$$

so that (the search for) $E(1) = 3/2$.