MSG800/MVE170 Basic Stochastic Processes

Written exam 14 January 2019 2–6 PM

TEACHER AND JOUR/ROND: Patrik Albin, telephone 031 7723512.

AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) <u>or</u> Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Calculate $P[X(2)=2\,|\,X(1)=1,\,X(3)=3]$ for a unit rate/intensity Poisson process $\{X(t)\}_{t\geq 0}$. **(5 points)**

Task 2. A discrete time Markov chain $\{X(n)\}_{n=0}^{\infty}$ has state space E and transition probability matrix P given by

$$E=\{0,1,2\} \qquad \text{and} \qquad P=\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/4 & 1/4 \\ 2/3 & 1/6 & 1/6 \end{bmatrix},$$

respectively. Find the conditional expected value $\mathbf{E}\{T_2|\,X(0)=0\}$ of the random variable $T_2=\min\{n\geq 0: X(n)=2\}$. [HINT: For a random variable ξ with probability mass function $\mathbf{P}\{\xi=k\}=(1-p)^{k-1}p$ for $k=1,2,3,\ldots$ it holds that $\mathbf{E}\{\xi\}=1/p$.]

(5 points)

Task 3. Let $\{W(t)\}_{t\geq 0}$ be a Wiener process [that is, a random process with stationary and independent zero-mean normal distributed increments and $W(0)=0$] such that $\mathbf{E}\{W(1)^2\}=1$. Show that $\{\int_0^t W(r)\,dr - tW(t)\}_{t\geq 0}$ is a martingale with respect to the filtration containing all information of the history of the process $F_s=\sigma(W(r): r\in[0,s])$.

(5 points)

Task 4. Consider an M/M/3/4 queueing system with $\lambda=\mu=1$. A busy time B for the system is the random time it takes from a customer comes into the system when it is empty until the system gets empty again. Write a computer programe that by means of stochastic simulation finds an approximative value of the probability $\mathbf{P}\{B>4\}$.

(5 points)

Task 5. For an LTI system the two axioms about linearity and time invariance ensure that if the system has impulse response $h(t)$ then the relation between the insignal $x(t)$ and the outsignal $y(t)$ of the system is $y(t)=(h\star x)(t)$. Prove this fact in continuous time. [HINT: Note that $x(t)=\int_{-\infty}^{\infty} x(s)\delta(t-s)\,ds$.] **(5 points)**

Task 6. A continuous time Markov chain $\{X(t)\}_{t\geq 0}$ has state space S, initial distribution $\boldsymbol{\mu}^{(0)}$ and generator G given by

$$S=\{0,1\}, \qquad \boldsymbol{\mu}^{(0)}=(1/2 \;\; 1/2) \qquad \text{and} \qquad G=\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

respectively. Calculate the autocorrelation function $\mathbf{E}\{X(t)X(t+\tau)\}$ for $t,\tau\geq 0$. [HINT: Note that one can manufacture $X(t)$ as $X(t)=\frac{1}{2}\,(1-(-1)^{X(0)+Y(t)})$, where $\{Y(t)\}_{t\geq 0}$ is a unit rate/intensity Poisson process independent of $X(0)$.] **(5 points)**

MSG800/MVE170 Solutions to written exam January 2019

Task 1. $P[X(2)=2\,|\,X(1)=1,\,X(3)=3] = P[X(1)=1,\,X(2)=2,\,X(3)=3]/P[X(1)=1,\,X(3)=3] = P[X(1)=1,\,X(2)-X(1)=1,\,X(3)-X(2)=1]/P[X(1)=1,\,X(3)-X(1)=2] = (\mathrm{e}^{-1})^3/(\mathrm{e}^{-1}\,\frac{2^2}{2!}\,\mathrm{e}^{-2}) = \frac{1}{2}$.

Task 2. The sought after expectation E satisfies the equation $E = 1/(2/3) + (1/2)\cdot(1/(3/4) + (2/3)\cdot E)$ giving $E = 13/4$.

Task 3. $\mathbf{E}\{\int_0^t W(r)\,dr - tW(t)\,|\,F_s\} = \mathbf{E}\{\int_s^t (W(r)-W(s))\,dr\,|\,F_s\} + (t-s)W(s) + \mathbf{E}\{\int_0^s W(r)\,dr\,|\,F_s\} - \mathbf{E}\{t(W(t)-W(s))\,|\,F_s\} - \mathbf{E}\{tW(s)\,|\,F_s\} = \mathbf{E}\{\int_s^t (W(r)-W(s))\,dr\} + (t-s)W(s) + \int_0^s W(r)\,dr - t\,\mathbf{E}\{W(t)-W(s)\} - tW(s) = \int_0^s W(r)\,dr - sW(s)$ for $0 \le s \le t$.

Task 4.

```
In[1]:= {repetitions, count} = {1000000, 0};
   For[i=1, i<=repetitions, i++, X=1; time=0;
     While[X>0 && time<=4,
       If[X==1, time=time+Random[ExponentialDistribution[2]];
         If[Random[UniformDistribution[{0,1}]]<1/2, X=0, X=2],
       If[X==2, time=time+Random[ExponentialDistribution[3]];
         If[Random[UniformDistribution[{0,1}]]<2/3, X=1, X=3],
       If[X==3, time=time+Random[ExponentialDistribution[4]];
         If[Random[UniformDistribution[{0,1}]]<3/4, X=2, X=4],
       If[X==4, time=time+Random[ExponentialDistribution[3]]; X=3]]]]];
   If[time>4, count=count+1]];
   N[count/repetitions]
Out[1]= 0.117121
```

Task 5. With the notation $y = Tx$ we have $(Tx)(t) = T\int_{-\infty}^{\infty}\delta(t-s)x(s)\,ds =$ [linearity] $= \int_{-\infty}^{\infty} x(s)(T\delta)(t-s)\,ds =$ [time invariance] $= \int_{-\infty}^{\infty} x(s)h(t-s)\,ds = (h\star x)(t)$.

Task 6. $\mathbf{E}\{X(t)X(t+\tau)\} = \mathbf{P}\{X(t)=1,\,X(t+\tau)=1\} = \mathbf{P}\{X(0)+Y(t)=\text{odd},\,Y(t+\tau)-Y(t)=\text{even}\} = (1/2)\cdot\mathbf{P}\{\text{Po}(\tau)\text{ is even}\} = \frac{1}{2}\sum_{k=0}^{\infty}\tau^{2k}\mathrm{e}^{-\tau}/((2k)!)$.