MSG 810, Lösningar tenta 2019-01-18

1. (a) f täthetsfunktion om $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = c\int_0^2 \left(x - \tfrac{1}{2}x^2\right)dx = c\left[\frac{x^2}{2} - \frac{x^3}{2\cdot 3}\right]_0^2 =$$

$$= c\left(2 - \frac{4}{3}\right) = c\cdot\frac{2}{3} \overset{\text{vill}}{=} 1 \iff \underline{\underline{c = \frac{3}{2}}}$$

(b) $$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{3}{2}\int_0^2 \left(x^2 - \tfrac{1}{2}x^3\right)dx = \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^4}{2\cdot 4}\right]_0^2 =$$

$$= \frac{3}{2}\left(\frac{8}{3} - 2\right) = \frac{3}{2}\cdot\frac{2}{3} = \underline{\underline{1}}$$

$$\int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{3}{2}\int_0^2 \left(x^3 - \tfrac{1}{2}x^4\right)dx = \frac{3}{2}\left[\frac{x^4}{4} - \frac{x^5}{2\cdot 5}\right]_0^2 =$$

$$= \frac{3}{2}\left(4 - \frac{16}{5}\right) = \frac{3}{2}\cdot\frac{4}{5} = \frac{6}{5} \Rightarrow$$

$$\Rightarrow \mathrm{Var}(X) = \frac{6}{5} - 1^2 = \frac{1}{5} \Rightarrow \sigma = \sqrt{\mathrm{Var}(X)} = \underline{\underline{\frac{1}{\sqrt{5}}}} \ (\approx 0.4472)$$

(c) $$\mathbb{E}(Y) = \mathbb{E}(700X) = 700\,\mathbb{E}[X] \overset{(b)}{=} 700\cdot 1 = 700$$

$$\mathrm{Var}(Y) = \mathrm{Var}(700X) = 700^2\,\mathrm{Var}(X) \overset{(b)}{=} \frac{700^2}{5} \Rightarrow \sigma = \frac{700}{\sqrt{5}}$$

Låt $Y_1, \ldots, Y_{365}$ där Y_i = Kickans utgifter dag nr. $i = 1, \ldots, 365$

Totala utgifter: $\zeta = Y_1 + \ldots + Y_{365}$

Centrala gränsvärdessatsen:

$$\zeta \text{ appr. } N\left(365\cdot 700, \frac{700}{\sqrt{5}}\sqrt{365}\right) = N\left(255\,500, 700\sqrt{73}\right)$$

$$P(\zeta \leq 245000) = P\left(\frac{\zeta - 255500}{700\sqrt{73}} \leq \frac{245000 - 255500}{700\sqrt{73}}\right) \approx$$

$$\approx P(Z \leq -1.76) = \underline{\underline{0.0392}}$$

2. $\bar{x} = 34.24$, $s = 4.46$, $n = 5$

(a) $\bar{x} \pm \lambda_{0.01} \frac{\sigma}{\sqrt{n}} = 34.24 \pm 2.33 \cdot \frac{4}{\sqrt{5}} =$

$= [30.07197, 38.40803] \approx \underline{[30.07, 38.41]}$

(b) $\bar{x} \pm t_{0.01}(4) \frac{s}{\sqrt{n}} = 34.24 \pm 3.747 \cdot \frac{4.46}{\sqrt{5}} =$

$= [26.76634, 41.71366] \approx \underline{[26.76, 41.72]}$

(c) $\left[\sqrt{\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}(n-1)}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}} \right] =$

$= \left[\frac{2 \cdot 4.46}{\sqrt{\chi^2_{0.01}(4)}}, \frac{2 \cdot 4.46}{\sqrt{\chi^2_{0.99}(4)}} \right] = \left[\frac{8.92}{\sqrt{13.3}}, \frac{8.92}{\sqrt{0.297}} \right] =$

$= [2.4459, 16.3677] \simeq \underline{[2.44, 16.37]}$

3 (a)

X	0	1	2	3
p_X	0.2	0.4	0.3	0.1

$$\mathbb{E}[X] = 0\cdot 0.2 + 1\cdot 0.4 + 2\cdot 0.3 + 3\cdot 0.1 = 1.3$$

$$\mathrm{Var}(X) = 0^2\cdot 0.2 + 1^2\cdot 0.4 + 2^2\cdot 0.3 + 3^2\cdot 0.1 - 1.3^2 = 0.81$$

$$\Rightarrow \quad \sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{0.81} = 0.9$$

