THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Fatigue limit, inclusion and finite lives -a statistical point of view

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Abstract

An important design property of steel is the fatigue limit, i.e. the load level where a specimen has infinite life. For the material it can also be defined an endurance limit, which is the stress level at which the specimen has a certain life. It will not break before n_o cycles. Obviously it is of interest to estimate these limits. The maximum inclusion size in clean steels influences the fatigue limit behaviour. Therefore it is also interesting to estimate the distribution of the inclusion size which causes the failure. Here the results from staircase tests are used to estimate the fatigue limit, the endurance limit and the inclusion size distribution.

By combining the fatigue limit and the finite life distribution the whole S-N curve can be estimated at the same time. The uncertainties in the estimated parameters are taken into account by using the method of information matrix or the method of profile predictive likelihood.

The staircase test is combined with the $\sqrt{area_{max}}$-model to estimate the distribution of the inclusion size where the failure occurs. This can be done by using stress levels with or without inclusion measurements.

By combining the $\sqrt{area_{max}}$-model and the assumption that the inclusion size above a fixed threshold follows a generalized Pareto distribution, the stresses from the staircase test can be used to estimate the probability that a component will fail. Under these assumptions the probability that the component will fail under a rotating bending test can be estimated using data from a uniaxial test on a different component and vice versa.

Keywords: Censoring, Likelihood, Profile predictive likelihood, Staircase test, Extreme value

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LIST OF PUBLICATIONS

This thesis is based on the work contained in the following papers, referred to by Roman numerals in the text:

I. Lorén, S. Fatigue limit estimated using finite lives, *Fatigue & Fracture of Engineering Materials & Structures*, 2003:26:757-766.

II. Lorén, S. Estimating inclusion distributions of hard metal using fatigue tests, *International Journal of Fatigue*, 2003:25:129-137.

III. Lorén, S. Estimating fatigue limit distributions under inhomogeneous stress conditions, to appear in *International Journal of Fatigue*.

IV. Lorén, S. Lundström, M. Estimating curved S-N curves, Submitted.

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1 Introduction

The fatigue limit is often an important characteristic in fatigue design for metallic materials. Components that are expected to survive millions of repeated loads must be designed against this limit. When the specimen is exposed to a load level below the fatigue limit it will never break. It will have an infinite life. As a specimen can never be tested for an infinite time, a load level will be defined, and below this limit the specimens will never break before some specified finite number of cycles. This limit is the endurance limit. In the literature some authors use endurance limit, others fatigue limit. Here the fatigue limit corresponds to infinite life and endurance limit to some specified finite life.

Fatigue properties of hard steels are strongly influenced by inclusions in the material. Inclusions can be microscopic particles of foreign material, for instance oxides [1]. The endurance limit can also be determined by the inclusions in the specimens. According to [2] the endurance limit is a function of the largest inclusion and the hardness of the material, the $\sqrt{area_{max}}$-model. Therefore it is also interesting to estimate the inclusion size distribution, as the endurance limit distribution can then also be estimated.

The S-N curve (Wöhler curve) describes the relation between a given stress amplitude S and the number of cycles N until a specimen breaks. The stress or the logarithm of the stress is plotted against the logarithm of the life. For some materials the empirical S-N curve has an asymptotic behaviour when $N \to \infty$. The corresponding nonzero stress level is then called the fatigue limit. For stress levels well below the ultimate strength and well above the fatigue limit the S-N curve follows the Basquin function $S = aN^b$, which means that it is linear in a log-log diagram. If S or the logarithm of S is plotted against the logarithm N, the corresponding S-N curve consists in principal of three parts, one linear, one horizontal and a third part in between. This curve will be modelled by assuming that both the fatigue limit and the finite life are random variables. Under the assumption that the fatigue limit is a random variable the asymptote is the median of the fatigue limit distribution.

Here the fatigue limit, endurance limit and inclusion size are all assumed to be random variables and the parameters in their distribution functions are estimated using the method of maximum likelihood. The likelihood function is the "probability" of observing the data as a function of the parameter we want to estimate. The maximum likelihood estimate of the parameter is the value that maximises the likelihood function.

The fatigue limit, endurance limit and inclusion size distributions are estimated by combining different test and measurement procedures and

applying statistical methods. The procedures are the staircase test and inclusion size measurements. The statistical methods are likelihood methods for censored data, and the models for the inclusions are based on the extreme value theory.

In Section 2 there is a description of the staircase test. The endurance limit distribution can be estimated from the stresses in the staircase test. This is described in Section 3. When the stress is combined with the finite life, the fatigue limit distribution can be estimated, and this is done in Section 4. In Section 5 the inclusion distribution is estimated with and without inclusion size measurements. In Section 5.3 inhomogeneous stresses are treated with different test procedures. In Section 6 percentiles of the fatigue limit are calculated. The conclusions are in Section 7, and in Section 8 there is a summary of the different papers included in the thesis.

2 The staircase test

One way to estimate the fatigue limit distribution is the staircase test (or the up-and-down method) [3]. A start load level, the difference between the test levels and a life n_o are chosen. If the specimen fails before n_o cycles the next specimen has to be tested at a lower load level. If the specimen does not fail within n_o cycles, the next specimen is tested at a higher load level. Therefore, each test is dependent on the previous test results. The number of specimens that are tested is quite small, 15 to 30. In Figure 1 there is an example of a staircase test where the start level is 540 MPa and the difference between the stress levels is 20 MPa. The first specimen is tested at 540 MPa and is unbroken before $n_o = 2 \cdot 10^6$, which is denoted by '$\triangle$'. The second specimen is therefore tested at 560 MPa. This specimen is broken and is denoted by '$\triangledown$'.

3 Estimating endurance limit distribution

The endurance limit distribution can be estimated using the stresses from the staircase test and the likelihood method. Assume that the endurance limit S_{el} is a random variable and that $X_{el} = \ln(S_{el})$ and $x = \ln(s)$. Let X_{el} have the distribution function $F_{X_{el}}\big(x;(\mu_{xel},\sigma_{xel})\big)$. The distribution can for example be normal with the mean μ_{xel} and the variance σ^2_{xel}. Assume also that the specimens break if the tested stress level is above the endurance limit. The probability that a specimen breaks is therefore $P\big(X_{el} \leq x\big) = F_{X_{el}}\big(x;(\mu_{xel},\sigma_{xel})\big)$. If the specimen survives it is assumed that the tested stress level is below the endurance limit. The probability that a tested specimen does not break is $P\big(X_{el} > x\big) =$

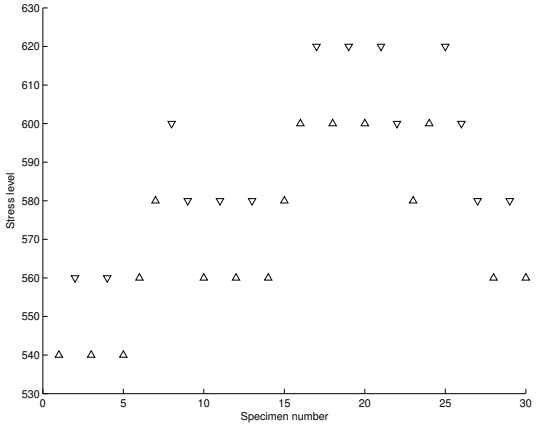

Figure 1: ' $\triangle$ ' *censored specimen,* ' $\triangledown$ ' *uncensored specimen.*

$1 - F_{Xel}\big(x;(\mu_{xel},\sigma_{xel})\big)$. Splitting the observations into two disjoint sets, U corresponding to the uncensored (broken) observations and C to the censored (unbroken) ones the likelihood function can be written as

$$L(\sigma_{xel},\mu_{xel}) = \prod_{i\in U} F_{Xel}\big(x_i;(\mu_{xel},\sigma_{xel})\big) \prod_{i\in C}\Big[1 - F_{Xel}\big(x_i;(\mu_{xel},\sigma_{xel})\big)\Big]. \tag{1}$$

To get the maximum likelihood estimate of the parameters σ_{xel} and μ_{xel}, the likelihood function is maximised numerically. In [4] a simplification of the likelihood is used to get the estimates.

