MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 6 April 2020 8.30–12.30

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AIDS: All aids are permitted. (See the Canvas announcement with instructions for Eastern reexam for clarification.)

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Consider an M/M/1/3 queueing system with one server and two queueing slots with $\lambda = 1$ and $\mu = 1/2$. Write a computer programme that by means of stochastic simulation finds an approximation of the mean of the time until the system becomes empty (when the queueing system is in steady state as usual). **(5 points)**

Task 2. Consider a Poisson process $X(t)$, $t \geq 0$. Is it possible to have $\mathbf{P}\{X(1) = 0, X(2) = 0, \ldots, X(n) = 0\} = \mathbf{P}\{X(1) = 2, X(2) = 4, \ldots, X(n) = 2n\}$? **(5 points)**

Task 3. Let X_n, $n = 0, 1, 2, \ldots$, be a discrete time Markov chain with transition probability matrix P. Is it true in general that $(P^{-1})_{ij} = \mathbf{P}\{X_n = i | X_{n+1} = j\}$? Can the same claim be true for some special case of P? **(5 points)**

Task 4. Let $W(t)$, $t \geq 0$, be a Wiener process that is independent of a unit intensity (/rate) Poisson process $N(t)$, $t \geq 0$. Show that $M(t) = \mathrm{e}^{2t}(-1)^{N(t)}W(t)$, $t \geq 0$, is a martingale with respect to the filtration F_t containing information of all process values $\{W(s)\}_{s \in [0,t]}$ and $\{N(s)\}_{s \in [0,t]}$. **(5 points)**

Task 5. Find the frequency response $H(f)$ of the integrator $(TX)(t) = \int_{t-1}^{t} X(s)\, ds$. **(5 points)**

Task 6. (CONTINUOUS TIME RANDOM WALK ON THE CORNERS OF A SQUARE) Let $X(t)$, $t \geq 0$, be a continuous time Markov chain with state space $\{A, B, C, D\}$, starting at state (/corner) $X(0) = A$ and generator G with $G_{AB} = G_{AC} = G_{BA} = G_{BD} = G_{CA} = G_{CD} = G_{DB} = G_{DC} = 1$ and $G_{AD} = G_{BC} = G_{CB} = G_{DA} = 0$. Find an expression for the probability $\mathbf{P}\{X(t) \neq D \text{ for } t \in [0, 10]\}$ that $X(t)$ does not visit state (/corner) D during the first 10 time units. **(5 points)**

MSG800/MVE170 Solutions to written exam 6 April 2020

Task 1.

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In[1]:= reps=100000; totaltime=0;

In[2]:= For[i=1, i≤reps, i++, time=0; x=RandomReal[];
    If[x≤1/15, X=0, If[x≤3/15, X=1, If[x≤7/15, X=2, X=3]]];
    While[X≥1,
      If[X≤2, atime=Random[ExponentialDistribution[1]];
          stime=Random[ExponentialDistribution[1/2]];
          time=time+Min[atime,stime];
          If[atime≤stime, X=X+1, X=X-1],
        time=time+Random[ExponentialDistribution[1/2]]; X=2]];
    totaltime=totaltime+time];

In[3]:= totaltime/reps

Out[3]:= 18.9295
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Task 2. With λ the intensity of the process the asked for equality means that $\mathbf{P}\{X(1) = 0\}\,\mathbf{P}\{X(2)-X(1) = 0\}\cdot\ldots\cdot\mathbf{P}\{X(n)-X(n-1) = 0\} = \mathbf{P}\{X(1) = 2\}\,\mathbf{P}\{X(2)-X(1) = 2\}\cdot\ldots\cdot\mathbf{P}\{X(n)-X(n-1) = 2\}$ which holds if and only if $\mathbf{P}\{X(1) = 0\} = \mathrm{e}^{-\lambda}$ equals $\mathbf{P}\{X(1) = 2\} = \frac{\lambda^2}{2!}\,\mathrm{e}^{-\lambda}$ so that it is necessary and sufficient that $\lambda = \sqrt{2}$.

Task 3. As $\left[\begin{smallmatrix}1/2 & 1/2\\ 0 & 1\end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix}2 & -1\\ 0 & 1\end{smallmatrix}\right]$ the claim is not true in general. However, the claim is trivially true for P the identity matrix.

Task 4. $\mathbf{E}\{\mathrm{e}^{2t}(-1)^{N(t)}W(t)|F_s\} = \mathrm{e}^{2t}(-1)^{N(s)}\,\mathbf{E}\{(-1)^{N(t)-N(s)}(W(t)-W(s))\} + \mathrm{e}^{2t} \times (-1)^{N(s)}W(s)\,\mathbf{E}\{(-1)^{N(t)-N(s)}\} = 0 + \mathrm{e}^{2t}(-1)^{N(s)}W(s)\sum_{k=0}^{\infty}(-1)^k\,\frac{(t-s)^k}{k!}\,\mathrm{e}^{-(t-s)} = \mathrm{e}^{2s}(-1)^{N(s)}W(s)$ for $0 \le s \le t$.

Task 5. $h(t) = 1$ for $t \in [0,1]$ so that $H(f) = \int_0^1 \mathrm{e}^{-j\omega t}\,dt = (1-\mathrm{e}^{-j\omega})/(j\omega)$.

Task 6. The times between movements of $X(t)$ are exponentially distributed with mean $1/2$ so that the number of movements $N(t)$ up to time t is a Poisson process with intensity 2. After each pair of movements of $X(t)$ from A it happens that $X(t)$ is back to A with probability $1/2$ and that $X(t)$ reaches D with probability $1/2$. The event that $X(t)$ do not reaches D during first 10 time units means that each pair of movements from A takes $X(t)$ back to A. And so the sought after probability is

$$\sum_{k=0}^{\infty}\mathbf{P}\{X(t)\neq D \text{ for } t\in[0,10]\,|\,N(10)=k\}\,\mathbf{P}\{N(10)=k\} = \sum_{k=0}^{\infty}(1/2)^{[k/2]}\,\frac{(2\cdot 10)^k}{k!}\,\mathrm{e}^{-2\cdot 10}.$$