Basic stochastic processes 2007

Solution to Take home examination 1

Day assigned: November 13. **Due date:** November 15, 10:00 am

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Problem 1. Consider the Poisson process $\{N(t),\, t \geq 0\}$ of rate λ. Let S_1, S_2, ... be the arrival times of the process. Fix the time point t and consider the interval $S_{N(t)+1} - S_{N(t)}$, where we have $S_{N(t)} = 0$, if $N(t) = 0$. What is true for the average length of this interval, $\mu = E[S_{N(t)+1} - S_{N(t)}]$:

(a) $\mu < \frac{1}{\lambda}$

(b) $\mu = \frac{1}{\lambda}$

(c) $\mu > \frac{1}{\lambda}$

Explain your answer. 1p

Solution

$$\mu = E[S_{N(t)+1} - t] + E[t - S_{N(t)}] = E[\gamma_t] + E[t - S_{N(t)}] = \frac{1}{\lambda} + E[t - S_{N(t)}].$$

Since

$$P\{t - S_{N(t)} > 0\} \geq P\{t - S_{N(t)} = t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

we have $E[t - S_{N(t)}] > 0$. Thus

$$\boxed{\mu > \frac{1}{\lambda}}$$

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Remark. Many would expect that $\mu = \frac{1}{\lambda}$. The above result is sometimes referred to as *the waiting time paradox* related to the Poisson process.

Problem 2. You need 5 hours to complete a certain routine job. However, your work is interrupted by telephone calls that come according to a Poisson process with rate 3 talks per hour. The duration of a single telephone talk is a random variable which is uniformly distributed between 3 and 7 minutes. How long does it take on the average to complete the job? Explain the model and give a detailed solution. 2p

Solution

Let C be the time needed to complete the job. We have

$$C = 5 + \sum_{i=1}^{N(5)} X_i$$

where $N(t)$ is the Poisson process of rate 3 events per hour and X_1, X_2, ... are iid random variables uniformly distributed between 3/60 and 7/60 hours. By the Wald's equation

$$E[C] = 5 + E[N(5)] \times E[X_1] = 5 + 15 \times \frac{1}{12} = 6.25$$

On the average, you need 6 hours and 15 minutes to complete the job.

Problem 3. Customers with items to repair arrive at a repair facility according to a Poisson process with rate λ. The repair time of an item has a uniform distribution on $[a, b]$. The exact repair time can be determined upon arrival of the item. If the repair time of an item takes longer than $\frac{a+b}{2}$ time units then the customer gets a loaner for the defective item until the item returns from repair. A sufficiently large supply of loaners is available. In the long run, what is the average number of loaners that are out? Explain the model and give a detailed solution.

Hint. Recognize the $M/G/\infty$ model. 3p

Solution

The probability for a repair time longer than τ is $p = (b - \tau)/(b - a)$. Let $R \sim U(a, b)$ be the repair time of an item. The average repair time of an item that requires repair time longer than τ is then $E[R|R > \tau] = \frac{\tau+b}{2}$.

Let $N(t)$ be the Poisson process describing the arrivals of customers, and $N_1(t)$ be the process counting the number of customers with defective items requiring a repair time longer than τ. The process $N_1(t)$ is obtained by "marking" the arrivals of $N(t)$ with probability p and is then itself a Poisson process of rate λp. Every arrival from $N_1(t)$ is assigned a loaner, or a "server", and the average "service time" is $\frac{\tau+b}{2}$. In this situation, the $M/G/\infty$ model applies. In the long-run, the number of loaners ("busy servers") is described by a Poisson random variable with parameter $\lambda p \times \frac{\tau+b}{2}$. Hence

The long-run average number of loaners that are out $= \lambda p \times \dfrac{\tau + b}{2}$

Problem 4. A production process in a factory yields waste that is temporarily stored on the factory site. The amount of the waste that are produced in successive weeks are independent and identically distributed random variables with finite first two moments μ_1 and μ_2. Opportunities to remove the waste from the factory side occur at the end of each week. The following control rule is used. If at the end of a week the total amount of the waste present is larger than D, then all the waste present is removed; otherwise , nothing is removed. There is a fixed cost of $K > 0$ for removing the waste and a variable cost of $v > 0$ for each unit of excess of the amount D. Describe the regenerative process involved and determine the long-run average cost per time unit. 3p

Solution

Let X_1, $X_2, \ldots$ be the weekly amounts of produced waste. We have $E[X_1] = \mu_1$, $E[X_1^2] = \mu_2$. Let $N(t)$ be the renewal process with interarrival times X_1, $X_2, \ldots$, renewal epochs S_1, S_2, $\ldots$, and a renewal function $M(t)$.

The process describing the amount of waste present at the end of each week is a regenerative process. Its first regenerative epoch is week $N(D)+1$, hence

$$E[\text{length of one cycle}] = E[N(D)] + 1] = M(D) + 1$$

The amount of waste in excess in week $N(D)+1$ is

$$\gamma_D = S_{N(D)+1} - D = \sum_{k=1}^{N(D)+1} X_k - D$$

The event $\{N(D)+1 = n\}$ is equivalent to the event $\{S_{n-1} \leq D,\, S_n > D\}$ which is independent of the random variables X_{n+1}, X_{n+2} .... By Wald's equation we thus have

$$E\left[\sum_{k=1}^{N(D)+1} X_k\right] = [M(D)+1]E[X_1] = [M(D)+1]\mu_1$$

and then

$$E[\text{cost incurred in one cycle}] = K + vE[\gamma_D] = K + v\big[(M(D)+1)\mu_1 - D\big]$$

Hence the long-run average cost per week equals

$$\boxed{\frac{E[\text{cost incurred in one cycle}]}{E[\text{length of one cycle}]} = \frac{K + v\big[(M(D)+1)\mu_1 - D\big]}{M(D)+1}}$$

Assume the distribution of X_1 has a continuous part. When D is large, formula 2.1.6 gives the following approximation of $M(D)$:

$$M(D) \approx \frac{D}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1$$

Hence in this case

$$\boxed{\frac{E[\text{cost incurred in one cycle}]}{E[\text{length of one cycle}]} = \frac{K + v\frac{\mu_2}{2\mu_1}}{\frac{D}{\mu_1} + \frac{\mu_1}{2\mu_1^2}}}$$