4 Estimating fatigue limit distribution

When estimating the fatigue limit it is tempting to use not only the fact that a specimen has broken but also the actual finite life. This has been done in [5], [6] and [7]. A model that uses the finite lives and the stresses to estimate the fatigue limit distribution is described in this section.

When estimating the fatigue limit using finite lives it is important to think of the connection between the fatigue limit and the finite life. In [7] the finite life depends on the difference of the load and the fatigue limit. In contrast it is here assumed that the load itself determines the finite life through crack propagation. On the other hand if the life is infinite this is because the load is below the fatigue limit corresponding to crack arrest. Hence it is assumed that there are two different mechanisms, one which determines whether crack propagation occurs and one which determines the finite life given that crack propagation occurs. In this case the two mechanisms are uncoupled in contrast to the case considered in [7]. The rational for the case with uncoupled mechanisms is inspired by the work in [8], where it is claimed that finite life occurs for a given load if there is a large enough inclusion. The fatigue limit does not depend on the shape of the inclusion. However, the finite life may not be determined entirely by the inclusion size but also by the shape and chemical content of the inclusion. The model below is based on the fact that the fatigue limit and the finite life are governed by two different mechanisms.

From the staircase test some of the specimens lives are censored since they have not broken before n_o cycles, typically $2{\cdot}10^6$ cycles. The specimen can be censored in two ways. Either the stress level is below the fatigue limit and then the specimen really has infinite life or the stress level is above the fatigue limit and the life is finite but longer than n_o. By taking into account the two ways the specimens can be censored, the fatigue limit distribution can also be estimated using the likelihood method. To do so the fatigue limit S_∞ and the finite life N are assumed to be random variables. The distribution for S_∞ is $F_{S_\infty}\big(s;(\mu,\sigma)\big)$ and the distribution for the logarithm of the finite life N is $F_\tau\big(y;(\nu_x,\tau)\big)$ where $y = \ln(n)$ and $x = \ln(s)$. An example is that both distributions are normal, ν_x is the expectation which is a function of the stress value through its logarithm and τ^2 the variance. The probability that a specimen is censored is $\Big[1 - F_\tau\big(\ln(n_o);(\nu_x,\tau)\big)F_{S_\infty}\big(s;(\mu,\sigma)\big)\Big]$ and the likelihood for an uncensored specimen is $f\big(y;(\nu_x,\tau)\big)F_{S_\infty}\big(s;(\mu,\sigma)\big)$, where $f\big(y;(\nu_x,\tau)\big)$ is the density function for the logarithm of the finite life and $F_{S_\infty}\big(s;(\mu,\sigma)\big)$ the probability that the tested stress level is above the fatigue limit. The likelihood function can then be written as

$$L(\boldsymbol{\theta}) = \prod_{i\in U} f\big(y_i;(\nu_x,\tau)\big)F_{S_\infty}\big(s_i;(\mu,\sigma)\big)\cdot$$

$$\prod_{i\in C}\Big[1 - F_\tau\big(\ln(n_o);(\nu_x,\tau)\big)F_{S_\infty}\big(s_i;(\mu,\sigma)\big)\Big] \tag{2}$$

where $\boldsymbol{\theta} = (\mu,\sigma,\nu_x,\tau)$. Again the likelihood function is maximised numer-

ically to get the maximum likelihood estimate of $\boldsymbol{\theta}$.

In Figure 2 there is an example of the model described above. The almost vertical line is the median of the finite life distribution, when the stress is above the fatigue limit. Looking at the finite lives we have seven observations to the left and seven to the right so the data seems to agree well with the model. The horizontal line is the median of the fatigue limit distribution, the limit where half of the specimens will be censored and half uncensored. Looking at the stress level 580 MPa it seems as if there are too many uncensored specimens compared to the number of censored specimens since 580 MPa is below the median of the fatigue limit. On the other hand, at the stress level 600 MPa there is a majority of censored specimens, which indicates that the median should be above this level.

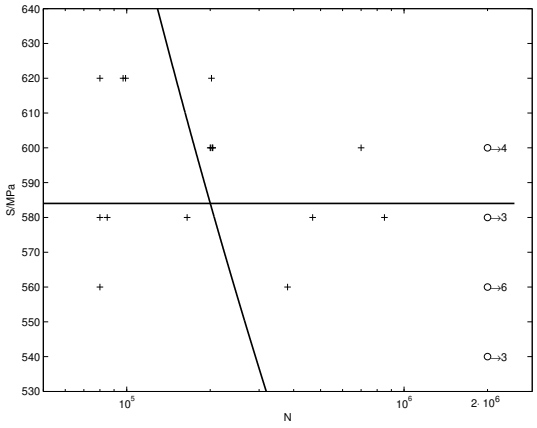

Figure 2: *The horizontal line is the median of the estimated fatigue limit distribution. The other line is the median of the finite life distribution, when the stress is above the fatigue limit. The censored specimen is denoted as 'o' and the uncensored specimen as '+'. The number denotes the number of censored specimens at each stress level.*

4.1 Estimating the curved S-N curve

From the likelihood function (2) it is possible to get the probability that the specimen will survive. The relation between life and stress can therefore be calculated. Given the stress level s the probability of not surviving a certain life is

$$P(Y \leq y) = F_\tau\big(y;(\nu_x,\tau)\big)F_{S_\infty}\big(s;(\mu,\sigma)\big), \qquad y < \ln(n_o). \tag{3}$$

By solving this equation for different stress levels but the same probability it is possible to get the curved S-N curve directly from the probability of not surviving a certain life. The curved S-N curve reflects the possibility of infinite life. For high stress levels the finite life is short and there is a remote possibility of infinite life. Hence the curved S-N curve coincides with the straight line.

In Figure 3 there is an example of the model described above, the curved S-N curve. The dashed line is the solution of equation (3) when the probability is 0.50. Given a stress value the predicted life will not exceed that line with probability 0.50. This line is the only difference from figure 2. Looking at stress levels 600 MPa and 620 MPa there are five observations below the dashed line and six observations above. Also here about half of the observations are below the line, which is in agreement with the model.

In order to take the uncertainty for the estimates into account the profile predictive likelihood can be used [11]. First a general description of the profile predictive likelihood is given. Let $Y = y$ be the data, and we want to predict the unobserved value z of a random variable Z. Let (Y, Z) have the probability density function $f_\theta(y, z; x)$. There are two unknown quantities z and θ, and we want to gain information about z with θ playing the role of a nuisance parameter. The joint likelihood for z and θ is $l_y(z, \theta) = f_\theta(y, z; x)$. By eliminating θ from l_y we get the predictive likelihood $L(z|y)$ for z. One way of eliminating θ is by maximizing $l_y(z,\theta)$ with respect to θ. Doing this we get the profile predictive likelihood $L_p(z|y) = sup_\theta f_\theta(y, z; x)$. This likelihood can often be normalised to be a probability density in z, $\bar{L}_p(z|y)$, and be used to construct prediction intervals for z.

