Basic stochastic processes 2007

Take home examination 2

Day assigned: December 11. **Due date:** December 13, 10:00 am

• The take home examination is a strictly individual assignment. Submissions that bear signs of being collective efforts will be disregarded

• Answers without explanations will be disregarded as well.

Problem 1. A Bernoulli trial results in a success with probability p and in a failure with probability $1-p$, where $0 < p < 1$. Suppose the Bernoulli trial is repeated indefinitely with each repetition independent of all others. Let X_n be a "success runs" Markov chain with a state space $I = \{0, 1, 2, \dots\}$, where $X_n = 0$ if the $n-th$ trial results in a failure and $X_n = j$ if $X_{n-j} = 0$ and trials $n-j+1, \dots, n$ have resulted in a success.

(a) Find the one-step transition matrix of the Markov chain. 0.5p

(b) Show that the state 0 is recurrent. 1.5p

Solution

(a) For $i,\, j \in I$ the one-step transition probabilities are

$$p_{ij} = \begin{cases} p & \text{if} \quad j = i+1 \\ 1-p & \text{if} \quad j = 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} f_{00}^{(n)} &= P\{X_n = 0,\, X_{n-k} \neq 0,\, 1 \leq k \leq (n-1) | X_0 = 0\} \\ &= P\{(n-1) \;\text{ successes followed by 1 failure }\} = p^{n-1}(1-p). \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} f_{00}^{(n)} = \sum_{n=1}^{\infty} p^{n-1}(1-p) = 1$$

the state 0 is recurrent.

Problem 2. Peter plays a game, where he either wins 1 EUR or loses 1 EUR. He is allowed to play even when his capital is not positive (a negative capital corresponds to a debt). In the game, there are two coins involved. One coin lands heads with probability $\frac{1}{10} - \varepsilon$ and the other coin lands heads with probability $\frac{3}{4} - \varepsilon$, where $0 < \varepsilon < \frac{1}{10}$. Peter must take the first coin when his capital is multiple by 3, and the second coin otherwise. He wins 1 EUR when the coin lands heads and loses 1 EUR otherwise. Our interest is in the long-run fraction of plays won by Peter.

(a) Define a discrete-time Markov chain with three states that can be used to analyse the problem. 1.5p

(b) Write the equilibrium equations and give a formula for the long-run fraction of plays won by Peter. 1.5p

Solution

(a) Consider a Markov chain $\{X_n\}$ with state space $I = \{0, 1, 2\}$, where $X_n = s$ if Peter's capital after the $n-th$ game equals $3k + s$ for some integer k, $s = 0, 1, 2$. The one-step transition probability matrix is then

$$\begin{bmatrix} 0 & 1/10-\varepsilon & 9/10+\varepsilon \\ 1/4+\varepsilon & 0 & 3/4-\varepsilon \\ 3/4-\varepsilon & 1/4+\varepsilon & 0 \end{bmatrix}$$

(b) The equilibrium equations are

$$\begin{aligned} \pi_0 &= (\frac{1}{4}+\varepsilon)\pi_1 + (\frac{3}{4}-\varepsilon)\pi_2 \\ \pi_1 &= (\frac{1}{10}-\varepsilon)\pi_0 + (\frac{1}{4}+\varepsilon)\pi_2 \\ \pi_2 &= (\frac{9}{10}+\varepsilon)\pi_0 + (\frac{3}{4}-\varepsilon)\pi_1 \end{aligned}$$

together with the normalizing equation

$$\pi_0 + \pi_1 + \pi_2 = 1.$$

The long-run fraction of games that are won by Peter is equal to

$$(\frac{1}{10}-\varepsilon)\pi_0 + (\frac{3}{4}-\varepsilon)\pi_1 + (\frac{3}{4}-\varepsilon)\pi_2$$

with probability one.

