Basic stochastic processes 2007

Take home examination 2

Day assigned: December 11. **Due date:** December 13, 10:00 am

Please indicate
Name and PN:
Education program:
The code of the course taken:

- The take home examination is a strictly individual assignment. Submissions that bear signs of being collective efforts will be disregarded.
- Answers without explanations will be disregarded as well.

Problem 1. A Bernoulli trial results in a success with probability p and in a failure with probability $1-p$, where $0<p<1$. Suppose the Bernoulli trial is repeated indefinitely with each repetition independent of all others. Let X_n be a "success runs" Markov chain with a state space $I=\{0,\,1,\,2,\,\ldots\}$, where $X_n=0$ if the $n-th$ trial results in a failure and $X_n=j$ if $X_{n-j}=0$ and trials $n-j+1,\ldots,n$ have resulted in a success.

(a) Find the one-step transition matrix of the Markov chain. 0.5p

(b) Show that the state 0 is recurrent. 1.5p

Problem 2. Peter plays a game, where he either wins 1 EUR or loses 1 EUR. He is allowed to play even when his capital is not positive (a negative capital corresponds to a debt). In the game, there are two coins involved. One coin lands heads with probability $\frac{1}{10}-\varepsilon$ and the other coin lands heads with probability $\frac{3}{4}-\varepsilon$, where $0\le\varepsilon<\frac{1}{10}$. Peter must take the first coin when his capital is multiple by 3, and the second coin otherwise. He wins 1 EUR when the coin lands heads and loses 1 EUR otherwise. Our interest is in the long-run fraction of plays won by Peter.

(a) Define a discrete-time Markov chain with three states, that can be used to analyse the problem. 1.5p

(b) Write the equilibrium equations and give a formula for the long-run fraction of plays won by Peter. 1.5p

Problem 3. An information centre has one attendant; people with questions arrive according to a Poisson process with rate λ. A person who finds n other customers present upon arrival joins the queue with probability $1/(n+1)$ for $n=0,\,1,\,\ldots$ and goes elsewhere otherwise. The service times of the persons are independent random variables having an exponential distribution with mean $1/\mu$.

(a) Verify that the equilibrium distribution of the number of persons present at the information centre is a Poisson distribution with mean λ/μ. 1.5p

(b) What is the long-run fraction of persons with request who actually join the queue? What is the long-run average number of persons served per time unit? Explain your answers. 1.5p

Problem 4. At a production facility orders arrive according to a renewal process with a mean interarrival time $1/\lambda$. A production is started only if N orders have accumulated. The production time is negligible. A fixed cost of $K > 0$ is incurred for each production set-up and holding costs are incurred at the rate of hj when j orders are waiting to be processed. What value of N minimizes the long-run average cost per time unit? 1p.