Take home examination 2 in Basic stochastic processes 2008

Day assigned: December 9, 10:00 am
Due date: December 11. 10:00 am

• The take home examination is a strictly individual assignment. Submissions that bear signs of being collective efforts will be disregarded.

• Students are supposed to give a precise description of the models used to solve the problem and rigorous explanations to the solution.

• Correct answers without proper explanations will be disregarded.

• *Please write the code of the course you are registered in on the upper left corner of the first page of your work.*

Problem 1. The diffusion of electrons and holes across a potential barrier in an electronic devise is modelled as follows. There are m black balls (electrons) in urn A and m white balls (holes) in urn B. We perform independent trials, in each of which a ball is selected at random from each urn and the selected ball from urn A is placed in urn B, while that from urn B is placed in A. Consider the Markov chain representing the number of black balls in urn A immediately after the $n-th$ trial.

(a) Describe the one-step transition probabilities of the process.

(b) Suppose $m = 3$. Compute the long-run fraction of time when urn A does not contain a black ball.

3p

Solution. Let X_n= the # of black balls in urn A just after the $n - th$ trial. $\{X_n,\ n \geq 0\}$ is a MC with state space $I = \{0, 1, \ldots m\}$.

(a) The one-step transition probabilities of the MC are

$$p_{i,i-1} = P\{X_{n+1} = i-1|X_n = i\} = \left(\frac{i}{m}\right)^2, \quad i \geq 1$$

$$p_{i,i+1} = P\{X_{n+1} = i|X_n = i\} = \left(\frac{m-i}{m}\right)^2, \quad i \leq m-1$$

$$p_{i,i} = P\{X_{n+1} = i-1|X_n = i\} = 1 - \left(\frac{i}{m}\right)^2 - \left(\frac{m-i}{m}\right)^2$$

$$p_{i,j} = 0 \quad otherwise.$$

(b) Let $m = 3$. The one-step transition probability matrix is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The equilibrium equations are

$$\begin{aligned} \pi_0 &= \frac{1}{9}\pi_1 \\ \pi_1 &= \pi_0 + \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 \\ \pi_2 &= \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 + \pi_3 \\ \pi_3 &= \frac{1}{9}\pi_2 \\ \sum_{j=0}^{3} \pi_j &= 1 \end{aligned}$$

The solution is

$$\pi_0 = \frac{1}{20} \quad \pi_1 = \frac{9}{20} \quad \pi_2 = \frac{9}{20} \quad \pi_3 = \frac{1}{20}$$

The long-run fraction of time when urn A does not contain a black ball is 1/20.

Problem 2. Peter takes the course Basic Stochastic Processes this quarter on Tuesday, Thursday, and Friday. The classes start at 10:00 am. Peter is used to work until late in the night and consequently, he sometimes misses the class. His attendance behaviour is such that he attends class depending only on whether or not he went to the latest class. If he attended class one day, then he will go to class next time it meets with probability 1/2. If he did not go to one class, then he will go to the next class with probability 3/4.

(a) Describe the Markov chain that models Peter's attendance. What is the probability that he will attend class on Tuesday if he went to class on Friday?

(b) Suppose the course has 30 classes altogether. Give an estimate of the number of classes attended by Peter and explain it.

3p

Solution.

(a) Let $X_n = 0$ if Peter goes to the $n-th$ class meeting and $X_n = 1$ if he skips it. The process $\{X_n,\, n \geq 1\}$ is a MC with state space $I = \{0, 1\}$ and one-step transition probability matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

$$P\{\text{Peter will attend on Tuesday}|\text{he went on Friday}\} = p_{11}^{(2)} = \frac{5}{8},$$

where $p_{11}^{(2)}$ is taken from the two-step transition matrix

$$P^2 = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{9}{16} & \frac{7}{16} \end{bmatrix}$$

(b) The state space has two states which communicate, so the equilibrium probabilities exist. From

$$\pi_0 = \pi_0 \frac{1}{2} + \pi_1 \frac{3}{4}$$
$$\pi_0 + \pi_1 = 1$$

we obtain $\pi_0 = 0.6$, $\pi_1 = 0.4$. The process reaches equilibrium before the end of the course, since

$$P^8 = \begin{bmatrix} 0.5998 & 0.3999 \\ 0.5998 & 0.3999 \end{bmatrix}.$$

Thus the long-run fraction of classes attended is ≈ 0.6, so the estimation for the number of classes attended is $0.6 \times 30 = 18$.

Problem 3. In an inventory system for a single product the depletion of stock is due to demand and deterioration. The demand process for the product is the Poisson process with rate λ. The lifetime of each product is exponentially distributed with mean $1/\mu$. The stock control is exercised as follows. Each time the stock drops to zero an order for Q is placed. The lead time of each order is negligible. We are interested in the long run average number of orders placed per time unit.

