MSG810, Lösningar tenta 2019-04-26

1 (a) Definition: Antag att det finns m möjliga utfall i S ($m \neq \infty$) och att alla utfall är lika sannolika. Om händelsen A består av g av utfallen, så är sannolikheten för A, $P(A)$, lika med

$$P(A) = \frac{g}{m}$$

Låt A_k = Man får exakt k felmärkta motstånd

(b) $$P(A_0) = \frac{\binom{6}{0}\binom{34}{5}}{\binom{40}{5}} = \frac{34 \cdot 33 \cdot 32 \cdot 31 \cdot 30}{40 \cdot 39 \cdot 38 \cdot 37 \cdot 36} = 0.422876$$

$\Rightarrow$ Ca 42%

(c) $$P(A_1) = \frac{\binom{6}{1}\binom{34}{4}}{\binom{40}{5}} = \frac{6 \cdot 34 \cdot 33 \cdot 32 \cdot 31}{8 \cdot 39 \cdot 38 \cdot 37 \cdot 36} = 0.422876$$

$$P(A_2) = \frac{\binom{6}{2}\binom{34}{3}}{\binom{40}{5}} = \frac{5 \cdot 17 \cdot 33 \cdot 32}{13 \cdot 38 \cdot 37 \cdot 36} = 0.136412$$

$$\Rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2) = 0.559288$$

$\Rightarrow$ Ca 56%

(d) $$P(A_3 \cup A_4 \cup A_5) = 1 - P(A_0 \cup A_1 \cup A_2) =$$

$$= 1 - P(A_0) - P(A_1) - P(A_2) = 1 - 2 \cdot 0.422876 - 0.136412 =$$

$$= 0.017836$$

$\Rightarrow$ Ca 1.8%

2. (a) $\int_{-\infty}^{\infty} f(x)\,dx = c\int_0^1 (x^2 - x^3)\,dx = c\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 =$

$= c\left(\frac{1}{3} - \frac{1}{4}\right) = c \cdot \frac{1}{12} \overset{\text{vill}}{=} 1 \iff \underline{\underline{c = 12}}$

(b) $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = 12\int_0^1 (x^3 - x^4)\,dx = 12\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 =$

$= 12\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{12}{20} = \frac{3}{5} = \underline{\underline{0.6}}$

$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 12\int_0^1 (x^4 - x^5)\,dx = 12\left[\frac{x^5}{5} - \frac{x^6}{6}\right]_0^1 =$

$= 12\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{12}{30} = \frac{2}{5}$

$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{5} - \frac{9}{25} = \frac{1}{25}$

$\Rightarrow \sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{1}{25}} = \frac{1}{5} = \underline{\underline{0.2}}$

(c) $P(0.5 \leq X \leq 0.8 \mid 0.4 \leq X \leq 0.7) =$

$= \frac{P(0.5 \leq X \leq 0.8 \text{ och } 0.4 \leq X \leq 0.7)}{P(0.4 \leq X \leq 0.7)} =$

$= \frac{P(0.5 \leq X \leq 0.7)}{P(0.4 \leq X \leq 0.7)} = \frac{12\int_{0.5}^{0.7}(x^2 - x^3)\,dx}{12\int_{0.4}^{0.7}(x^2 - x^3)\,dx} =$

$= \frac{\left[4x^3 - 3x^4\right]_{0.5}^{0.7}}{\left[4x^3 - 3x^4\right]_{0.4}^{0.7}} = \frac{0.3392}{0.4725} \approx \underline{\underline{0.72}}$

3. $\bar{x} = \frac{9.33}{5} = 1.866$

$$s^2 = \frac{1}{4}\left(\sum_{i=1}^{5} x_i^2 - 5\cdot\bar{x}^2\right) = \frac{1}{4}(31.2035 - 17.40978) = 3.44843$$

$$\Rightarrow s = \sqrt{3.44843} \approx 1.85699$$

(a) $$\bar{x} \pm \lambda_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} = \left\{ \begin{matrix} \alpha = 0.05 \\ n = 5,\ \sigma = 2 \end{matrix} \right\} = 1.866 \pm 1.96 \frac{2}{\sqrt{5}} =$$