Given a logarithm of the stress, x^*, the logarithm of the predicted life at x^* is denoted as Z. Since the observed distribution is censored at y_o we avoid extrapolation if we restrict ourselves to predict outcomes less than or equal to y_o. Hence the likelihood only needs to be maximized for each z-value up to y_o.

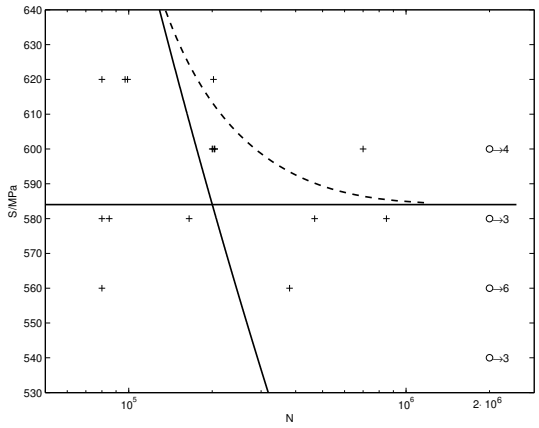

Figure 3: *The horizontal line is the median of the estimated fatigue limit. The other solid line is the median of the finite life distribution, when the stress is above the fatigue limit. The censored specimen is 'o' and the uncensored specimen is '+'. The number denotes the number of censored specimens at each stress level. The dashed line is the life the specimen will survive with probability 0.5 given a certain stress level.*

The density function for an observation y given x can be written in the following way

$$g(y;x,\theta) = f\big(y;(\nu_x,\tau)\big)\,F_{S_\infty}\big(s;\mu,\sigma\big)1_{\{y<y_{ro}\}}$$

$$+\Big(1-F_\tau\big(y_o;(\nu_x,\tau)\big)\,F_{S_\infty}\big(s;\mu,\sigma\big)\Big)1_{\{y=y_{ro}\}}.$$

The profile predictive likelihood can then be written as

$$L_p\big(z|\mathbf{y}\big) = sup_\theta\Big[g\big(z;x^*,\theta\big)\prod_{i=1}^{n} g\big(y_i;x_i,\theta\big)\Big] \tag{4}$$

where y_i $i=1,\cdots,n$ are both the censored and uncensored observations. The product $\prod_{i=1}^{n} g\big(y_i;x_i,\theta\big)$ corresponds to the likelihood in (2).

For a fixed logarithmic stress value x^*, $L_p\big(z|\mathbf{y}\big)$ in equation (4) is calculated for all z-values. Then the profile predictive likelihood is normalised to be a probability density function in z in the following way

$$\bar{L}_p\big(z|\mathbf{y}\big) = \frac{L_p\big(z|\mathbf{y}\big)}{\int_{u=0}^{y_o} L_p\big(u|\mathbf{y}\big)\,du + L_p\big(y_o|\mathbf{y}\big)}. \tag{5}$$

In Figure 4 the life is predicted given a certain stress level by using the normalised profile predictive likelihood $\bar{L}_p\big(z|\mathbf{y}\big)$ in equation (5) and is shown as '+'. The uncertainty in the estimated parameters is also taken into account. The solid lines are the estimated curved S-N curve using equation (3) for probabilities 0.10 and 0.50. In this prediction the uncertainty in the estimated parameters is not taken into account. For small quantiles and low stress values the '+' is far from the curved S-N curve so in this case there is a rather great uncertainty in the estimated curved S-N curve.

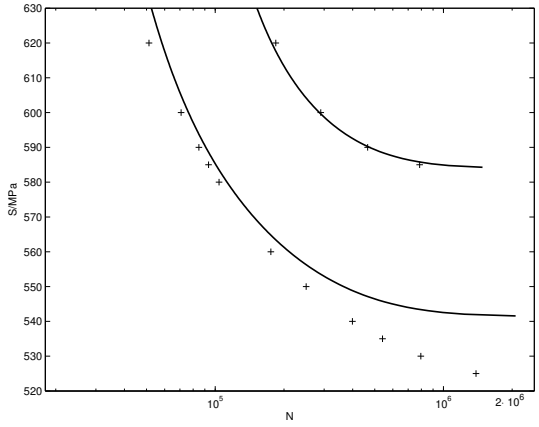

Figure 4: *The predicted life using $\bar{L}_p\Big(z|\mathbf{y}\Big)$ is denoted by '+'. The solid line is the estimated curved S-N curve using equation (3) for probabilties 0.1 and 0.5.*

5 Estimating inclusion size distribution

The mechanical properties of hard and clean metals are in part governed by the inclusion size distribution, e.g. the fatigue limit is determined by the size of the maximum projected area on a plane perpendicular to the load [2]. Therefore we can use relation (6) below, the so called $\sqrt{area_{max}}$-model to determine the endurance limit, where HV is Vickers hardness in Hv, R is the stress ratio S_{min}/S_{max}, and $area_{max}$ is the largest inclusion areas in μm^2 projected onto a plane perpendicular to the applied load in the volume of the material that is subjected to the maximum stress. The constant K depends on the location of the inclusion, K=1.43 for a surface inclusion and K=1.56 for an internal inclusion. The exponent α is given as $\alpha = (HV \cdot 10^{-4} + 0.226)$. The endurance limit S_{el} is given by

$$S_{el} = K(HV + 120)\sqrt{area_{max}}^{-1/6}\Big(\frac{1-R}{2}\Big)^{\alpha}. \qquad (6)$$

Letting $q = K(HV + 120)(\frac{1-R}{2})^{\alpha}$, equation (6) can be written as $S_e = h(D) = qD^{-1/6}$ where D is the square root of the largest projected inclusion area perpendicular to the load, $\sqrt{area_{max}}$.

Direct observations of inclusions sizes are difficult since the inclusions are rare and often occur in the interior of the specimen. A usual way of observing is the following. A number of areas of the same size are observed on polished cross-sections by optical microscopy. The largest inclusion in each area is identified and measured and then the extreme value theory is used to extrapolate the maximum inclusion size distribution of the specimen of interest.

One way of making direct observations of the inclusion in the specimen is to assume that for a staircase test the specimen will break in the largest inclusion and measure the inclusion sizes where the specimens have broken. By this method only one inclusion in each specimen has to be measured and it is not necessary to search for the inclusions.

5.1 Estimating the inclusion size distribution without measuring inclusion sizes

Using the $\sqrt{area_{max}}$-model we get a relation connecting the endurance limit and the square root of the largest projected inclusion area perpendicular to the load. This relation makes it possible to estimate the inclusion size distribution just using the stress from the staircase test without any inclusion size measurement. From now on the inclusion size means the square root of the inclusion size area perpendicular to the load.

