MSG800/MVE170 Basic Stochastic Processes

Written exam Friday 26 August 2016 $2-6$ pm

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Task 1. Find the autocorrelation function of the discrete time random process $X(n) = \sum_{i=1}^{m} a\cos(\omega n+\Theta_i)$ for $n\in\mathbb{Z}$, where $a,\omega\in\mathbb{R}$ are constants and $\Theta_1,\ldots,\Theta_n$ are independent random variables that are all uniformly distributed over $[0,2\pi]$. [Hint: $2\cos(x)\cos(y)=\cos(x+y)+\cos(x-y)$.] **(5 points)**

Task 2. Prove that for a discrete time homogeneous Markov chain X_n, $n\in\mathbb{N}$, it holds that $P(X_{n+1}=k_1,\ldots,X_{n+m}=k_m|X_0=i_0,\ldots,X_{n-1}=i_{n-1},X_n=i) = P(X_1=k_1,\ldots,X_m=k_m|X_0=i)$ for $n,m\geq 1$ and all $k_1,\ldots,k_m,i_0,\ldots,i_{n-1},i$. **(5 points)**

Task 3. A continuous time Markov chain has state space S and generator G given by

$$S=\{0,1,2\} \qquad \text{and} \qquad G=\begin{bmatrix}-1&0&1\\1&-1&0\\1&1&-2\end{bmatrix},$$

respectively. Find the characteristic function for the random time it takes the chain to move from state 2 to state 0. **(5 points)**

Task 4. Let $M(t)=(X(t)-\lambda t)^2-\lambda t$ for $t\geq 0$, where $X(t)$ is a Poisson process with intensity $\lambda>0$. Show that $M(t)$ is a martingale with respect to the information (filtration) $F_s=\sigma(X(r),\, r\leq s)$ of historic values of the Poisson process. **(5 points)**

Task 5. Let $N(t)$, $t\in\mathbb{R}$, be a continuous time WSS white noise process with power spectral density $S_N(f)=N_0$ for $f\in\mathbb{R}$. Find the autocorrelation function $R_W(t_1,t_2)$ for the integrated white noise process $W(t)=\int_0^t N(r)\,dr$, $t\geq 0$. **(5 points)**

Task 6. Consider an M/M/1/3 queueing system with $\lambda=\mu=1$ that is started full at time zero. Show how one can find an approximation of the probability that the queueing system gets empty some time during the first three time units of service by means of computer simulations. **(5 points)**

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Solutions to written exam 26 August 2016

Task 1. $R_X(n,n{+}\tau) = E[X(n)X(n{+}\tau)] = \sum_{i=1}^m \sum_{j=1}^m a^2\, E[\cos(\omega n{+}\Theta_i)\cos(\omega(n{+}\tau){+}\Theta_j)] = (a^2/2)\sum_{i=1}^m \sum_{j=1}^m E[\cos(\omega(2n{+}\tau)+\Theta_i+\Theta_j)] + (a^2/2)\sum_{i=1}^m \sum_{j=1}^m E[\cos(\omega\tau - \Theta_i+\Theta_j)] = m\,(a^2/2)\cos(\omega\tau)$.

Task 2. Both probabilities are equal to $P_{ik_1}P_{k_1k_2}\cdot\ldots\cdot P_{k_{m-1}k_m}$.

Task 3. $\Psi(\omega) = E[\mathrm{e}^{j\omega\exp(2)}]\big(1/2+(1/2)\cdot E[\mathrm{e}^{j\omega\exp(1)}]\big)$.

Task 4. $E[M(t)|F_s] = E[(X(t){-}\lambda t)^2{-}\lambda t|F_s] = E[(X(t){-}X(s){+}X(s){-}\lambda t)^2|F_s]-\lambda t = E[(X(t)-X(s))^2|F_s] + 2\,E[(X(t)-X(s))\,(X(s)-\lambda t)|F_s] + E[(X(s)-\lambda t)^2|F_s] - \lambda t = E[(X(t){-}X(s))^2] + 2\,E[X(t){-}X(s)]\,(X(s){-}\lambda t) + (X(s){-}\lambda t)^2 - \lambda t = (\lambda(t{-}s))^2+\lambda(t{-}s) + 2\,\lambda(t{-}s)\,(X(s){-}\lambda t) + (X(s){-}\lambda t)^2 - \lambda t = \ldots = (X(s){-}\lambda s)^2{-}\lambda s = M_s$ for $s\le t$.

Task 5. We have $R_W(t_1,t_2) = E[(\int_0^{t_1} N(r)\,dr)\,(\int_0^{t_2} N(s)\,ds)] = \int_0^{t_1}\int_0^{t_2} E[N(r)N(s)]\,dr\,ds = \int_0^{t_1}\int_0^{t_2} N_0\,\delta(s{-}r)\,drds = \int_0^{\min(t_1,t_2)} N_0\,dr = N_0\min(t_1,t_2)$.

Task 6.

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In[1]:= For[i=1; Reps=10000; SumSucc=0, i<=Reps, i++, T=0; X=3; Succ=0;
          While[T<=3,
            If[X==0, Succ=1; T=T+Random[ExponentialDistribution[1]]; X=1,
            If[X==1, T=T+Random[ExponentialDistribution[2]];
              If[Random[]<=1/2, X=0, X=2],
            If[X==2, T=T+Random[ExponentialDistribution[2]];
              If[Random[]<=1/2, X=1, X=3],
            If[X==3, T=T+Random[ExponentialDistribution[1]]; X=2]]]]];
          SumSucc=SumSucc+Succ]; N[SumSucc/Reps]

Out[1]= 0.3282
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