Solution to Take home examination 1 in Basic stochastic processes 2008

Problem 1. Taxis are waiting in a queue for passengers to come. Passengers for those taxis arrive according to a Poisson process with an average of 60 passengers per hour. A taxi departs as soon as two passengers have been collected or 3 minutes have expired since the first passenger has got in the taxi. Suppose you get in the taxi as first passenger. What is your average waiting time for the departure? 3p

Hint: Condition on the first arrival after you get in the taxi.

Solution

Let X denote your waiting time in minutes, and let $N(t)$ be the process counting the arrivals of passenger from the moment you get in the taxi. $N(t)$ is a Poisson process of parameter $\lambda = 1$ passenger per minute. Let S_1 denote the first arrival time of the process. We have

$$E[X] = E[X|S_1 \geq 3]P\{S_1 \geq 3\} + E[X|S_1 < 3]P\{S_1 < 3\}$$

$$E[X|S_1 \geq 3]P\{S_1 \geq 3\} = 3e^{-3}$$

$$E[X|S_1 < 3]P\{S_1 < 3\} = E[S_1|S_1 < 3]P\{S_1 < 3\} = \int_0^3 x f_{S_1}(x)dx$$

$$= \int_0^3 xe^{-x}dx = -xe^{-x}|_0^3 + \int_0^3 e^{-x}dx = 1 - 4e^{-3}$$

$$E[X] = 1 - e^{-3} = 0.95 \approx 57sec.$$

Problem 2. Oil tankers with destination Rotterdam leave from harbours in the Middle East according to a Poisson process with an average of two tankers per day. The sailing time to Rotterdam has a Gamma distribution with an expected value of 10 days and a standart deviation of 4 days. Estimate the probability that the number of oil tunkers that are under way from the Middle East to Rotterdam at an arbitrary point of time exceeds 30. 3p

Solution. We recognize the $M/G/\infty$ system. The number of busy servers in the system at time t corresponds to the number $L(t)$ of oils tunkers that are under way at this time point. In the long run the distribution of $L(t)$ is approximately $Poisson(20)$ and the distribution of $\frac{L(t)-20}{\sqrt{20}}$ is then approximately $N(0,1)$. This gives

$$P\{L(t) > 30\} = P\left\{\frac{L(t) - 20}{\sqrt{20}} > \frac{10}{\sqrt{20}}\right\} \approx 1 - \Phi(2.236) = 0.0094$$

Exact computations give 0.0135.

Problem 3. The bus that takes you home from Chalmers arrives at the nearest bus station from early morning till late in the evening according to a renewal process with interarrival times that are uniformly distributed between 5 and 10 minutes. You come to the bus station at 5 pm. Estimate your average waiting time for the bus to arrive. 3p

Solution. Set the time unit to be a minute. Your waiting time for the bus to come is the excess variable γ associated with the time you come to the bus station. Since at this time the process has reached statistical equilibrium we can use the approximation

$$E[\gamma] \approx \frac{\mu_2}{2\mu_1}$$

where

$$\mu_1 = \frac{5+10}{2} = 7.5 \quad \mu_2 = \frac{1}{5}\int_5^{10} x^2 dx = \frac{1}{15}x^3|_5^{10} = 58.33$$

$$E[\gamma] \approx \frac{58.33}{15} \approx 3.89\ min.$$

Problem 4. Messages arrive at a communication channel according to a renewal process with a mean interarrival time $1/\lambda$. The messages are stored in a buffer. The buffer is emptied when M messages have accumulated. The time needed to empty the buffer is negligible. A holding cost of $h > 0$ is incurred for each time unit the message has to wait in the buffer and a fixed cost of $K > 0$ is incurred for each time the buffer is empted. Describe the regenerative process involved. Determine the long-run average cost per time unit and find a value of M that minimizes this cost. 3p

Solution. Let $S_1,\ S_2,\ +, \ldots$ denote the arrival times of the messages and $X(t)$ be the number of messages in the buffer at time t. $X(t)$ is a regenerative process with an average cycle length $E[S_M] = M/\lambda$. The total waiting time per a cycle is $\sum_{k=1}^{M-1}[S_M - S_k]$ and the average waiting time is then

$$\sum_{k=1}^{M-1} E[S_M - S_k] = \sum_{k=1}^{M-1}\left[\frac{M}{\lambda} - \frac{k}{\lambda}\right] = \frac{M(M-1)}{2\lambda}.$$

The average cost incurred for one cycle is

$$K + \frac{h\,M(M-1)}{2\lambda}$$

and the long-run average cost per time unit is then

$$\frac{average\ \ cost\ \ per\ \ cycle}{average\ \ cycle\ \ length} = \frac{\lambda K}{M} + \frac{h\,(M-1)}{2}.$$

Consider the function $g(x) = \frac{\lambda K}{x} + \frac{h\,(x-1)}{2}$ for $x > 0$. From

$$g'(x) = -\frac{\lambda K}{x^2} + \frac{h}{2}$$

we see that the function has an absolute minimum at $x_0 = \sqrt{\frac{2\lambda K}{h}}$. Hence the optimal value of N is one of the integers nearest to x_0.

Problem 5. Consider a Poisson arrival process with rate λ. For each fixed $t > 0$, define the random variable δ_t as the time elapsed since the last arrival before t and set $\delta_t = t$ if there are no arrivals before t. Compute the distribution of δ_t and draw its graph. 3p

Solution

$$P\{\delta_t = t\} = e^{-\lambda t}$$

Let $0 \leq x < t$.

$$\{\delta_t > x\} = \cup_{n=0}^{\infty}\{S_n \leq t - x,\ S_{n+1} > t\}$$

$$P\{S_n \leq t - x,\ S_{n+1} > t\} = \frac{[\lambda(t-x)]^n}{n!} e^{-\lambda t}$$

Hence

$$P\{\delta_t > x\} = \sum_{0}^{\infty} \frac{[\lambda(t-x)]^n}{n!} e^{-\lambda t} = e^{-\lambda t} e^{\lambda(t-x)} = e^{-\lambda x}.$$

The distribution of δ_t is the truncated Exponential distrobution on $[0,\ t]$.