$$= [0.112922, 3.619077] \approx [0.11, 3.62]$$

(b) $$\bar{x} \pm t_{\frac{\alpha}{2}}(n-1) \cdot \frac{s}{\sqrt{n}} = \left\{ \begin{matrix} \alpha = 0.05 \\ n = 5 \\ s = 1.85699 \end{matrix} \right\} =$$

$$= 1.866 \pm 2.7764 \cdot \frac{1.85699}{\sqrt{5}} = [-0.43973, 4.171726] \approx$$

$$\approx [-0.44, 4.18]$$

(c) $$\left[\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}(n-1)}, \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)} \right] = \left\{ \begin{matrix} \alpha = 0.01,\ n = 5 \\ s^2 = 3.44843 \end{matrix} \right\} =$$

$$= \left[\frac{4 \cdot 3.44843}{14.860}, \frac{4 \cdot 3.44843}{0.207} \right] =$$

$$= [0.928245, 66.636329] \approx [0.92, 66.64]$$

4. Direktmetoden. Vi vill testa:

H_0: Myntet är symmetriskt

mot

H_1: Myntet är ej symmetriskt

med $\alpha = 5\%$

Låt x = antal krona vid de 136 kasten = 57

Om H_0 sann så x observ. på $X \in Bin(136, 0.5)$

$$\Rightarrow P = P(X \leq 57 \mid X \in Bin(136, 0.5)) =$$

$$= \Big\{ np(1-p) = 136 \cdot 0.5 \cdot 0.5 = 34 > 10 \Rightarrow$$

$$\Rightarrow X \text{ appr. } N(np, \sqrt{np(1-p)}) = N(68, \sqrt{34}) \Big\} =$$

$$= P(X \leq 57) = P\left(\frac{X-68}{\sqrt{34}} \leq \frac{57-68}{\sqrt{34}}\right) \approx$$

$$\approx P(Z \leq -1.89) = 0.0294$$

$P = 0.0294 > 0.025 = \frac{\alpha}{2} \Rightarrow$ förkasta <u>inte</u> H_0!

$\therefore$ Myntet är med 5% felrisk symmetriskt!

5 (a) $P(Y_1 \geq 12000) = P(100X_1 \geq 12000) = P(X_1 \geq 120) =$

$= 1 - P(X_1 < 120) = 1 - P\left(\frac{X_1 - 100}{10} < \frac{120-100}{10}\right) =$

$= 1 - P(Z < 2) = 1 - 0.9772 = \underline{\underline{0.0228}}$

(b) $P(Y_2 \geq 12000) = P(50X_1 + 50X_2 \geq 12000) =$

$= P(50(X_1 + X_2) \geq 12000) = P(X_1 + X_2 \geq 240) =$

$= \{X_1 + X_2 \in N(200, \sqrt{200})\} = 1 - P(X_1 + X_2 < 240) =$

$= 1 - P\left(Z < \frac{40}{\sqrt{200}}\right) \approx 1 - P(Z < 2.83) =$

$= 1 - 0.9977 = \underline{\underline{0.0023}}$

(c) $P(Y_1 > Y_2 + 1000) = P(Y_1 - Y_2 > 1000) =$

$= P(100X_1 - (50X_1 + 50X_2) > 1000) = P(50(X_1 - X_2) > 1000) =$

$= 1 - P(X_1 - X_2 \leq 20) = \{X_1 - X_2 \in N(0, \sqrt{200})\} =$

$= 1 - P\left(Z \leq \frac{20}{\sqrt{200}}\right) \approx 1 - P(Z \leq 1.41) =$

$= 1 - 0.9207 = \underline{\underline{0.0793}}$

6. Låt X = livslängd tvättmaskin

Vet att : $X \in Exp(\lambda) \Rightarrow \mathbb{E}[X] = \frac{1}{\lambda} \overset{\text{vet}}{=} 2 \Leftrightarrow \lambda = 0.5$

Alltså : $X \in Exp(0.5)$

Låt nu Y = kostnad reparation

Sökt : $\mathbb{E}[Y]$

Vet att Y har fördelningen :

Y	0	2000	4000
p_Y	$P(X<1)$	$P(1 \leq X < 5)$	$P(X \geq 5)$

$P(X<1) = F_X(1) = 1 - e^{-0.5 \cdot 1} \approx 0.39347$

$P(1 \leq X < 5) = F(5) - F(1) = 1 - e^{-0.5 \cdot 5} - (1 - e^{-0.5 \cdot 1}) =$