Using the information that if the specimen is broken the stress is above the endurance limit and vice versa for the unbroken specimen the inclusion

size distribution can be estimated without any inclusion size measurements. Here both D and S_{el} are random variables and the relation between their cumulative probability distribution functions is $F_{S_{el}}(s) = 1 - F_D(h^{-1}(s))$. The inclusion sizes for the censored specimens are all assumed to be below $h^{-1}(s)$, the critical size. The probability that a specimen is censored (unbroken) is therefore $P(D \le h^{-1}(s)) = F_D(h^{-1}(s))$ and the probability that a specimen is uncensored (broken) is $1 - F_D(h^{-1}(s))$. The likelihood function for this model is

$$L = \prod_{i\in U}\Big(1 - F_D\big(h^{-1}(s_i)\big)\Big)\prod_{i\in C}F_D\big(h^{-1}(s_i)\big). \qquad (7)$$

Since D is the square root of the maximum inclusion size it is reasonable to let D follow an extreme value distribution. If the likelihood in (7) is used it is possible to estimate the inclusion size distribution without including any inclusion size measurement.

5.2 Estimating inclusion size distribution based also on inclusion size measuring

The square root of the inclusion size area where the specimen breaks is measured and is assumed to be the largest inclusion in that specimen. The test data is then the stress levels for broken and unbroken specimens and for the broken specimens also the inclusion sizes where the specimen has broken. Again let $F_D(d)$ be the cumulative distribution for the inclusion size and $f_D(d)$ the corresponding density function. The probability that the specimen is censored is that all inclusion sizes are below the critical size, $F_D(h^{-1}(s))$. The likelihood function then is

$$L = \prod_{i\in U} f_D(d_i)\prod_{i\in C}F_D\big(h^{-1}(s_i)\big) \qquad (8)$$

where D follows an extreme value distribution.

From now on denote the critical size $h^{-1}(s)$ as d_c. In the likelihood in (8) the specimens are assumed to break in the largest inclusion. Assume now that there are more than one inclusion size over the critical size d_c and that the specimen breaks in one of these with the same probability for each of them. Assume also that the number of inclusions in the specimen that are greater than d_c is Poisson distributed with the parameter ν. The conditional probability that an inclusion is not greater than d, given that it is at least d_c, follows a generalized Pareto distribution with the cumulative distribution function

$$F_{D|D\ge d_c}(d) = 1 - \Big(1 - \frac{\xi(d-d_c)}{\sigma}\Big)^{1/\xi} \qquad (9)$$

where $\sigma > 0$, $0 < d - d_c < \frac{\sigma}{\xi}$ if $\xi > 0$, and $0 < d - d_c < \infty$ if $\xi \leq 0$.

Using the law of total probability the density function of the inclusion size for $d > d_c$ can be written as

$$f_D(d) = \frac{1}{\sigma}(1 - e^{-\nu})\Big(1 - \frac{\xi(d - d_c)}{\sigma}\Big)^{1/\xi - 1}. \tag{10}$$

The critical size d_c, the expected number of inclusions ν over that size and the scale parameter σ for the inclusion size distribution depend on the stress level s.

For the unbroken specimen the number of inclusions with a size greater than d_c is zero. Using this and $f_D(d)$ in (10) we get another likelihood

$$L = \prod_{i \in U} f_D(d_i) \prod_{i \in C} P(N_i = 0) \tag{11}$$

which is the likelihood of finding at least one inclusion exceeding the critical size, for each broken component.

In Figure 5 there is an example of estimating the inclusion size distribution from likelihood functions (7), (8) and (11). The solid line is from likelihood function (7) and the dotted line is from using likelihood (8), both using the Gumbel assumption on the inclusion sizes. The dashed line is from likelihood (11), where ξ is fixed to 0 so that the maximum inclusion size follows a Gumbel distribution [10]. The '$\times$' is the estimated empirical distribution using the method of Kaplain Meier [12]. In this example there is no big difference between the models in (8) and (11) but this example has to do with a very clean steel with few inclusions.

5.3 Inhomogeneous stress conditions

When the stress distribution in the component is homogeneous all inclusion are exposed to the same stress, but when there is an inhomogeneous stress distribution a large inclusion may not be dangerous since the local stress may be small. This can also be handled by the different stress levels from the staircase test by using the $\sqrt{area_{max}}$-model, the Poisson and generalised Pareto assumptions.

Denote the volume of the component by V. From [13] we get that the probability that a component will fail is

$$\mathrm{P}\big(\text{The component fails}\big) = 1 - exp\Big(-\int_V \lambda(\boldsymbol{x})d\boldsymbol{x}\Big).$$

The assumption used is that inclusions with a size greater than d_o appear as the Poisson process with the intensity λ_o, and the sizes of inclusions

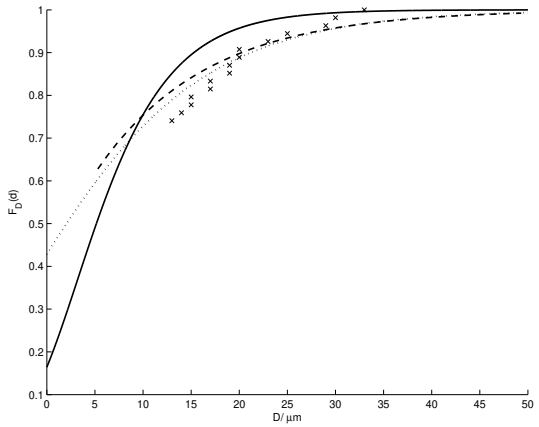

Figure 5: *Estimated inclusion size distribution. The '$\times$' is the empirical distribution. The solid line shows an estimate of the inclusion size distribution using only stress levels and assumes that the inclusion size follows a Gumbel distribution. The dotted line uses both stress levels and inclusion sizes. The inclusion size again follows a Gumbel distribution. The dashed line shows an estimate of the inclusion size distribution assuming that the specimen breaks in some inclusion over a threshold and the maximum inclusion size follows a Gumbel distribution.*

are random and independent of each other and their locations. The intensity of inclusions that can cause failure at the location $\boldsymbol{x}$ in the component is $\lambda(\boldsymbol{x})$. The failure intensity $\lambda(\boldsymbol{x})$ satisfies

$$\lambda(\boldsymbol{x}) = \lambda_o \cdot \mathrm{P}\big(s(\boldsymbol{x}) > h(D) | D > d_o\big).$$

Assume that the inclusion size above a fixed threshold d_o follows a generalised Pareto distribution. The probability that there is a failure in the point $\boldsymbol{x}$ given that there is an inclusion bigger than d_o is then the following

$$\mathrm{P}\big(s(\boldsymbol{x}) > h(D) | D > d_o\big) = \mathrm{P}\big(D > q^6 s(\boldsymbol{x})^{-6} | D > d_o\big) =$$

$$\Big(1 - \frac{\xi(q^6 s(\boldsymbol{x})^{-6} - d_o)}{\sigma_o}\Big)^{1/\xi}$$

where σ_o is a scale parameter, ξ is a shape parameter and d_o is a threshold.

The probability that component number i will break depends on the tension s_i on which it is tested. The tension $s_i(\boldsymbol{x})$ is the local tensile stress which gives rise to the failure intensity $\lambda_i(\boldsymbol{x})$. The likelihood in this case is

$$L = \prod_{i \in U} \Big[1 - exp\Big(-\int_V \lambda_i(\boldsymbol{x}) d\boldsymbol{x}\Big)\Big] \prod_{i \in C} exp\Big(-\int_V \lambda_i(\boldsymbol{x}) d\boldsymbol{x}\Big). \qquad (12)$$

Again we only have the stresses from the staircase test but we can estimate the inclusion intensity and size.

