MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 13 January 2020 2 PM–6 PM

TEACHER AND JOUR: Patrik Albin, telephone 031 7723512.

AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) <u>or</u> Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Consider an M/M/1/3 queueing system with one server and two queueing slots with $\lambda = \mu = \rho = 1$. Write a computer programme that by means of stochastic simulation finds an approximation of the probability that the system doesn't get full during ten straight time units (when the queueing system is in steady state as usual). [HINT: $p_0 = p_1 = p_2 = p_3 = 1/4$.] **(5 points)**

Task 2. Let $X(t)$, $Y(t)$ and $Z(t)$, $t \geq 0$, be three independent Poission processes with intensity (/rate) 1. Find the probability $\mathbf{P}\{4\,X(1) = 2\,Y(2) = Z(4)\}$. **(5 points)**

Task 3. (DISCRETE TIME RANDOM WALK ON THE CORNERS OF A SQUARE) Consider a discrete time Markov chain $X(n)$, $n \in \mathbb{N}$, with possible values $\{A, B, C, D\}$ and transition matrix P with $P_{AB} = P_{AC} = P_{BA} = P_{BD} = P_{CA} = P_{CD} = P_{DB} = P_{DC} = \frac{1}{2}$. Find the mean $\mathbf{E}\{T_{A\to D}\}$ of the time $T_{A\to D}$ it takes $X(t)$ to move from state (/corner) A to state (/corner) D. **(5 points)**

Task 4. (POLYA'S URN) An urn contains initially a read and a black ball. At each time $n = 1, 2, 3, \ldots$ a ball is drawn randomly from the urn, its color noted and both the drawn ball and another ball of the same color are put back into the urn. Let X_n denote the number of black balls of the total number $n+2$ balls that are in urn after n draws. Show that $M_n = X_n/(n+2)$ is a martingale with respect to the filtration $F_n = \sigma(X_0, \ldots, X_n)$. **(5 points)**

Task 5. In a digital communication system 1 is represented by sending an analogue continuous time deterministic signal $s(t)$, $t \in \mathbb{R}$, while 0 i represented by sending 0. The signal is sent on a noisy channel with a zero-mean Gaussian WSS continuous time white noise process $N(t)$, $t \in \mathbb{R}$, having $R_N(\tau) = (N_0/2)\,\delta(\tau)$ and $S_N(f) = N_0/2$. Hence the recived signal is $X(t) = s(t) + N(t)$ if 1 is sent and $X(t) = N(t)$ if 0 is sent.

To obtain best signal detection at detection time t_0 the recived signal $X(t)$ is processed through a matched filter LTI system with impulse response $h(u) = s(t_0 - u)$ for $u \in \mathbb{R}$ designed to maximize the signal to noise ratio

$$\begin{aligned}\mathrm{SNR} &= \frac{(h \star s)(t_0)^2}{\mathrm{Var}\{(h \star N)(t_0)\}} = \frac{(h \star s)(t_0)^2}{\mathbf{E}\{(h \star N)(t_0)^2\}} = \frac{(\int_{-\infty}^{\infty} s(t_0 - u)^2\, du)^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(f)|^2 S_N(f)\, df} \\ &= \frac{2}{N_0}\, \frac{(\int_{-\infty}^{\infty} s(u)^2\, du)^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(f)|^2\, df} = \frac{2}{N_0}\, \frac{(\int_{-\infty}^{\infty} s(u)^2\, du)^2}{\int_{-\infty}^{\infty} h(u)^2\, du} = \frac{2}{N_0} \int_{-\infty}^{\infty} s(u)^2\, du.\end{aligned}$$

At detection it is decided that 1 was sent if $(h \star X)(t_0) > \frac{1}{2}\,(h \star s)(t_0)$ while it is decided that 0 was sent if $(h \star X)(t_0) \leq \frac{1}{2}\,(h \star s)(t_0)$. Find the probabilities for erroneous detection $\mathbf{P}\{\text{decision 1 when 0 was sent}\}$ and $\mathbf{P}\{\text{decision 0 when 1 was sent}\}$. [Hint: As $(h \star N)(t_0)$ is normal distributed so is $(h \star X)(t_0)$.] **(5 points)**

Task 6. (Continuous time random walk on the corners of a triangle) Let $X(t)$, $t \geq 0$, be a continuous time Markov chain with state space $S = \{0, 1, 2\}$ and generator G with $G_{ij} = 1$ for $i \neq j$. Show that the time it takes $X(t)$ to move from state (/corner) 0 to state (/corner) 2 is unit mean exponential distributed. **(5 points)**

MSG800/MVE170 Solutions to exam 13 January 2020

Task 1.

