MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 8 January 2018 $2-6$ pm

TEACHER AND JOUR: Patrik Albin telephone 031 7723512.

AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Consider an M/M/2/3 queueing system with parameters $\lambda = \mu = 1$ that is started up empty at time 0. Write a computer programme that by means of stochastic simulation finds an approximation of the probability that there are three consequtive arrivals of new customers (so that the systems gets full) before any customer is being finished served. [**Note:** An analytic calculation gives no points as it is a computer programme that is asked for.] **(5 points)**

Task 2. Let $\{W(n)\}_{n\in\mathbb{Z}}$ be a discrete time white noise process with $\mathbf{E}\{W(n)^2\} = 1$. Find the autocorrelation function $R_X(k,l)$ of the process $X(k) = W(k) + W(k-1) + W(k-2)$. **(5 points)**

Task 3. Give four examples of discrete time random processes where the first one is neither a Markov chain or a martingale, the second a Markov chain but not a martingale, the third a martingale but not a Markov chain and the fourth both a Markov chain and a martingale. **(5 points)**

Task 4. Calculate $\mathbf{P}\{X(0)+X(1) \geq Y(1)+Y(2)+1\}$ when $X(t)$ and $Y(t)$ are independent zero-mean continuous time WSS Gaussian processes with autocorrelation functions $R_X(\tau) = \mathrm{e}^{-|\tau|}$ and $R_Y(\tau) = 1/(1+\tau^2)$. **(5 points)**

Task 5. For a continuous time LTI system with WSS input and output $X(t)$ and $Y(t)$, respectively, with PSD's $S_X(\omega)$ and $S_Y(\omega)$, respectively, and with frequency response $H(\omega)$ it holds that $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$. Prove this fact. **(5 points)**

Task 6. Consider a continuous time Markov chain $\{X(t)\}_{t\geq 0}$ with state space $\{0,1\}$ and generator $G = \left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ that is started at time 0 according to its stationary distribution. Describe how $\mathbf{Cov}\{X(s), X(t)\}$ can be calculated. **Note:** The actual calculation does not have to be performed but only a working scheme to carry it out have to be described. **(5 points)**

MSG800/MVE170 Solutions to written exam 8 January

Task 1.

```
In[1]:= Rep=10000;

In[2]:= For[i=1; Succ=0, i<=Rep, i++,
  If[Random[ExponentialDistribution[1]]
    < Random[ExponentialDistribution[1]],
    If[Random[ExponentialDistribution[1]]
      < Random[ExponentialDistribution[2]],
        Succ = Succ + 1]]]

In[3]:= N[Succ/Rep]

Out[3]= 0.1669
```

Task 2. It is easy to see that $X(k)$ is WSS with $R_X(k) = R_X(l, k+l) = 3\,\delta(k) + 2\,\delta(k-1) + 2\,\delta(k+1) + \delta(k-2) + \delta(k+2)$.

Task 3. Take $W(k)$ as in Task 2 and let $X(0) = W(0)$, $X(1) = X(0) + W(1)$ and $X(n) = X(n-1) + X(n-2) + W(n)$ for $n \geq 2$. Then $X(k)$ is a martingale but not Markov while $Y(k) = X(k) + k$ i neither martingale or Markov. The process $Z(k) = 0$ is both martingale and Markov while $Z(k) + k$ is Markov but not martingale.

Task 4. The probability is $1 - \Phi(1/\sigma)$ where $\sigma^2 = \mathbf{Var}\{X(0)+X(1)-Y(1)-Y(2)\} = \mathbf{Var}\{X(0)+X(1)\} + \mathbf{Var}\{Y(1)+Y(2)\} = 2\,(R_X(0)+R_X(1)) + 2\,(R_Y(0)+R_Y(1))$.

Task 5. See Chapter 6 in Hsu's book.

Task 6. As $\pi = (1/2\ 1/2)$ we have $\mathbf{Cov}\{X(s), X(t)\} = \mathbf{E}\{X(s)X(t)\} - 1/4 = \mathbf{P}\{X(s) = X(t) = 1\} - 1/4 = p_{11}(|t-s|)\,\pi_1 - 1/4 = p_{11}(|t-s|)/2 - 1/4$ where $p_{11}(t) = (P_t)_{11} = (\mathrm{e}^{tG})_{11}$ can be calculated e.g., by diagonalizing G.