BASIC STOCHASTIC PROCESSES 2007

# THEORETICAL QUESTIONS FOR THE WRITTEN EXAMINATION

*Chapter 1. The Poisson process and related processes*

**Question 1.**

(a) Give a definition of the Poisson process $\{N(t),\, t \geq 0\}$ with rate $\lambda$. 1p

(b) State and prove the memoryless property of the process (Theorem 1.1.2). 5p

**Question 2.**

(a) State and prove the result about the conditional distribution

$$P(S_k \leq x | N(t) = n)$$

and the conditional expectation

$$E[S_k | N(t) = n]$$

of the arrival times $S_k$, $1 \leq k \leq n$, of the Poisson process $\{N(t),\, t \geq 0\}$ with rate $\lambda$ (Lemma 1.1.4). 5p

(b) State without proof the result about the conditional joint distribution

$$P(S_1 \leq x_1, \ldots S_n \leq x_n | N(t) = n)$$

(Theorem 1.1.5). 1p

**Question 3.** Let $\{N(t),\, t \geq 0\}$ be a Poisson process of rate $\lambda$ which is independent of the *iid* random variables $D_1,\, D_2,\, \ldots$.

(a) Define the compound Poisson process $\{X(t),\, t \geq 0\}$ related to the Poisson process and the random variables above (Definition 1.2.1). Give a formula for $E[X(t)]$ (formula (1.2.1)) and explain it. 1p

(b) Suppose the random variables $D_1,\, D_2,\, \ldots$ are integer-valued. Let

$$a_j = P\{D_1 = j\} \quad \text{and} \quad r_j(t) = P\{X(t) = j\},\ j = 0,\, 1,\, \ldots$$

Consider the generating functions

$$A(z) = \sum_{j=0}^{\infty} a_j z^j, \ \ R(z,t) = \sum_{j=0}^{\infty} r_j(t) z^j, \quad |z| \leq 1.$$

Prove the following result:

**Theorem 1.2.1 (a).** For any fixed $t > 0$ it holds that

$$R(z,t) = e^{-\lambda t[1-A(z)]}, \quad |z| \leq 1.$$

5p

**Question 4.**

(a) Give a definition of the renewal process $\{N(t),\, t \geq 0\}$ with interoccurrence times $X_1,\, X_2,\, \ldots$, and of its renewal function $M(t)$. 1p

(b) Let $S_n$, $n \geq 1$, be the renewal epochs of the process with corresponding distribution functions $F_n(t) = P(S_n \leq t),\ n \geq 1$. State and prove Lemma 2.1.1 about the relationship between $M(t)$ and the functions $F_n(t),\ n \geq 1$. 5p

**Question 5.**

(a) Give a definition of a regenerative stochastic process $\{X(t),\, t \geq 0\}$. 1p

(b) Suppose $C_1,\, C_2,\, \ldots$ are the lengths of the renewal cycles of the regenerative process and assume that $E[C_1] < \infty$. Give a proof to

**Lemma 2.2.2.** For any $t > 0$, let $N(t)$ be the number of cycles completed up to the moment $t$. Then

$$\lim_{t \to \infty} \frac{1}{t} N(t) = \frac{1}{E[C_1]}$$

State without proof the renewal-reward theorem (Theorem 2.2.1). 5p

*Chapter 3. Discrete-time Markov chains*

**Question 6.**

(a) Give a definition of a discrete-time Markov chain $\{X_n,\, n = 0,\, 1,\, \ldots\}$ with state space $I$, and of the n-step transition probabilities $p_{ij}^{(n)}$. 1p

(b) Give a proof to

**Theorem 3.2.1 (Chapman-Kolmogoroff equations).** For all $n,\, m = 0,\, 1,\, \ldots$,

$$p_{ij}^{(n+m)} = \sum_{k \in I} p_{ik}^{(n)} p_{kj}^{(m)}, \quad i,\, j \in I.$$

5p

**Question 7.** Consider a a discrete-time Markov chain $\{X_n,\, n = 0,\, 1,\, \ldots\}$ with a finite state space $I$ and one-step transition probabilities $\{p_{ij},\, i,\, j \in I\}$. Assume there is some state $r \in I$ such that for each state $i \in I$, there is an integer $n_i \geq 1$ such that $p_{ir}^{(n_i)} > 0$.