5.3.1 Exchange between different test procedures

Using the likelihood in (12) it is possible to predict the fatigue limit in rotating bending based on data from uniaxial tests and vice versa. This is exemplified in Figure 6. The solid line is the probability that a cylindrical component exposed to a rotating bending test will fail. This estimate is then used to estimate the failure probability using a uniaxial test procedure instead, still on a cylindrical component which is the dotted line in Figure 6. Here we see that the estimated fatigue limit for the rotating bending test is much higher than for the uniaxial test, 434.5 MPa versus 390 MPa, as it should be, because for the uniaxial test on a cylindrical shape the whole component is exposed to the same stress, but for the rotating bending case the stress decreases with the radius. For the same material the estimated parameters from likelihood (12) should be similar to each other. In [III] it is shown that the model is consistent with the data.

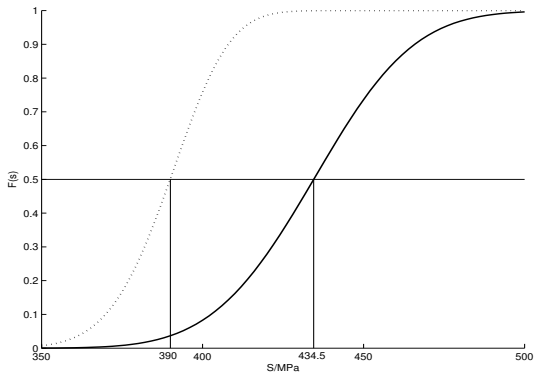

Figure 6: *The solid line is the estimated fatigue limit distribution function from a rotating bending test. The dotted line is the estimated fatigue limit distribution under uniaxial condition using the estimated parameters from the rotating bending test.*

6 Percentiles of the fatigue limit distribution

The median of the fatigue limit is the stress level where 50 % of the specimens will survive. In this section a lower limit for the fatigue limit is estimated at which $(1-p)100$ % of the specimen will survive, $p < 0.5$.

We have mainly estimated the median of the fatigue limit in the preceding sections. However, it is also interesting to estimate a lower percentile of the fatigue limit distribution. This non-random quantity will be denoted S_{∞_p}, where p is the probability of the lower tail, $P(S_\infty \leq S_{\infty_p}) = p$. To handle the uncertainty in the estimated parameters, a random lower tolerance limit is estimated, $S_{\infty_{pl}}$. The lower tolerance limit satisfies that with probability $1 - \alpha_1$ the p percentile of the fatigue limit distribution lies above the limit $S_{\infty_{pl}}$, $P(S_{\infty_{pl}} \leq S_{\infty_p}) = 1 - \alpha_1$. This is done by using the estimated information matrix or the method of profile likelihood. From these two constructions we get two probabilities which influence each other. A higher value on p can be compensated by a lower value on α_1 to get the same lower tolerance limit, $S_{\infty_{pl}}$. The two probabilities p and α_1 are illustrated in Figure 7. The disadvantage using the information matrix

is that the normality assumption does not work well here. The advantage is that it is not as computatively heavy as the method of profile likelihood.

An alternative approach to estimate a lower limit is to calculate a prediction interval for S_{∞_p}. We will get a lower limit $S_{\infty_{pr}}$ which the fatigue limit will be below with a certain small probability p_2, $P(S_\infty < S_{\infty_{pr}}) = p_2$. This can be done by the method of profile predicted likelihood described in Section 4.1. The profile predicted likelihood, $L_{p_s}(s|\mathbf{y})$ for the fatigue limit, is normalized to be a probability density in s and then used to construct prediction intervals for S_∞ in the following way

$$P(S_\infty \leq S_{\infty_{pr}}) = \int_{s=0}^{S_{\infty_{pr}}} \tilde{L}_p(s|\mathbf{y})ds = p_2\,.$$

By this method we get a lower limit which S_∞ will not be below with a small probability and where the uncertainty in the estimated parameters is taken into account. The profile predicted likelihood method gives accurate results but the computations are demanding, see [IV].

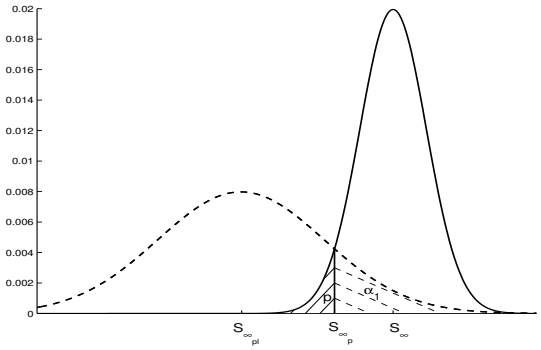

Figure 7: *The density function for S_∞, the fatigue limit, is the solid line. The dashed line is the probability density of the fatigue limit percentile estimate.*

7 Conclusion

The staircase test has been analysed in different ways. The test has been combined with inclusion size measurements under different assumptions. Using the stresses and the finite lives we get an estimate of the fatigue limit distribution. The fatigue limit distribution and the finite life distribution can be combined to estimate a curved S-N curve. The uncertainties in the estimated parameters are taken into account by using the method of information matrix or the method of profile predictive likelihood.

When we combined the staircase test with the $\sqrt{area_{max}}$-model, the distribution of the inclusion size where the failure occurs can be estimated. This can be done by using the stress levels with or without inclusion measurement. The inclusion size is measured where the specimen breaks, which means that we do not have to search for them. As expected the models which use the inclusion size for the broken specimens better estimate the wanted distribution. From the measured inclusion size on the broken specimens it can also be concluded whether or not the $\sqrt{area_{max}}$-model fits the measurements.

By combining the $\sqrt{area_{max}}$-model and the assumption that the inclusion size above a fixed threshold follows a generalized Pareto distribution the stresses from the staircase test can be used to estimate the probability that the component will fail for different test procedures and component shapes. Note that although the test data only contains the number of broken and unbroken components at each stress level and no inclusion sizes, it is possible to estimate the inclusion intensity and size.

The variance for the finite life is not constant but depends on the stress level. Further work is needed to enlarge the model to take this into account when the curved S-N curve is estimated. The uncertainties in the models which estimate the inclusion size distributions under homogeneous stress conditions remain to be analyzed.

8 Summary of Papers

8.1 Paper I: Fatigue limit estimated using finite lives

A model which estimates the endurance limit distribution and a model which estimates the fatigue limit distribution are compared.

The model which estimates the endurance limit is a variant of a model in [4]. Here the endurance limit means the limit where the specimens have a specified finite life. The logarithm of the endurance limit is assumed to be normally distributed. When the specimen breaks it is assumed that the stress level is above the endurance limit and if the specimen survives the

stress level is below the endurance limit.

The other model estimates the fatigue limit, below which the specimen has infinite life. The logarithm of the fatigue limit is assumed to follow a normal distribution. For this model one also uses the information when the specimen has broken, i.e. number of cycles to failure.

It is shown that it is possible to estimate the distribution of the fatigue limit, although the fatigue limit itself is impossible to observe. When the two models give very different estimates this is an indication that the chosen run-out level is too low.

8.2 Paper II: Estimating inclusion distributions of hard metal using fatigue tests

Different approaches of estimating the inclusion size distribution from a staircase test are presented. The main assumption for all the models is that for each load level there is a critical inclusion size. This critical size is such that if the maximum inclusion size in the specimen is smaller than the critical size, the specimen will not break and if the maximum inclusion size is bigger than the critical size the specimen will break.

