MVE055 2018 Lecture 2

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Random variables

- A random variable is a (measurable) function on the sample space;
- There are three types of random variables:
  - categorical: if they assume non-numerical values, e.g. Blood type;
  - discrete: if the possible values it can assume are at most finite or countably many (example: number of earthquakes in a year);
  - continuous: it can assume values in an interval or the entire real line (example: height, time)
- Notation: X typically denotes a random variable and x a realisation (observed value) of the random variable.

What is a random variable?

Discrete random variables

Definition

Let X be a discrete random variable. The function

$$f(x) = \Pr[X = x], \; x \in \mathbb{R}$$

is called the density function of X. The cumulative distribution function of X, denoted by F, is defined as

$$F(x) = \Pr[X \leq x] = \sum_{y \leq x} f(y), \; x \in \mathbb{R}$$

Theorem

A function f is a density function of some discrete random variable if and only if

- $f(x) \geq 0$ *for all* x
- $\sum_{all\ x} f(x) = 1$

Expected value, variance and standard deviation

Definition (Expected value)

Let X be a discrete random variable with density function f. We define the expected value of X (denoted as $\mathbb{E}[X]$) as

$$\mathbb{E}[X] = \sum_{\text{all } x} x f(x)$$

provided that $\sum_{\text{all } x} |x| f(x)$ is finite.
In general, if $H(X)$ is a random variable, the expected value of $H(X)$ is given by

$$\mathbb{E}[H(X)] = \sum_{\text{all } x} H(x) f(x)$$

provided that $\sum_{\text{all } x} |H(x)| f(x)$ is finite.

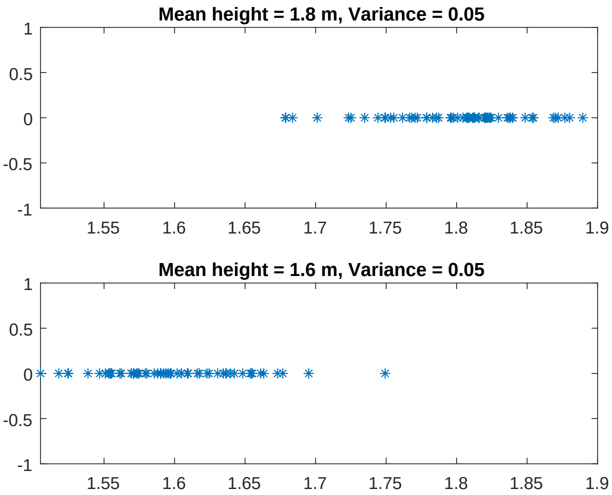

Figure: Top: Height of 60 males; Bottom: Height of 60 females.

Properties of the expected value

- For a constant a, b, c and random variables X, Y

$$\mathbb{E}[c] = c$$

$$\mathbb{E}[cX] = c\mathbb{E}[X]$$

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\rightarrow \mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

- Notation: we typically use the letter μ to denote the expected value.

Variance and standard deviation

Definition (Variance, standard deviation)

Let X be a random variable with expected value μ. The variance of X (denoted as $\mathrm{Var}[X]$ or σ^2) is

$$\mathrm{Var}[X] = \mathbb{E}[(X-\mu)^2].$$

The standard deviation of X (denoted by σ) is given by

$$\mathrm{Sd}[X] = \sqrt{\mathrm{Var}[X]}$$

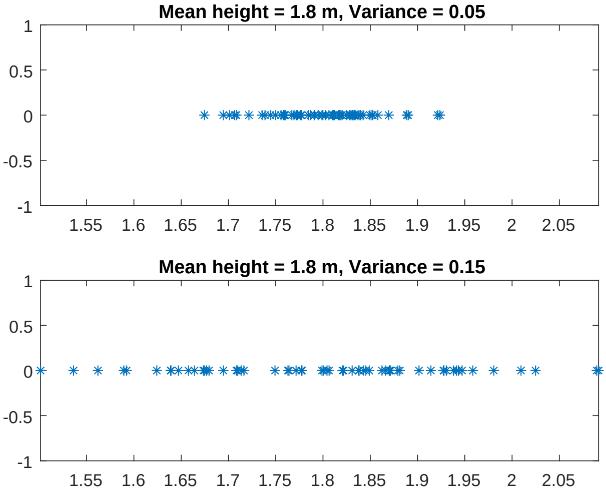

Figure: Top: Height of 60 Swedes; Bottom: Height of 60 Norwegians.

