MSG800/MVE170 Basic Stochastic Processes

Written exam Wednesday 24 April 2019 8.30–12.30 AM

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $\{X(t)\}_{t\geq 0}$ be a unit rate/intensity Poisson process. Find a function $f:[0,\infty)\to\mathbb{R}$ such that $\{f(t)2^{X(t)}\}_{t\geq 0}$ is a martingale with respect to the filtration containing all information of the history of the process $F_s=\sigma(X(r):r\in[0,s])$.

(5 points)

Task 2. A discrete time Markov chain $\{X(k)\}_{k=0}^{\infty}$ has state space E, initial distribution $\boldsymbol{p}(0)$ and transition probability matrix P given by

$$E=\{0,1,2\},\qquad \boldsymbol{p}(0)=[1/3\quad 1/3\quad 1/3]\qquad\text{and}\qquad P=\begin{bmatrix}1/3 & 1/3 & 1/3\\ 1/3 & 1/3 & 1/3\\ 1/3 & 1/3 & 1/3\end{bmatrix},$$

respectively. Calculate the autocorrelation function $\mathbf{E}\{X(k)X(k+n)\}$ for $k,n\in\mathbb{N}$ with $n\geq 1$. **(5 points)**

Task 3. Give example of two random processes $X(t)$ and $Y(t)$ that have common mean function $E[X(t)]=E[Y(t)]$ and common autocorrelation function $E[X(s)X(t)]=E[Y(s)Y(t)]$ for all s and t but that are different processes (that is, they have different probabilistic properties). **(5 points)**

Task 4. Consider an M/M/3/4 queueing system with $\lambda=\mu=1$ and let $X(t)$ denote the total number of customers in the queueing system at time $t\geq 0$. The queueing system is started empty at time zero $X(0)=0$. system. Write a computer programe that by means of stochastic simulation finds an approximative value of the probability $\mathbf{P}\{\max_{0\leq t\leq 4}X(t)=4\}$. **(5 points)**

Task 5. Prove that the power spectral density $S_X(\omega)$ of a continuous time WSS random process $X(t)$ is always both real valued and symmetric. **(5 points)**

Task 6. Is it possible for a continuous time Markov chain with only two possible values not to have a stationary distribution? **(5 points)**

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Task 1. $\mathbf{E}\{2^{X(t)}|F_s\} = \mathbf{E}\{2^{X(s)}2^{X(t)-X(s)}|F_s\} = 2^{X(s)}\,\mathbf{E}\{2^{X(t)-X(s)}\} = 2^{X(s)}\sum_{k=0}^{\infty} 2^k \times (t-s)^k \mathrm{e}^{-(t-s)}/(k!) = 2^{X(s)}\,\mathrm{e}^{t-s}$ so that we must take $f(t) = \mathrm{e}^{-t}$.

Task 2. As $\boldsymbol{p}(k) = \boldsymbol{p}(0)$ is the stationary distribution and $P^n = P$ for $n \geq 1$ we have $\mathbf{E}\{X(k)X(k+n)\} = 1\cdot 1\cdot p_1(k)\,p_{11}^{(n)} + 1\cdot 2\cdot p_1(k)\,p_{12}^{(n)} + 2\cdot 1\cdot p_2(k)\,p_{21}^{(n)} + 2\cdot 2\cdot p_2(k)\,p_{22}^{(n)} = \frac{1}{9} + \frac{2}{9} + \frac{2}{9} + \frac{4}{9} = 1$ for $n \geq 1$.

Task 3. Let $\{X(t)\}_{t\in\mathbb{Z}}$ be independent standard normal random variables while $\{Y(t)\}_{t\in\mathbb{Z}}$ are independent random variables with $P(Y(t)= 1) = P(Y(t)=-1) = 1/2$.

Task 4.

```
In[1]:= {repetitions, count} = {1000000, 0};
   For[i=1, i<=repetitions, i++, X=0; time=0;
     While[X<4 && time<=4,
       If[X==0, time=time+Random[ExponentialDistribution[1]]; X=1,
       If[X==1, time=time+Random[ExponentialDistribution[2]];
         If[Random[UniformDistribution[{0,1}]]<1/2, X=0, X=2],
       If[X==2, time=time+Random[ExponentialDistribution[3]];
         If[Random[UniformDistribution[{0,1}]]<2/3, X=1, X=3],
       If[X==3, time=time+Random[ExponentialDistribution[4]];
         If[Random[UniformDistribution[{0,1}]]<3/4, X=2, X=4]]]]]];
   If[time<=4, count=count+1]];
   N[count/repetitions]
```

Task 5. By symmetry of R_X we have $S_X(\omega) = (\mathcal{F}R_X)(\omega) = (\mathcal{F}R_X(-\cdot))(\omega) = (\mathcal{F}R_X)(-\omega) = S_X(-\omega)$ so that $S_X(\omega)$ is symmetric. Similarly $\overline{S_X(\omega)} = \overline{(\mathcal{F}R_X)(\omega)} = (\mathcal{F}R_X)(-\omega) = (\mathcal{F}R_X(-\cdot))(\omega) = (\mathcal{F}R_X)(\omega) = S_X(\omega)$ so that $S_X(\omega)$ is real valued.

Task 6. For the state space S and generator G given by

$$S = \{0,1\} \qquad \text{and} \qquad G = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix},$$

respectively, where $\alpha, \beta \geq 0$ a stationary distribution $\pi = (\pi_0 \;\; \pi_1)$ exists if and only if $\pi G = 0$. This gives the equations $\alpha\,\pi_0 = \beta\,\pi_1$ and $\pi_0 + \pi_1 = 1$ which can always be solved. Hence a stationary distribution always exists.