(a) Define the mean return time $\mu_{rr}$ and the mean visit times $\mu_{ir}$, $i \in I$, $i \neq r$. 1p

(b) Prove

**Lemma A** (equation (3.2.3) on p. 92). The mean return time $\mu_{rr}$ and the mean visit times $\mu_{ir}$, $i \neq r$ satisfy the equation

$$\mu_{rr} = 1 + \sum_{j \in I, j \neq r} p_{rj} \mu_{jr}$$

5p

*Proof.* Introduce the time for the first visit to $r$

$$\tau = \min\{n \geq 1 : X_n = r\}.$$

The possible values of $\tau$ are $k = 1, 2, \ldots$ . For $k = 1$ we have

$$P\{\tau = 1 | X_0 = r\} = p_{rr} \tag{1}$$

For $k \geq 2$, from the representation

$$\{\tau = k\} = \cup_{j \in I, j \neq r} \{\tau = k,\ X_1 = j\}$$

where the events in the right-hand side are disjoint, we obtain by using the total probability formula

$$\begin{aligned}
P\{\tau = k \mid X_0 = r\} &= \sum_{j \in I, j \neq r} P\{\tau = k,\ X_1 = j \mid X_0 = r\} \\
&= \sum_{j \in I, j \neq r} \frac{P\{\tau = k,\ X_1 = j,\ X_0 = r\}}{P\{X_0 = r\}} \\
&= \sum_{j \in I, j \neq r} P\{\tau = k | X_1 = j,\ X_0 = r\} \frac{P\{X_1 = j,\ X_0 = r\}}{P\{X_0 = r\}} \qquad (2) \\
&= (\text{the markovian property}) = \sum_{j \in I, j \neq r} P\{\tau = k | X_1 = j\} p_{rj} \\
&= (\text{the definition of } \tau) = \sum_{j \in I, j \neq r} P\{\tau = k - 1 | X_0 = j\} p_{rj}
\end{aligned}$$

To compute $\mu_{rr} = E[\tau | X_0 = r]$, we use (1) and (2):

$$\begin{aligned}
\mu_{rr} &= \sum_{k \geq 1} k P\{\tau = k \mid X_0 = r\} \\
&= 1 \times p_{rr} + \sum_{k \geq 2} k \sum_{j \in I, j \neq r} P\{\tau = k - 1 | X_0 = j\} p_{rj} \\
&= p_{rr} + \sum_{j \in I, j \neq r} p_{rj} \sum_{k \geq 2} k P\{\tau = k - 1 | X_0 = j\} \\
&= p_{rr} + \sum_{j \in I, j \neq r} p_{rj} \Big[ \sum_{k-1 \geq 1} (k-1) P\{\tau = k - 1 | X_0 = j\} + \sum_{k-1 \geq 1} P\{\tau = k - 1 | X_0 = j\} \Big] \\
&= p_{rr} + \sum_{j \in I, j \neq r} p_{rj} \Big[ E[\tau | X_0 = j] + 1 \Big] \\
&= p_{rr} + \sum_{j \in I, j \neq r} p_{rj} \mu_{jr} + \sum_{j \in I, j \neq r} p_{rj} = 1 + \sum_{j \in I, j \neq r} p_{rj} \mu_{jr}
\end{aligned}$$

**Question 8.** Give a proof to the first part of Theorem 3.3.1.

**Theorem 3.3.1** (first part). Let $\{X_n,\, n = 0,\, 1,\, \dots\}$ be a discrete-time Markov chain with state space $I$ and one-step transition probabilities $p_{ij},\, i,\, j \in I$. For any $j \in I$, it holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p_{jj}^{(k)} = \begin{cases} \frac{1}{\mu_{jj}}, & \text{if state j is recurrent} \\ 0, & \text{if state j is transient} \end{cases}$$

where $\mu_{jj}$ denotes the mean return time from state $j$ to itself. 6p

*Chapter 4. Continuous-time Markov chains*
*Chapter 5. Markov chains and queues*

**Question 9.** Analysis of the M/M/1 queueing system by using a continuous-time Markov chain model.

(a) Describe the M/M/1 queueing system 1p

(b) Let $X(t) =$ the number of customers present at time $t$. Under what assumption is the process $\{X(t),\, t \geq 0\}$ a continuous-time Markov chain? 1p

(c) Sketch the transition rate diagram and write the balance equations of the process. 1p

(d) Compute the equilibrium probabilities $\{\pi_j,\, j \in I\}$ of $\{X(t),\, t \geq 0\}$. 1p

(e) Give a formula for the long-rum average number of customers in the system. Explain in words how the formula is derived. 1p

(f) What is the long-run fraction of customers who find $j$ other customers present upon arrival? Explain your answer. 1p