Properties of the variance and standard deviation

- It holds
$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
- For a constant c and a random variable X
$$\mathrm{Var}[c] = 0$$
$$\mathrm{Var}[cX] = c^2\,\mathrm{Var}[X]$$
$$\mathrm{Sd}[cX] = c\,\mathrm{Sd}[X]$$

Bernoulli random variable

- A trial has two possible outcomes: success (1) and failure (0) with density function

$$f(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

- $\mathbb{E}[X] = p$
- $\text{Var}[X] = p(1 - p)$
- Example: a coin toss where we consider success if the outcome is "H"

Important series

For $|s| < 1$ it holds

- $$\sum_{k=0}^{\infty} s^k = \frac{1}{1-s}$$
- $$\sum_{k=0}^{n} s^k = \frac{1-s^{n+1}}{1-s}$$

and

- $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Geometric random variable

- Possible values $0, 1, 2, ...$ with density function

$$f(x) = \begin{cases} p(1-p)^{x-1} & \text{if } x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

- It can be associated to a sequence of independent experiments, each one with probability p of "success". The geometric random variable models the trial in which we obtain the first "success" (example: toss a coin infinite times and check the first time where we obtain "head");
- $\mathbb{E}[X] = \frac{1}{p}$
- $\text{Var}[X] = \frac{1-p}{p^2}$

Binomial random variable

- A trial has two possible outcomes: success (1) and failure (0) with probability p of success. The random variable X which counts the number of successes in n independent trial is called Binomial and has the following density function

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{if } x = 0, 1, 2, ..., n \\ 0 & \text{otherwise} \end{cases}$$

- $\mathbb{E}[X] = np$
- $\mathrm{Var}[X] = np(1-p)$
- **Exercise**: Show that f above is a density function, compute expected value and variance (Hint: use the series in the previous slide).

Binomial density function

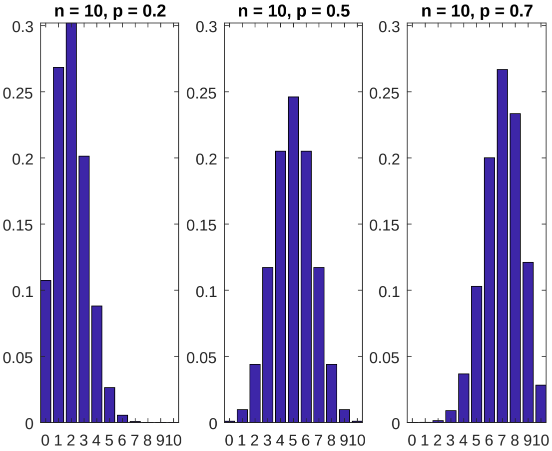

Figure: Binomial density function for different values of the parameters n and p.

Moment generating function

The **moments** of a random variables are $\mathbb{E}[X], \mathbb{E}[X^2], \mathbb{E}[X^3], \ldots$

Definition (Moment generating function)

Let X be a random variable with density f. The moment generating function for X is given by

$$m_X(t) = \mathbb{E}[e^{tX}]$$

provided the right hand side is finite for all t in some open interval.

Moment generating function

Theorem

Let $m_X(t)$ be the moment generating function of X. Then

$$\frac{d^k m_X(t)}{dt^k}\Big|_{t=0} = \mathbb{E}[X^k]$$

We will use the moment generating function later in the course.