One model uses the number of survivors and failures at each load level and the relation between the endurance limit and the inclusion size distribution. The other two models use the inclusion size for the inclusion which has caused the failure and the load level for the unbroken specimens. One of these models uses the assumption that the specimen breaks in the largest inclusion, and the other one is based on the assumption that the specimens break in some inclusion over a critical size, not necessarily the largest one.

As expected the models which use the inclusion size for the broken specimen give a better estimate of the inclusion size distribution. The advantage of using the load level and observed inclusion size from a staircase test instead of the inclusion sizes for all specimens is that the inclusion size is measured only where the specimen breaks, which means that we do not have to search for them.

8.3 Paper III: Estimating fatigue limit distribution under inhomogeneous stress conditions

Here the non-metallic inclusion size is used to estimate the endurance limit distribution under inhomogeneous stress conditions from a staircase test. This is done by combining the $\sqrt{area_{max}}$-model [2] and the assumption that the inclusion size above a fixed threshold follows a generalized Pareto distribution [9].

The $\sqrt{area_{max}}$-model gives a relation between the maximum inclusion size and the endurance limit under homogeneous stress conditions. For each load level there is a critical inclusion size, but when the stress condition is inhomogeneous the critical size of the inclusion depends on the location of the inclusion in the component. A model that can be used to estimate the probability that the component will fail for different test procedures and component shapes is described.

It is shown that there is a possibility to translate between different test procedures and component shapes. Although the test data only contain the number of broken and unbroken components at each stress level and no inclusion sizes, it is possible to estimate the inclusion intensity and size.

8.4 Paper IV: Estimating curved S-N curves

The curved S-N curve is estimated from fatigue data. This is done by combining the finite life distribution and the fatigue limit distribution.

By combining the finite life distribution and the fatigue limit distribution it is possible to get the probability of not surviving a certain life. This probability is used to estimate the whole curved S-N curve. The life is predicted in two ways. One approach directly uses the probability of not surviving, where the uncertainties in the estimated parameters are not taken into account. The other approach uses the method of profile predictive likelihood to deal with the parameter uncertainties. There is quite a large uncertainty when the curved S-N curve is estimated from a staircase test, which one must be aware of when predicting the life.

The quantile of the fatigue limit distribution is also predicted using predictive profile likelihood.

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Fatigue limit estimated using finite lives

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ABSTRACT For metallic material one is interested in estimating the fatigue limit although it is impossible to test if a specimen has infinite life. In this paper two models are compared. One model estimates the fatigue limit and another estimates the endurance limit. The model which estimates the endurance limit is a model which only uses the information that a specimen has broken or not broken at a certain stress level. The model which estimates the fatigue limit also uses the information about when the specimen is broken, i.e., after how many cycles the specimen breaks. The two models are compared, both on real and simulated staircase tests. It is shown that it is possible to estimate the distribution of the fatigue limit although the fatigue limit itself is impossible to observe. When the two models give very different estimates it indicates that the chosen run-out level is too low.

Keywords censoring; endurance limit; information matrix; maximum likelihood; staircase test.

NOMENCLATURE

1 = indicator function
C = set of censored observations
Cov = covariance
E = expectation
f = density function
F = distribution function
I = information matrix
I_e = number of times the confidence interval contains X_{e_p}
N = endurance life
$N(\cdot, \cdot)$ = normal distribution
n_o = number of cycles
n = number of observations
P = probability
q_1, q_2, p, α = probability
S = stress level
S_e = fatigue limit
S_{el} = endurance limit
S_{el_p} = lower bound for S_{el}
$\underline{S}_{el_p}$ = lower confidence bound of S_{el_p}
S_{e_p} = lower bound for S_e
$\underline{S}_{e_p}$ = lower confidence bound of S_{e_p}
$S^*_{e_q}$ = lower bound for S_e
$S^*_{el_q}$ = lower bound for S_{el}
U = set of uncensored observations
Var = variance
$x = \ln(S)$
$X_e = \ln(S_e)$
$X_{el} = \ln(S_{el})$

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$X_{el_p} = \ln(S_{el_p})$
$\underline{X}_{el_p} = \ln(\underline{S}_{el_p})$
$X_{e_p} = \ln(S_{e_p})$
$\underline{X}_{e_p} = \ln(\underline{S}_{e_p})$
$X^*_{e_q} = \ln(S^*_{e_q})$
$\underline{X}^*_{el_q} = \ln(\underline{S}^*_{el_q})$
$Y(x)$ = logarithm of life, $\ln(N)$
$y_o = \ln(n_o)$
$Z(x)$ = censored value of $Y(x)$
$1 - \alpha$ = confidence
$\lambda_\alpha, \lambda_p, a_e$ = percentiles, λ_α is the 100αth percentile of $N(0,1)$
μ_{xe} = mean of fatigue limit distribution
μ_{xel} = mean of endurance limit distribution
$\nu(x)$ = mean of logarithm of the finite life distribution
σ^2_{xe} = variance of fatigue limit distribution
σ^2_{xel} = variance of endurance limit distribution
σ^2_ν/w^2 = variance of logarithm of the finite life distribution
Φ = standard normal cumulative distribution
$\hat{\cdot}$ = estimate

INTRODUCTION

The S–N curve describes the relations between a given stress amplitude and the number of cycles until a specimen has broken. To get this curve each specimen is exposed to a stress amplitude at some given level, and the number of cycles until the specimen breaks is observed. The test is interrupted if the specimen has not broken before some specified number of cycles n_o (often 2×10^6). Here n_o denotes the endurance life.

For some materials the S–N curve has an asymptote at some stress level. When this asymptote exists the corresponding stress level is called the fatigue limit and this implies that if the specimen is exposed at a lower stress level than the fatigue limit it will have an infinite life. For the material we can also define an endurance limit for a specified endurance life of n_o cycles. The endurance life is typically $n_o = 10^6$ or 10^7 cycles. If the specimen is exposed at a lower stress level than the endurance limit it will have a life exceeding n_o cycles.

The data used here are from a staircase test. Before the test is started one has to choose the start stress level s_1 and the endurance life n_o. The first specimen is exposed to a stress level s_1. If the specimen breaks before n_o cycles, the next specimen is tested at a lower stress level, i.e., $s_2 < s_1$. If the specimen survives n_o cycles then s_2 is higher than s_1. This procedure is continued until all specimens are tested.

The median endurance limit has been estimated from a staircase test in Refs. [1] and [2]. A number of broken and unbroken specimens for each stress level are used. To estimate the endurance limit, the maximum likelihood method under different distribution assumptions on the endurance limit is used. The extreme value distribution is used in Ref. [2] and here, and in Ref. [1] the normal distribution is used instead. The endurance limit in this paper is estimated by a variant of the method given in Ref. [1]. The model here is called Model 2. The difference between these models is that in Ref. [1] the likelihood is simplified before it is maximized and here the likelihood function is maximized directly without approximation.

From the staircase test we also have information from the finite lives of the broken specimens. Using this extra information one can estimate the fatigue limit. This has been done in Refs [3–5]. The model which estimates the fatigue limit here is denoted as Model 1.

When estimating fatigue limit using finite lives it is important to think of the connection between the fatigue limit and the finite life. In Refs [3] and [5], the finite life depends on the difference between stress and the fatigue limit. In Ref. [4] they are presuming that the endurance limit follows a continuous distribution, which is a function of the endurance life.

