MSG800/MVE170 Basic Stochastic Processes

Written exam 28 August 2018 2–6 PM

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Task 1. A pair of WSS random processes $X(t)$ and $Y(t)$ are called jointly WSS if $\mathbf{E}\{X(t)Y(t+\tau)\}$ do no depend on t (but on τ only). Find two WSS random processes $X(t)$ and $Y(t)$ that are not jointly WSS. **(5 points)**

Task 2. A discrete time Markov chain $\{X(n)\}_{n=0}^{\infty}$ has state space E, initial distribution $\mathbf{p}(0)$ and transition probability matrix P given by

$$E = \{0, 1, 2\}, \qquad \mathbf{p}(0) = [1/6 \;\; 1/3 \;\; 1/2] \qquad \text{and} \qquad P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/4 & 1/4 \\ 2/3 & 1/6 & 1/6 \end{bmatrix},$$

respectively. Find the expected value $\mathbf{E}\{T\}$ of the time $T = \min\{n \geq 1 : X(n) \neq X(0)\}$ the chain spends in it first state. [Hint: For a random variable ξ with probability mass function $\mathbf{P}\{\xi = k\} = (1-p)^{k-1}p$ for $k = 1, 2, 3, \ldots$ it holds that $\mathbf{E}\{\xi\} = 1/p$.].

(5 points)

Task 3. A zero-mean WSS random signal (process) $\{X(t)\}_{t\in\mathbb{R}}$ with power spectral density $S_X(\omega)$ is transmitted on a noisy channel where an independent zero-mean random noise process $\{N(t)\}_{t\in\mathbb{R}}$ with power spectral density $S_N(\omega)$ is added to the signal so that the recived signal (process) is $Y(t) = X(t)+N(t)$. In an attempt to approximately reconstruct the originally transmitted signal $X(t)$ the recived signal $Y(t)$ is processed as insignal to an LTI system with frequency response $H(\omega) = S_X(\omega)/(S_X(\omega)+S_N(\omega))$ and output $Z(t)$. Show that the mean-square reconstruction error $\mathbf{E}\{(Z(t)-X(t))^2\}$ is given by $\int_{-\infty}^{\infty} S_X(\omega)S_N(\omega)/(S_X(\omega)+S_N(\omega))\, d\omega$. **(5 points)**

Task 4. Let $\{W(t)\}_{t\geq 0}$ be a Wiener process [that is, a random process with stationary and independent zero-mean normal distributed increments and $W(0) = 0$] such that $\mathbf{E}\{W(1)^2\} = 1$. Find a function $f : [0,\infty) \to \mathbb{R}$ such that $\{W(t)^3 + f(t)W(t)\}_{t\geq 0}$ is a martingale with respect to the filtration containing all information of the history of the process $F_s = \sigma(W(r) : r \in [0, s])$. **(5 points)**

Task 5. An M/M/3/6 queueing system with $\lambda = \mu = 1$ is started up empty. Find the probability that there are six straight arrivals to the queueing system (so that the system gets full) without a single customer being finished with service meanwhile.

(5 points)

Task 6. Consider a continuous time random walk on the eight corners of a cube. The walk stays a unit mean exponentially distributed time at a corner after which one of the three neighbouring corners is selected as the next occupation of the random walk with equal probability 1/3. Find a set of equations from which the characteristic function for the time it takes the walk to move from the starting corner to the corner most distant from the starting corner (that is, three edges away). NOTE: The equations need not be solved. **(5 points)**

MSG800/MVE170 Solutions to written exam August 2018

Task 1. It is enough to put $Y(t)=X(-t)$ to arrive at $\mathbf{E}\{X(t)Y(t+\tau)\}=\mathbf{E}\{X(t)X(-t-\tau)\}=R_X(2t+\tau)$ which will depend on t for all non-degenerate WSS processes $X(t)$.

Task 2. $E[T]=(1/6)\cdot(1/(2/3))+(1/3)\cdot(1/(3/4))+(1/2)\cdot(1/(5/6))$.

Task 3. Writing $h(t)$ for the impulse response corresponding to the frequency response $H(\omega)$ [noting that $H(\omega)$ is real and positive] we have $\mathbf{E}\{(Z(t)-X(t))^2\}=\mathbf{E}\{((h\star X)(t)+(h\star N)(t)-X(t))^2\} = \mathbf{E}\{(h\star X)(t)^2+(h\star N)(t)^2+X(t)^2+2(h\star X)(t)(h\star N)(t)-2(h\star X)(t)X(t)-2(h\star N)(t)X(t)\} = \int_{-\infty}^{\infty}H(\omega)^2S_X(\omega)\,d\omega+\int_{-\infty}^{\infty}H(\omega)^2S_N(\omega)\,d\omega+\int_{-\infty}^{\infty}S_X(\omega)\,d\omega+0-2\int_{-\infty}^{\infty}H(\omega)S_X(\omega)\,d\omega-0=\int_{-\infty}^{\infty}S_X(\omega)S_N(\omega)/(S_X(\omega)+S_N(\omega))\,d\omega$.

Task 4. As $\mathbf{E}\{W(t)^3|F_s\}=\mathbf{E}\{(W(t)-W(s)+W(s))^3|F_s\}=\mathbf{E}\{(W(t)-W(s))^3|F_s\}+3\,\mathbf{E}\{(W(t)–W(s))^2W(s)\,|F_s\}+3\,\mathbf{E}\{(W(t)–W(s))\,W(s)^2|F_s\}+\mathbf{E}\{W(s)^3|F_s\}=\mathbf{E}\{(W(t)-W(s))^3\}+3W(s)\,\mathbf{E}\{(W(t)–W(s))^2\}+3W(s)^2\,\mathbf{E}\{W(t)–W(s)\}+W(s)^3=0+3W(s)\times(t-s)+0+W(s)^3$ for $0\le s\le t$ so we must take $f(t)=-3t$.

Task 5. By symmetry reasons the probability is $1\cdot(1/2)\cdot(1/3)\cdot(1/4)\cdot(1/4)\cdot(1/4)$.

Task 6. Let $\Psi_1(\omega)$, $\Psi_2(\omega)$ and $\Psi_3(\omega)$ denote the characteristic function for the time it takes to move to a particular one of the three corners one edge away from the starting corner, the characteristic function for the time it takes to move to a particular one of the three corners two edges away from the starting corner, and the sought after characteristic function for the time it takes to move to the corner three edges away from the starting corner, respectively. Then, with obvious notation, it holds that

$$\begin{cases}\Psi_1(\omega)=\Psi_{\exp(1)}(\omega)\,\big((1/3)+(2/3)\cdot\Psi_2(\omega)\big)\\ \Psi_2(\omega)=\Psi_{\exp(1)}(\omega)\,\big((2/3)\cdot\Psi_1(\omega)+(1/3)\cdot\Psi_3(\omega)\big)\\ \Psi_3(\omega)=\Psi_{\exp(1)}(\omega)\,\Psi_2(\omega)\end{cases}.$$