(a) Introduce an appropriate continuous-time Markov chain to analyse the system and compute the infinitesimal transition rates.

(b) Give a recursive algorithm for computing the equilibrium probabilities and a formula for the long run average number of orders placed per time unit.

4p

Solution. Let $X(t)$ be the stock on hand at time t. The process $X(t)$, $t \geq 0$ is a continuous-time MC with state space $I = \{1, 2, \dots Q\}$ and transitions rates

(a)

$$\begin{aligned} q_{i,i-1} &= \lambda + i\mu, \quad i = 2, \dots Q \\ q_{1Q} &= \lambda + \mu \\ q_{ij} &= 0 \quad otherwise. \end{aligned}$$

(b) By equating the rate out of state i to the rate into state i we obtain

$$\begin{aligned} (\lambda + i\mu)p_i &= [\lambda + (i+1)\mu]p_{i+1}, \quad 1 \leq i \leq Q-1 \\ (\lambda + Q\mu)p_Q &= [\lambda + \mu]p_1. \end{aligned}$$

Starting from $\overline{p}_q = 1$ we can recursively compute $\overline{p}_i$, $1 = q-1, \dots 1$. The normalizing equation then gives

$$p_i = \frac{\overline{p}_i}{\sum_{j=1}^{Q} \overline{p}_j}.$$

The long run average number of orders placed per time unit equals the long run average number of transitions from state 1 to state Q per time unit, which is $p_1(\lambda + \mu)$.

Computation of the equilibrium probabilities (not required in the problem).

We have

$$
\begin{aligned}
p_2 &= p_1 \frac{\lambda+\mu}{\lambda+2\mu} \\
p_3 &= p_2 \frac{\lambda+2\mu}{\lambda+3\mu} = p_1 \frac{\lambda+\mu}{\lambda+3\mu} \\
&\vdots \\
p_Q &= p_1 \frac{\lambda+\mu}{\lambda+Q\mu} \\
1 &= \sum_1^Q p_1 \frac{\lambda+\mu}{\lambda+i\mu} \\
p_1 &= \frac{1}{(\lambda+\mu)\sum_1^Q \frac{1}{\lambda+i\mu}} \\
p_i &= \frac{1}{(\lambda+i\mu)\sum_1^Q \frac{1}{\lambda+i\mu}}, \quad i \le Q
\end{aligned}
$$

The long-run average number of orders placed per time unit is thus

$$
\frac{1}{\sum_1^Q \frac{1}{\lambda+i\mu}} \to 0 \; as \; Q \to \infty,
$$

as expected.

Problem 4. In each time unit a job arrives at a conveyor with a single workstation. The workstation can process only one job at a time and has a buffer to store the arriving jobs that find the workstation busy. However, the buffer can hold at most K jobs, so that any arriving job that finds the buffer full is lost. The processing times of the jobs are independent random variables having a common $Erlang(r, \mu)$ distribution. It is assumed that $r/\mu < 1$.

(a) Find an appropriate Markov chain to analyse the number of jobs in the buffer just prior to the arrival epochs of new jobs.

(b) Show how to calculate the long-run fraction of lost jobs and the long-run fraction of the time the workstation is busy.

5p

Solution.Consider an arriving job as r arriving subtask with independent processing times that are $Exp(\lambda)$ distributed. Let X_n be the number of uncompleted subtask just before the $n-th$ arrival epoch.

(a) The process $\{X_n,\ n \geq 0\}$ is a MC with state space $I = \{0,\, 1,\, \ldots\, (K+1)r)\}$. The one-step transitions probabilities are as follows. For $0 \leq i \leq Kr$ and $1 \leq j \leq i+r$

$$p_{ij} = e^{-\mu}\frac{\mu^{i+r-j}}{(i+r-j)!}, \quad p_{i0} = 1 - \sum_{j=1}^{i+r} p_{ij}$$

and for $Kr < i \leq (K+1)r$ and $1 \leq j \leq i$

$$p_{ij} = e^{-\mu}\frac{\mu^{i-j}}{(i-j)!}, \quad p_{i0} = 1 - \sum_{j=1}^{i} p_{ij}$$

(b) Since $r < \mu$ the equilibrium probabilities $\{\pi_i,\ 0 \leq i \leq (K+1)r\}$ exist. The long-run fraction of lost jobs equals the long run fraction of time the system is in a state from $Kr+1, \ldots (K+1)r$, thus

the long-run fraction of jobs lost=$\sum_{i=Kr+1}^{(K+1)r} \pi_i$.

The arrival rate of accepted jobs is then $\sum_{i=0}^{Kr} \pi_i$.

By Little's formula

the long-run fraction of the time the workstation is busy=$\frac{r}{\mu}\sum_{i=0}^{Kr} \pi_i$.