In this paper, it is assumed that stress itself determines the finite life through crack propagation. On the other hand if the life is infinite this is because the stress is below the fatigue limit. Hence, it is assumed that there are two different mechanisms, one which determines whether crack propagation occurs or not and one which determines the finite life given that crack propagation occurs. In this case the two mechanisms are uncoupled in contrast to the case considered in Ref. [5]. The rationale for the case with uncoupled mechanisms is inspired by Ref. [6]. They claim that finite life occurs for a given stress if there is a big enough inclusion. The actual life is not determined entirely by the inclusion size but also by the shape and

chemical content of the inclusion, and the most important quantity determining the infinite life is the applied stress according to the S–N curve.

On the other hand, a physical argument for the approach given in Ref. [5] is to assume that the fatigue limit acts as a threshold and only the part of the stress above the threshold drives the crack. This may be justified by the Elber crack closure concept (see Ref. [7]).

The question studied in this paper is if the actual finite life carries any information on the fatigue limit though the fatigue limit and the finite life are governed by two different mechanisms. The answer to that question here is that it depends on the result of the staircase test. If the finite life is close to the specified number of cycles where the test is interrupted, it contains information on the actual fatigue limit.

MODEL 1

Here the fatigue limit means that if the specimen is exposed to a lower stress level than the fatigue limit it will have infinite life. To estimate the distribution of the fatigue limit the following assumptions are made. The fatigue limit S_e is a random variable whose logarithm is normally distributed with the mean μ_{xe} and the variance σ_{xe}^2. The logarithm of the finite life N is also a random variable which is normally distributed with an expectation $\nu(x)$ which is a function of the stress value S through its logarithm $x = \ln(S)$. The variance of the finite logarithmic life is not constant. Instead, it increases, the closer the stress level is to the fatigue limit, and is denoted by σ_v^2/w^2, where w depends on the stress value. Different relationships between σ_v and the logarithm of the stress can be used exponentially variable, linearly variable and constant (see Ref. [8]).

An interpretation of the model is that there will be no crack propagation at all with a certain probability, $P(S_e > S)$ on the applied stress S, but with the complementary probability $P(S_e \le S)$, crack propagation will take place. In the latter case the conditional expected logarithmic life is $\nu(x)$, where as above $x = \ln(S)$. The graph of $\nu(x)$ is assumed to be a straight line with a negative slope, or more generally, a decreasing continuous function of the stress values.

One way of summarizing the situation is to assume that we have a latent variable $Y(x)$ which is either observed or censored. If there are n observations, write

$$Y(x_i) = \ln(N_i) \sim N\left(\nu(x_i), \frac{\sigma_v^2}{w_i^2}\right), \qquad i = 1, \ldots, n$$

$$X_{e_i} = \ln(S_i) \sim N(\mu_{xe}, \sigma_{xe}^2), \qquad i = 1, \ldots, n.$$

There are two ways in which $Y(x_i)$ can be censored. The first one is used if the specimen has survived the test, i.e., the specified number of cycles, n_o, but actually has a finite life. The second way in which the specimen can be censored is when the stress level is below the fatigue limit. In both cases the value of the observation $Z(x_i)$ is put to y_o:

$$\begin{aligned} Z(x_i) &= Y(x_i)1_{\{X_{e_i} < x_i\}}1_{\{Y(x_i) \le y_o\}} + y_o\left(1 - 1_{\{X_{e_i} < x_i\}}1_{\{Y(x_i) \le y_o\}}\right) \\ &= \begin{cases} y_o & \text{if censored} \\ Y(x_i) & \text{otherwise,} \end{cases} \end{aligned}$$

where $1_{\{\cdot\}}$ is the indicator function and $y_o = \ln(n_o)$.

The probability that $Y(x_i)$ is censored is

$$\begin{aligned} &P(Y(x_i) \text{ is censored}) = P(Z(x_i) = y_o) \\ &= P(Z(x_i) = y_o \mid X_{e_i} \le x_i)P(X_{e_i} \le x_i) \\ &\quad + P(Z(x_i) = y_o \mid X_{e_i} > x_i)P(X_{e_i} > x_i) \\ &= \left(1 - \Phi\left(\frac{y_o - \nu(x_i)}{\sigma_v/w_i}\right)\right)\Phi\left(\frac{x_i - \mu_{xe}}{\sigma_{xe}}\right) \\ &\quad + 1 \cdot \left(1 - \Phi\left(\frac{x_i - \mu_{xe}}{\sigma_{xe}}\right)\right) \\ &= 1 - \Phi\left(\frac{y_o - \nu(x_i)}{\sigma_v/w_i}\right)\Phi\left(\frac{x_i - \mu_{xe}}{\sigma_{xe}}\right) \end{aligned}$$

and the likelihood for an uncensored observation is

$$f(y(x_i); (\sigma_v, \nu(x_i)))\Phi\left(\frac{x_i - \mu_{xe}}{\sigma_{xe}}\right),$$

where

$$f(y(x_i); (\sigma_v, \nu(x_i))) = \frac{w_i}{\sqrt{2\pi}\sigma_v} e^{-(y(x_i) - \nu(x_i))^2 w_i^2/2\sigma_v^2}$$

and $\Phi(\cdot)$ standard normal cumulative distribution.

Likelihood function

To estimate unknown parameters we maximize the likelihood function for our model. This is done by splitting the observations into two disjoint sets, U corresponding to the uncensored observations and C to the censored ones. The likelihood function then is

$$\begin{aligned} lik(\boldsymbol{\theta}) &= \prod_{i \in U} f(y_i; (\sigma_v, \nu(x_i)))\Phi\left(\frac{x_i - \mu_{xe}}{\sigma_{xe}}\right) \\ &\quad \times \prod_{i \in C} P(Z(x_i) = y_o) \\ &= \prod_{i \in U} f(y_i; (\sigma_v, \nu(x_i)))\Phi\left(\frac{x_i - \mu_{xe}}{\sigma_{xe}}\right) \\ &\quad \times \prod_{i \in C}\left(1 - \Phi\left(\frac{y_o - \nu(x_i)}{\sigma_v/w_i}\right)\Phi\left(\frac{x_i - \mu_{xe}}{\sigma_{xe}}\right)\right), \end{aligned} \quad (1)$$

where

$$\boldsymbol{\theta} = (\sigma_v, \nu(\cdot), \sigma_{xe}, \mu_{xe}).$$

The maximum likelihood estimate of $\boldsymbol{\theta}$ is that value of $\boldsymbol{\theta}$ that maximizes the likelihood function.

Variance estimation

The maximum likelihood estimator for the parameter $\boldsymbol{\theta}$ is denote by $\hat{\boldsymbol{\theta}}$. One approximative variance estimated of $\hat{\boldsymbol{\theta}}$ is obtained from the inverse of the observed information matrix $\hat{I}$

$$\hat{I}(j,k) = -\frac{\partial^2}{\partial\theta_j'\partial\theta_k'}\log(lik(\boldsymbol{\theta}))|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}; \qquad j = 1, \ldots, l \quad k = 1, \ldots, l,$$

where l is the number of parameters in θ. If $\nu(\cdot)$ is assumed to be a straight line, l is five. These derivates are calculated by a formula manipulating software (Mathematica).

The approximate variance estimates of $\hat{\mu}_{xe}$ and $\hat{\sigma}_{xe}$ are $\widehat{\mathrm{Var}}(\hat{\mu}_{xe}) \approx I^{-1}(5,5)$, $\widehat{\mathrm{Var}}(\hat{\sigma}_{xe}) \approx I^{-1}(4,4)$ and $\widehat{\mathrm{Cov}}(\hat{\mu}_{xe}, \hat{\sigma}_{xe}) \approx I^{-1}(4,5)$, where $I^{-1}(j,k)$ denotes the (j,k) element of the inverse of the information matrix.