(b) Låt $X_1, \dots, X_{1000}$ s.v. där X_i = antal limpor kund nr. $i = 1, \dots, 1000$

Totala antalet eftersökta limpor:

$$Y = X_1 + \dots + X_{1000}$$

Centrala gränsvärdessatsen + (a):

Y appr. $N(1000\cdot 1.3,\ 0.9\sqrt{1000}) = N(1300,\ 0.9\sqrt{1000})$

Vill bestämma $k \in \mathbb{R}$ så att: $P(Y > k) = 0.05$

$$P(Y>k) = 1 - P(Y \leq k) = 1 - P\left(\frac{Y-1300}{0.9\sqrt{1000}} \leq \frac{k-1300}{0.9\sqrt{1000}}\right) \approx$$

$$\approx 1 - P\left(Z \leq \frac{k-1300}{0.9\sqrt{1000}}\right) \overset{\text{vill}}{=} 0.05 \iff$$

$$\iff P\left(Z \leq \frac{k-1300}{0.9\sqrt{1000}}\right) = 0.95$$

$$P(Z \leq 1.65) = 0.95 \Rightarrow \frac{k-1300}{0.9\sqrt{1000}} = 1.65 \iff$$

$$\iff k = 1300 + 1.65\cdot 0.9\sqrt{1000} \approx \underline{1347 \text{ limpor}}$$

4. Två stickprov med samma σ

$\underline{1.:}$ $n = 7$, $\bar{x} = 2.637143$, $s_1^2 = 0.0001230$

$\underline{2.:}$ $m = 7$, $\bar{y} = 2.681429$, $s_2^2 = 0.0001783$

$$s^2 = \frac{(n-1)s_1^2 + (m-1)s_2^2}{n+m-2} = \frac{1}{2}(s_1^2 + s_2^2) = 0.0001506$$

$$\bar{y} - \bar{x} \pm t_{\frac{\alpha}{2}}(n+m-2)\, s \sqrt{\frac{1}{n} + \frac{1}{m}} = \{\alpha = 0.01\} =$$

$$= 0.044286 \pm 3.0545 \cdot 0.012273 \cdot 0.534522 =$$

$$= [0.0242479,\ 0.0643241] \approx [0.024,\ 0.065]$$

$\therefore$ Maskin 2 tillverkar i 99% av fallen underlägg som är mellan 0.024 och 0.065 mm tjockare än Maskin 1.

5 (a) Vill testa: $H_0: \mu = 60$ mot $H_1: \mu < 60$ med $\alpha = 1\%$

Vill bestämma $k \in \mathbb{R}$ så att:

Om $\bar{x} < k$ så förkasta H_0

Om $\bar{x} \geq k$ så förkasta inte H_0

Om H_0 sann så $\bar{x}$ observ. på $\bar{X} \in N\left(60, \frac{6.2}{\sqrt{12}}\right)$

$$\alpha = P(\text{förkasta } H_0 \mid H_0 \text{ sann})$$

$$\Longleftrightarrow$$

$$0.01 = P\left(\bar{X} < k \mid \bar{X} \in N\left(60, \frac{6.2}{\sqrt{12}}\right)\right) =$$

$$= P\left(\frac{\bar{X} - 60}{6.2/\sqrt{12}} < \frac{k - 60}{6.2/\sqrt{12}}\right) = P\left(Z < \frac{k-60}{6.2/\sqrt{12}}\right)$$

$$P(Z \leq -2.33) = 0.01 \Rightarrow \frac{k-60}{6.2/\sqrt{12}} = -2.33 \Longleftrightarrow$$

$$\Longleftrightarrow k = 60 - 2.33 \frac{6.2}{\sqrt{12}} \approx 55.8$$

$\therefore$ Om medelvikten av de 12 äggen är mindre än 55.8 g kan Kickan med 1% felrisk påstå att äggen väger mindre än vad som utlovats.