Problem 3. An information centre has one attendant; people with questions arrive according to a Poisson process with rate λ. A person who finds n other customers present upon arrival joins the queue with probability $1/(n+1)$ for $n = 0, 1, \ldots$ and goes elsewhere otherwise. The service times of the persons are independent random variables having an exponential distribution with mean $1/\mu$.

(a) Verify that the equilibrium distribution of the number of persons present at the information centre is a Poisson distribution with mean λ/μ. 1.5p

(b) What is the long-run fraction of persons with request who actually join the queue? What is the long-run average number of persons served per time unit? Explain your answers. 1.5p

Solution

(a) Let $X(t)$ be the number of persons present at time t. The process $\{X(t),\, t \geq 0\}$ is a continuous-time Markov chain with state space $I = \{0,\, 1,\, 2,\, \ldots\}$. The transition rate diagram is given by

$$\begin{tikzcd} \boxed{0} \arrow[r, shift left, "\lambda/1"] & \boxed{1} \arrow[l, shift left, "\mu"] \quad \cdots \quad \boxed{(i-1)} \arrow[r, shift left, "\lambda/i"] & \boxed{i} \arrow[l, shift left, "\mu"] \quad \cdots \end{tikzcd}$$

By equating the rate out of the set $\{i,\, i+1,\, \ldots\}$ to the rate into this set, we find the recurrence relations

$$\mu p_i = \frac{\lambda}{i}\, p_{i-1}, \;\; i = 1,\, 2,\, \ldots$$

These equations lead to

$$p_i = \frac{(\lambda/\mu)^i}{i!} p_0, \quad i \geq 1.$$

Using the normalizing equation $\sum p_i = 1$ we obtain

$$p_i = e^{-\lambda/\mu} \frac{(\lambda/\mu)^i}{i!}, \quad i \geq 0.$$

(b) By the PASTA property, the long-run fraction of arrivals that actually join the queue is

$$\sum_{i=0}^{\infty} p_i \frac{1}{i+1} = \frac{\mu}{\lambda}\big(1 - e^{-\lambda/\mu}\big).$$

The long-run average number of persons served per time unit is

$$\lambda \left[\frac{\mu}{\lambda}\big(1 - e^{-\lambda/\mu}\big) \right] = \mu(1 - p_0),$$

in agreement with the Little's formula.

Problem 4. At a production facility orders arrive according to a renewal process with a mean interarrival time $1/\lambda$. A production is started only if N orders have accumulated. The production time is negligible. A fixed cost of $K > 0$ is incurred for each production set-up and holding cost are incurred for at the rate of hj when j orders are waiting to be processed. What value of N minimizes the long-run average cost per time unit? 1p.

Solution

Let $\{X(t),\, t \geq 0\}$ be the renewal process representing the number of orders present at time t. This process regenerates each time a production is started. The expected length of one cycle is N/λ. The total waiting time in one cycle is

$$(S_N - S_{N-1}) + (S_N - S_{N-2}) + \ \ldots (S_N - S_1)$$

and the average waiting time is then

$$\frac{1}{\lambda} + \frac{2}{\lambda} + \ldots + \frac{N-1}{\lambda} = \frac{N(N-1)}{2\lambda}$$

The expected cost incurred in one cycle is

$$K + h\frac{N(N-1)}{2\lambda}$$

and the long-run average cost is then

$$\frac{K + h\frac{N(N-1)}{2\lambda}}{N/\lambda} = \frac{\lambda K}{N} + \frac{h(N-1)}{2}$$

The function $g(t) = \dfrac{\lambda K}{t} + \dfrac{h(t-1)}{2}$ obtains minimum in the point $\sqrt{\dfrac{2K\lambda}{h}}$.

If one of the integers

$$\left\lceil \sqrt{\frac{2K\lambda}{h}} \right\rceil, \quad \left\lfloor \sqrt{\frac{2K\lambda}{h}} \right\rfloor$$

delivers a smaller value to $g(t)$ than the other one does, the first integer is the solution; otherwise both integers solve the problem.