# MSG800/MVE170 Basic Stochastic Processes

# Written exam 27 Tuesday August 2019 2–6 PM

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

**Task 1.** Write a computer programme in some easy to understand code or pseudo-code that determines by means of simulations an approximative value for the probability $\mathbf{P}\{X(t) > t \text{ for some } t \in [0,10]\}$ for a unit rate Poisson process $\{X(t)\}_{t\geq 0}$.

**(5 points)**

**Task 2.** Let $Y_n = ((1-p)/p)^{S_n}$ for $n \geq 0$, where $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$ and $X_1, X_2, \ldots$ are independent identically distributed r.v.'s such that $P(X_i = -1) = 1-p$ and $P(X_i = 1) = p$ where $0 < p < 1$ is a constant. Show that $\{Y_n\}_{n=0}^{\infty}$ is a martingale with respect to the filtration $F_n = \sigma(X_1, \ldots, X_n)$. **(5 points)**

**Task 3.** Give an example of a discrete-parameter Markov chain $\{X(n),\, n \geq 0\}$ that has both periodic and aperiodic states. **(5 points)**

**Task 4.** Let $\{X(t),\, t \in \mathbb{R}\}$ be a WSS process and $\alpha > 0$ a constant. Show that $\{X(\alpha t),\, t \in \mathbb{R}\}$, $\{X(t-\alpha),\, t \in \mathbb{R}\}$ and $\{X(-t),\, t \in \mathbb{R}\}$ are also WSS processes. **(5 points)**

**Task 5.** A continuous time random process $\{Y(t),\, t \in \mathbb{R}\}$ is defined by $Y(t) = A\,X(t) \times \cos(\omega_c t + \Theta)$ for $t \in \mathbb{R}$, where $A, \omega_c \in \mathbb{R}$ are constants, $\Theta$ is a random variable that is uniformly distributed over the interval $(-\pi, \pi)$ and $\{X(t),\, t \in \mathbb{R}\}$ is a zero-mean WSS random process with power spectral density $S_X(\omega)$ that is independent of $\Theta$. Show that $Y(t)$ is a WSS process and find its power spectral density $S_Y(\omega)$. **(5 points)**

**Task 6.** Calculate the limit $\lim_{s,t\to\infty} R_X(s, s+t) = \lim_{s,t\to\infty} \mathbf{E}\{X(s)X(s+t)\}$ for a continuous time Markov chain $\{X(t)\,;\, t \geq 0\}$ with state space (possible values) $S$ and generator $G$ given by

$$S = \{0, 1\} \qquad \text{and} \qquad G = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix},$$

respectively, where $\alpha, \beta > 0$ are given constants. **(5 points)**

## MSG800/MVE170 Solutions to written exam August 2019

**Task 1.**

```
rep=100000;For[i=1;count=0,i<=rep,i++,X=0;succ=0;
                    t=Random[ExponentialDistribution[1]];
While[(t<=10)&&(succ=0),t=t+Random[ExponentialDistribution[1]];
X=X+1;If[X>t,succ=1]];count=count+succ];N[count/rep]
```

**Task 2.** $\mathbf{E}\{Y_{n+1}|F_n\} = Y_n\,\mathbf{E}\{((1-p)/p)^{X_{n+1}}\} = Y_n\,[((1-p)/p)\cdot p+(p/(1-p))\cdot(1-p)] = Y_n$.

**Task 3.** For example the chain with state space $E$ and transition probability matrix $P$ given by

$$E=\{0,1,2,3\} \qquad \text{and} \qquad P=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix},$$

respectively, where the states $\{0,1\}$ have period 2 while the states $\{2,3\}$ are aperiodic.

**Task 4.** Clearly, $X(\alpha t)$, $X(t-\alpha)$ and $X(-t)$ all have the same constant (time-invariant) mean $\mu_X$ as has $X(t)$ while $\mathbf{E}\{X(\alpha t)X(\alpha(t+\tau))\} = R_X(\alpha\tau)$, $\mathbf{E}\{X(t-\alpha)X(t+\tau-\alpha)\} = R_X(\tau)$ and $\mathbf{E}\{X(-t)X(-(t+\tau))\} = R_X(-\tau) = R_X(\tau)$ do not depend on $t\in\mathbb{R}$.

**Task 5.** $\mathbf{E}\{Y(t)Y(t+\tau)\} = A^2\,R_X(\tau)\,\frac{1}{2}\,\mathbf{E}\{\cos(\omega_c(2t+\tau)+2\Theta)+\cos(\omega_c\tau)\} = A^2\,R_X(\tau)\,\frac{1}{2}$ $\times\cos(\omega_c\tau)$ which gives $S_Y(\omega) = \frac{1}{4}\,A^2\,[S_X(\omega-\omega_c)+S_X(\omega+\omega_c)]$.

**Task 6.** The chain is irreducible with stationary distribution $\pi = (\frac{\beta}{\alpha+\beta}\ \ \frac{\alpha}{\alpha+\beta})$ (as this gives $\pi G = 0$). Noting that $X(s)X(t) = 1$ when both $X(s)$ and $X(t)$ are 1 while $X(s)X(t) = 0$ otherwise we see that $\mathbf{E}\{X(s)X(s+t)\} = (\mu^{(s)})_1\,p_{11}(t) = (\mu^{(0)}P_s)_1\,p_{11}(t)$ $= \big((\mu^{(0)})_0\,p_{01}(s)+(\mu^{(0)})_1\,p_{11}(s)\big)\,p_{11}(t) \to \big((\mu^{(0)})_0\,\pi_1+(\mu^{(0)})_1\,\pi_1\big)\,\pi_1 = \pi_1^2 = \alpha^2/(\alpha+\beta)^2$ as $s,t\to\infty$.