(b) (Styrka $\mu = 57$) $\overset{(a)}{=} P\left(\bar{X} < 55.8 \mid \bar{X} \in N\left(57, \frac{6.2}{\sqrt{12}}\right)\right) =$

$$= P\left(\frac{\bar{X} - 57}{6.2/\sqrt{12}} < \frac{55.8 - 57}{6.2/\sqrt{12}}\right) \approx P(Z < -0.67) \approx \underline{\underline{25\%}}$$

6 (a) $\bar{x} = \frac{5500}{10} = 550$, $\bar{y} = \frac{589}{10} = 58.9$

$$SS_{xx} = \sum_{i=1}^{10} x_i^2 - 10 \cdot \bar{x}^2 = 3710\,000 - 10 \cdot 550^2 = 685\,000$$

$$SS_{yy} = \sum_{i=1}^{10} y_i^2 - 10 \cdot \bar{y}^2 = 37685 - 10 \cdot 58.9^2 = 2992.9$$

$$SS_{xy} = \sum_{i=1}^{10} x_i y_i - 10 \bar{x}\bar{y} = 364700 - 10 \cdot 550 \cdot 58.9 = 40750$$

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx}}\sqrt{SS_{yy}}} = \frac{40750}{\sqrt{685000}\sqrt{2992.9}} \approx 0.90$$

r nära 1, så det verkar finnas ett linjärt samband

(b) $$b_1 = \frac{SS_{xy}}{SS_{xx}} = \frac{40750}{685000} \approx 0.0595$$

$$b_0 = \bar{y} - b_1 \bar{x} = 58.9 - 0.0595 \cdot 550 \approx 26.181$$

$$\Rightarrow y = 26.181 + 0.0595x$$

(c) $$s^2 = \frac{SS_{xx}SS_{yy} - SS_{xy}^2}{(n-2)SS_{xx}} = \frac{685000 \cdot 2992.9 - 40750^2}{8 \cdot 685000} \approx 71.090$$

$$\Rightarrow s = \sqrt{71.090} \approx 8.4315$$

$$x_0 = 300 \Rightarrow b_0 + b_1 x_0 \approx 44.031$$

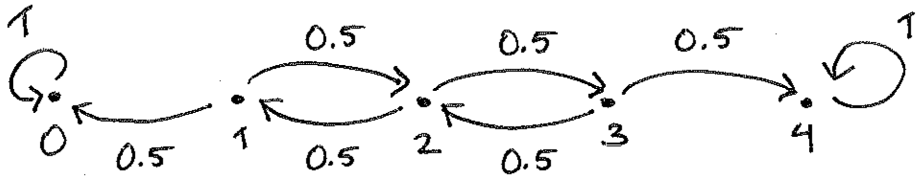
$$b_0 + b_1 x_0 \pm t_{\frac{\alpha}{2}}(n-2) \cdot s \cdot \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{SS_{xx}}} = \{\alpha = 0.05\} =$$

$$= 44.031 \pm 2.306 \cdot 8.4315 \sqrt{1 + 0.1 + \frac{250^2}{685000}} =$$

$$= 44.031 \pm 21.2209 \approx [22, 66]$$

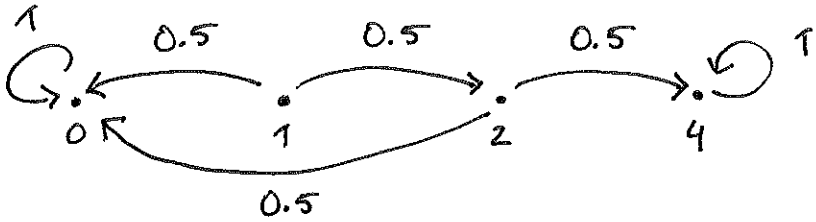
$\therefore$ Mellan 22 och 66 varor

7. (a) (i):



$$\mathbb{P}_1 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 0 & 4 \\ \hline 1 & 0 & 0.5 & 0 & 0.5 & 0 \\ 2 & 0.5 & 0 & 0.5 & 0 & 0 \\ 3 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

(ii):



$$\mathbb{P}_2 = \begin{array}{c|cccc} & 1 & 2 & 0 & 4 \\ \hline 1 & 0 & 0.5 & 0.5 & 0 \\ 2 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