$= e^{-0.5} - e^{-2.5} \approx 0.52445$

$P(X \geq 5) = 1 - F(5) = e^{-2.5} \approx 0.082085$

$\Rightarrow \mathbb{E}[Y] = 0 \cdot 0.39347 + 2000 \cdot 0.52445 + 4000 \cdot 0.082085 \approx$

$\approx \underline{\underline{1377 \text{ kr}}}$

7. Vill använda gen. fkn. för $\{a_n\}_{n=0}^{\infty}$ dvs $A(x)=\sum_{n=0}^{\infty} a_n x^n$

$\Rightarrow$ Mult. båda leden i $a_n = \frac{1}{2}a_{n-1}+2$ med x^n

och summera över $n \geq 1$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty}\left(\frac{1}{2}a_{n-1}+2\right)x^n$$

$$VL = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n x^n + a_0 - a_0 = \sum_{n=0}^{\infty} a_n x^n - a_0 =$$

$$= A(x) - 3$$

$$HL = \frac{1}{2}\sum_{n=1}^{\infty} a_{n-1}x^n + 2\sum_{n=1}^{\infty} x^n =$$

$$= \frac{1}{2}x\sum_{n=1}^{\infty} a_n x^n + 2\left(\sum_{n=0}^{\infty} x^n - 1\right) =$$

$$= \frac{1}{2}x\,A(x) + 2\left(\frac{1}{1-x} - \frac{1-x}{1-x}\right) = \frac{1}{2}x\,A(x) + \frac{2x}{1-x}$$

$$\Rightarrow A(x) - 3 = \frac{1}{2}x\,A(x) + \frac{2x}{1-x} \Leftrightarrow A(x)\left(1-\frac{1}{2}x\right) = 3 + \frac{2x}{1-x}$$

$$\Leftrightarrow A(x)\,\frac{2-x}{2} = \frac{3-x}{1-x} \Leftrightarrow A(x) = 2\cdot\frac{3-x}{(1-x)(2-x)}$$

$$\frac{3-x}{(1-x)(2-x)} = \frac{A}{1-x} + \frac{B}{2-x} \Leftrightarrow 3-x = A(2-x) + B(1-x)$$

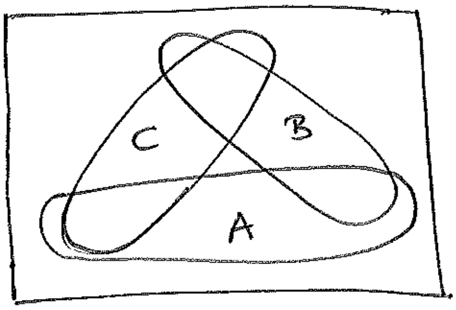
$\underline{x=1}$: $2 = A$, $\underline{x=2}$: $1 = -B \Leftrightarrow B = -1$

$$A(x) = 2\left(\frac{2}{1-x} - \frac{1}{2(1-\frac{x}{2})}\right) = 2\left(2\sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}x^n\right) =$$

$$= \sum_{n=0}^{\infty} 4x^n - \sum_{n=0}^{\infty}\frac{1}{2^n}x^n = \sum_{n=0}^{\infty}\left(4-\frac{1}{2^n}\right)x^n \overset{\text{vill}}{=} \sum_{n=0}^{\infty} a_n x^n$$

$$\therefore \underline{a_n = 4 - \frac{1}{2^n}} \quad n \geq 0$$

8. Vet att $P(A \cap B \cap C) = 0$, så vi har följande figur:



Vet att $P(X=3) = P(A \cap B \cap C) = 0$

$P(X=2) = P(A \cap B) + P(A \cap C) + P(B \cap C) = \{A, C \text{ ober.}\} =$

$= 0.003 + P(A) \cdot P(C) + 0.003 =$

$= 0.006 + 0.01 \cdot 0.02 = 0.0062$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(X=2) =$

$= 0.01 + 0.01 + 0.02 - 0.0062 = 0.0338$

$\Rightarrow P(X=0) = 1 - P(A \cup B \cup C) = 0.9662$

$P(X=1) = P(A \cup B \cup C) - P(X=2) = 0.0276$

$\therefore$

X	0	1	2	3
P_X	0.9662	0.0276	0.0062	0