# MSG800/MVE170 Basic Stochastic Processes

# Written exam Tuesday 25 August 2020 2 PM–6 PM

TEACHER: Patrik Albin 031 7723512 palbin@chalmers.se .

AIDS: All aids are permitted. (See the Canvas course "MVE170 Re-Exam MVE170/ MSG800" with instructions for this reexam for clarifications.)

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

**Task 1.** Consider a discrete time Markov chain with three states and transition probability matrix

$$P = \begin{pmatrix} 1-p & p & 0 \\ 0 & 1-q & q \\ r & 0 & 1-r \end{pmatrix}$$

for some constants $p, q, r \in (0, 1)$.

**(a)** Explain without any computations why the chain must have a unique stationary distribution. **(2,5 points)**

**(b)** Is the chain reversible? (Remember: Answers must be motivated!) **(2,5 points)**

**Task 2.** Let $\{W(t)\}_{t\geq 0}$ be a Wiener process with $\mathbf{E}\{W(t)^2\} = 1$. Find the probability density function of the random variable $X = W(1) + \int_0^1 W(s)\, ds$. **(5 points)**

**Task 3.** Let $\{W(t)\}_{t\geq 0}$ be a Wiener process with $\mathbf{E}\{W(t)^2\} = 1$. Find necessary and sufficient conditions on a function $g : [0, \infty) \to \mathbb{R}$ to make $\{g(t) \sin(W(t))\}_{t\geq 0}$ a martingale wrt. $F_t = \sigma(W(s) : s \in [0, t])$. [HINT: $\mathbf{E}\{\cos(\mathrm{N}(0, t))\} = \mathrm{e}^{-t/2}$.] **(5 points)**

**Task 4.** Let $\{X(t)\}_{t\geq 0}$ be a continuous time Markov chain with state space $\{0, 1, 2, 3\}$. What generator $G$ makes the chain spend exponentiallly distributed with parameter 1 times at each state and have jump chain with transition probability matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 1/4 & 1/4 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 & 0 \end{pmatrix} ?$$

**(5 points)**

**Task 5.** Find the frequency response $H(\omega)$ of the continuous time LTI system with insignal $X(t)$ and outsignal $\int_0^1 X(t-s)\, ds$. **(5 points)**

**Task 6.** Consider an M/M/1/3 queueing system in steady state with one server and two queueing slots and with $\lambda = 1$ and $\mu = 2$. Write a computer programme that by means of stochastic simulation finds an approximation of the probability that the server is busy but the queueing system never full during five straight time units. **(5 points)**

# MSG800/MVE170 Solutions to exam August 2020

**Task 1.** **(a)** As the chain is finite it must have at least one non-null persistent state. As the state is irreducible all states must have the same classification and thus be non-null persistent. The stationary distribution is given by one over the mean reccurence times.

**(b)** No, as writing $\pi$ for the stationary distribution each balance equation $\pi_i p_{ij} = \pi_j p_{ji}$ with $i \neq j$ will have one zero and one non-zero side and is thus not solvable.

**Task 2.** Clearly, $X$ is $\mathrm{N}(0,\sigma^2)$ with $\sigma^2 = \mathbf{E}\{W(1)^2\} + 2\,\mathbf{E}\{W(1)\,[\int_0^1 W(s)\,ds]\} + \mathbf{E}\{[\int_0^1 W(s)\,ds]^2\} = 1 + 2\int_0^1 R_W(1,s)\,ds + \int_0^1\int_0^1 R_W(s,t)\,dsdt = \ldots = \frac{7}{3}$.

**Task 3.** As $\mathbf{E}\{\sin(W(t))|F_s\} = \mathbf{E}\{\sin(W(t)-W(s))\cos(W(s))|F_s\} + \mathbf{E}\{\cos(W(t)-W(s))\sin(W(s))|F_s\} = \cos(W(s))\,\mathbf{E}\{\sin(W(t)-W(s))\} + \sin(W(s))\,\mathbf{E}\{\cos(W(t)-W(s))\}$ $= 0 + \mathrm{e}^{-(t-s)/2}$ for $0 \leq s \leq t$ it is necessary and sufficient that $g(t) = C\,\mathrm{e}^{t/2}$ for some constant $C \in \mathbb{R}$.

**Task 4.** $G = P - I$.

**Task 5.** $H(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-j\omega s}\,h(s)\,ds = \int_0^1 \mathrm{e}^{-j\omega s}\,ds = (1-\mathrm{e}^{-j\omega})/(j\omega)$.

**Task 6.**

```
In[1]:= Reps=1000000;

In[2]:= For[i=1; Busy=0, i≤Reps, i++,
        time=Random[ExponentialDistribution[3]]; x=RandomReal[];
    If[x≤8/15, X=0, If[x≤12/15, X=1, If[x≤14/15, X=2, X=3]]];
    While[(X≥1) && (X≤2) && (time≤5),
      time=time+Random[ExponentialDistribution[3]];
      If[RandomReal[]≤2/3, X=X-1, X=X+1];
    If[(X≥1) && (X≤2), Busy=Busy+1]];

In[3]:= N[Busy/Reps]

Out[3]:= 0.000129
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