(b) $\mathbb{Q}_1 = \begin{pmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 \end{pmatrix} \Rightarrow I - \mathbb{Q}_1 = \begin{pmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 1 & -1/2 & 0 & 1 & 0 \\ 0 & -1/2 & 1 & 0 & 0 & 1 \end{array}\right) 1/2 \sim \begin{pmatrix} 1 & -1/2 & 0 & 1 & 0 & 0 \\ 0 & 3/4 & -1/2 & 1/2 & 1 & 0 \\ 0 & -1/2 & 1 & 0 & 0 & 1 \end{pmatrix} \cdot \frac{2}{3} \sim$$

$$\sim \begin{pmatrix} 1 & -1/2 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & -1/3 & 1/3 & 2/3 & 0 \\ 0 & -1/2 & 1 & 0 & 0 & 1 \end{pmatrix} 1 \sim \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 2/3 & 0 \\ 0 & 1/2 & -1/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & 2/3 & 1/3 & 2/3 & 1 \end{pmatrix} 1/2 \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 2/3 & 1/3 & 2/3 & 1 \end{pmatrix} \begin{matrix} \\ \cdot 2 \\ \cdot \frac{3}{2} \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1.5 & 1 & 0.5 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0.5 & 1 & 1.5 \end{array}\right)$$

$$\Rightarrow \mathbb{N}_1 = (I - \mathbb{Q}_1)^{-1} = \begin{pmatrix} 1.5 & 1 & 0.5 \\ 1 & 2 & 1 \\ 0.5 & 1 & 1.5 \end{pmatrix}$$

$$\mathbb{B}_1 = \mathbb{N}_1 \mathbb{R}_1 = \begin{pmatrix} 1.5 & 1 & 0.5 \\ 1 & 2 & 1 \\ 0.5 & 1 & 1.5 \end{pmatrix} \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \\ 0 & 0.5 \end{pmatrix} = \begin{array}{cc} \begin{array}{cc} \quad 0 & \quad\; 4 \end{array} & \\ \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

Sann. att gå från 1 kr till 4 kr $= (\mathbb{B}_1)_{12} = 0.25$

$$\mathbb{Q}_2 = \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix} \Rightarrow I - \mathbb{Q}_2 = \begin{pmatrix} 1 & -0.5 \\ 0 & 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \mathbb{N}_2 = (I - \mathbb{Q}_2)^{-1} = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$$

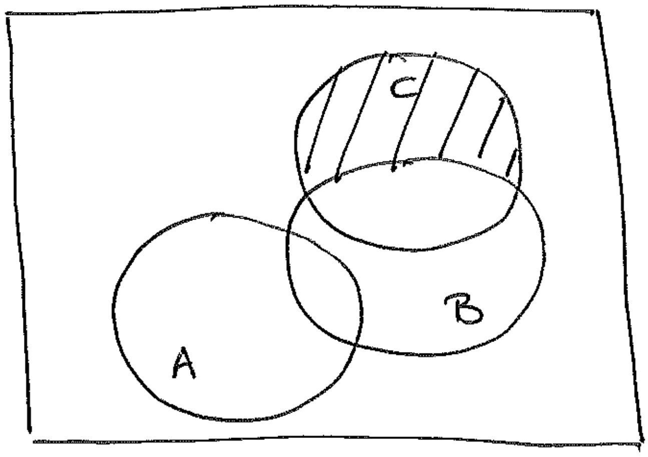
$$\mathbb{B}_2 = \mathbb{N}_2 \mathbb{R}_2 = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{pmatrix} = \begin{array}{cc} \begin{array}{cc} \quad 0 & \quad\; 4 \end{array} & \\ \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

Sann. att gå från 1 kr till 4 kr $= (\mathbb{B}_2)_{12} = 0.25$

$\therefore$ Spelar ingen roll vilken strategi Pelle väljer!

8 (a) Se kursboken

(b) A och C disjunkta $\Rightarrow P(A \cap C) = 0$, så vi har följande figur:



Ser att: $P(B^c \cap C) = P(A \cup B \cup C) - P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \{A, B \text{ ober.}\} =$$

$$= P(A) + P(B) - P(A) \cdot P(B) =$$

$$= 0.4 + 0.5 - 0.4 \cdot 0.5 = 0.7$$

$$\therefore P(B^c \cap C) = 0.9 - 0.7 = \underline{\underline{0.20}}$$