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In[1]:= Reps=1000000; NonFull=0;

In[2]:= For[i=1, i≤Reps, i++, X=RandomInteger[{0,3}];

            If[X≥1, time=Random[ExponentialDistribution[2]],

              If[X==1, time=Random[ExponentialDistribution[1]]]];

    While[(X≤2) && (time<10),

      If[X≥1, time=time+Random[ExponentialDistribution[2]];

          X=X+2*(RandomInteger[{0,1}]-1/2),

        time=time+Random[ExponentialDistribution[1]]; X=1]];

    If[X≤2, NonFull=NonFull+1]];

In[3]:= N[NonFull/Reps]

Out[3]:= 0.158513
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Task 2. $\mathbf{P}\{4\,X(1)=2\,Y(2)=Z(4)\}=\sum_{k=0}^{\infty}\mathbf{P}\{X(1)=k\}\,\mathbf{P}\{Y(2)=2k\}\,\mathbf{P}\{Z(4)=4k\}=\sum_{k=0}^{\infty}\frac{1^k}{k!}\,\mathrm{e}^{-1}\,\frac{2^{2k}}{(2k)!}\,\mathrm{e}^{-2}\,\frac{4^{4k}}{(4k)!}\,\mathrm{e}^{-4}$.

Task 3. $\mathbf{E}\{T_{A\to D}\}=1+\frac{1}{2}\,\mathbf{E}\{T_{B\to D}\}+\frac{1}{2}\,\mathbf{E}\{T_{C\to D}\}=1+\mathbf{E}\{T_{B\to D}\}$ where $\mathbf{E}\{T_{B\to D}\}=1+\frac{1}{2}\,\mathbf{E}\{T_{A\to D}\}$ so that $\mathbf{E}\{T_{A\to D}\}=4$.

Task 4. As $\mathbf{P}\{X_{n+1}=k+1|X_n=k\}=\frac{k}{n+2}$ and $\mathbf{P}\{X_{n+1}=k|X_n=k\}=\frac{n+2-k}{n+2}$ we have $\mathbf{E}\{M_{n+1}|F_n\}=\mathbf{E}\big\{\frac{X_{n+1}}{n+3}\,\big|\,F_n\big\}=\frac{1}{n+3}\big(X_n+\frac{X_n}{n+2}\big)=\frac{X_n}{n+2}=M_n$.

Task 5. Symmetry gives $\mathbf{P}\{\text{decision 0 when 1 sent}\}=\mathbf{P}\{\text{decision 1 when 0 sent}\}=\mathbf{P}\big\{(h\star N)(t_0)>\frac{1}{2}\,(h\star s)(t_0)\big\}=\mathbf{P}\big\{\mathrm{N}(0,\mathrm{Var}\{(h\star N)(t_0)\})>\frac{1}{2}\,(h\star s)(t_0)\big\}=1-\Phi(\frac{1}{2}\sqrt{\mathrm{SNR}}\,)$.

Task 6. With obvious notation symmetry gives that

$$\Psi_{T_{0\to 2}}(\omega)=\Psi_{\exp(2)}(\omega)\,\big[\tfrac{1}{2}+\tfrac{1}{2}\Psi_{T_{1\to 2}}(\omega)\big]=\Psi_{\exp(2)}(\omega)\,\big[\tfrac{1}{2}+\tfrac{1}{2}\Psi_{T_{0\to 2}}(\omega)\big]$$

so that, using that $\Psi_{\exp(\lambda)}(\omega)=\lambda/(\lambda-j\omega)$,

$$\Psi_{T_{0\to 2}}(\omega)=\frac{\frac{1}{2}\cdot 2/(2-j\omega)}{1-\frac{1}{2}\cdot 2/(2-j\omega)}=\frac{1}{1-j\omega}=\Psi_{\exp(1)}(\omega).$$