MODEL 2

Here the endurance limit means that if the specimens are exposed to a lower stress level than the endurance limit they will have lives exceeding n_o. One way of estimating the endurance limit distribution is to use a model which only uses the information that a specimen has broken or has not broken at a certain stress level. The only assumption which is needed for this model is that the endurance limit S_{el} is a random variable whose logarithm is normally distributed with the mean μ_{xel} and the variance σ_{xel}^2:

$$X_{el_i} = \ln(S_{el_i}) \sim N(\mu_{xel}, \sigma_{xel}^2), \qquad i = 1, \ldots, n.$$

The specimen in this model can only be censored in one way and this occurs when the stress level is below the endurance limit. For this model the probability that $Y(x_i)$ is censored is $1 - \Phi(\frac{x_i - \mu_{xel}}{\sigma_{xel}})$.

Likelihood function

The likelihood function for this model is the probability of obtaining the actual endurance limits below the applied stresses, $\prod_{i \in U}\Phi(\frac{x_i - \mu_{xel}}{\sigma_{xel}})$, times the probability of obtaining endurance limits above the stresses, $\prod_{i \in C}(1 - \Phi(\frac{x_i - \mu_{xel}}{\sigma_{xel}}))$. The parameters μ_{xel} and σ_{xel} in the distribution of the endurance limit depend on the definition of this limit, e.g., the longer the endurance life n_o, the lower the expected logarithmic endurance limit μ_{xel}. The following likelihood function is obtained:

$$\begin{aligned} lik(\sigma_{xel}, \mu_{xel}) &= \prod_{i \in U}\Phi\left(\frac{x_i - \mu_{xel}}{\sigma_{xel}}\right) \\ &\quad \times \prod_{i \in C}\left(1 - \Phi\left(\frac{x_i - \mu_{xel}}{\sigma_{xel}}\right)\right). \end{aligned} \quad (2)$$

One gets the estimates μ_{xel} and σ_{xel} by directly maximizing the likelihood function. In Ref. [1] a simplification of the likelihood is used to get the estimates.

Variance estimation

For this model the information matrix I is a 2×2 matrix. The (j,k) element of the observed information matrix is

$$\hat{I}(j,k) = -\frac{\partial^2}{\partial\theta_j''\partial\theta_k''}\log(lik(\boldsymbol{\theta}'', x))|_{\boldsymbol{\theta}''=\hat{\boldsymbol{\theta}}''}, \qquad j = 1, 2 \quad k = 1, 2,$$

where $\boldsymbol{\theta}'' = (\mu_{xel}, \sigma_{xel})$.

The inverse of the approximative information matrix is again used to get the approximative variance estimates of μ_{xel} and σ_{xel}

$$\widehat{\mathrm{Var}}(\hat{\mu}_{xel}) \approx I^{-1}(1,1), \widehat{\mathrm{Var}}(\hat{\sigma}_{xel}) \approx I^{-1}(2,2)$$
$$\text{and} \quad \widehat{\mathrm{Cov}}(\hat{\mu}_{xel}, \hat{\sigma}_{xel}) \approx I^{-1}(1,2).$$

MODELS 1 AND 2 UNDER GENERAL ASSUMPTIONS

The two models can be described in a more general form. Assume that the logarithm of the fatigue limit is distributed according to a certain distribution $F_{xe}(x; (\sigma_{xe}, \mu_{xe}))$. Assume also that the distribution of the endurance limit is given as $F_{xel}(x; (\sigma_{xel}, \mu_{xel}))$ and that the finite life is distributed according to $F_v(y; (\sigma_v, \nu(x)))$ with derivative $f_v(y; (\sigma_v, \nu(x)))$.

The likelihood for Model 1 under the general distribution assumptions is

$$\begin{aligned} lik(\boldsymbol{\theta}) &= \prod_{i \in U} f_v(y_i; (\sigma_v, \nu(x_i)))F_{xe}(x_i; (\sigma_{xe}, \mu_{xe})) \\ &\quad \times \prod_{i \in C}(1 - F_v(y_i; (\sigma_v, \nu(x_i)))F_{xe}(x_i; (\sigma_{xe}, \mu_{xe}))) \end{aligned}$$

and for Model 2 is

$$\begin{aligned} lik(\sigma_{xel}, \mu_{xel}) &= \prod_{i \in U} F_{xel}(x_i; (\sigma_{xel}, \mu_{xel})) \\ &\quad \times \prod_{i \in C}(1 - F_{xel}(x_i; (\sigma_{xel}, \mu_{xel}))). \end{aligned}$$

EXAMPLE 1

In this example the estimations are based on the data given in Table A1 of the Appendix. The data are from a real staircase test. The material in this example and the next one is stainless steel. For the rest of this example $\nu(x)$ is assumed to be a straight line of the stress values, and the different distributions are all assumed to be normal. The weight (w_i) is equal to one for the lowest stress level and for every 20 MPa increase in stress level the weight is increased by some constant (see Table 1). In Table 1 the estimated parameters from maximizing the likelihood function

numerically for Model 1 are shown. The estimates of the endurance limit from Model 2 are $\hat{\sigma}_{xel} = 0.0571$ and $\hat{\mu}_{xel} = 6.3695$.

In this case the two parameters for the fatigue limit and the endurance limit almost coincide; the fatigue limit is somewhat lower than the endurance limit as it should be according to its definition. The two medians of the distribution of the limits are approximately $e^{6.369} = 583.47$ (MPa) and their coefficient of variance is approximately 0.06.

The reason why the two limits coincide here is that the $\Phi(\frac{y_0 - v(x_i)}{\sigma_v/w_i})$ factor in (1) is approximately equal to one, i.e., the probability that the logarithm of the lives is below y_0 is approximately equal to one and the difference between the two models disappears. This occurs when the finite lives are far below the endurance life n_0.

Table 1 ML results for Example 1 with Model 1

Weight increase	$\hat{\sigma}_v$	$\hat{v}(x)$	$\hat{\sigma}_{xe}$	$\hat{\mu}_{xe}$
0	0.7920	$42.6965 - 4.7865\,x$	0.0571	6.3693
0.4	1.0943	$52.6207 - 6.3390\,x$	0.0577	6.3691
0.8	1.3305	$57.7412 - 7.1390\,x$	0.0582	6.3690

The result is shown in Fig. 1, where the '+' denotes an uncensored specimen and 'o' a censored one. The estimated fatigue and endurance limit is the horizontal line. In this example they coincide. The other lines are $\hat{v}(x)$ for different weights.

The approximative variance estimates in Model 1 are given in Table 2 and for Model 2 the corresponding estimates are $\widehat{\mathrm{Var}}(\hat{\mu}_{xel}) = 0.0002060$ and $\widehat{\mathrm{Var}}(\hat{\sigma}_{xel}) = 0.0006787$, $\widehat{\mathrm{Cov}}(\hat{\mu}_{xel}, \hat{\sigma}_{xel}) = 0.00004633$.

EXAMPLE 2

In this example the estimations are based on the data given in Table A2 of the Appendix. The material is the same as in Example 1. The weights, w_i, and the line, $v(x)$, in this example are the same as in the previous example. The

Table 2 Approximative variance estimates for Example 1

Weight increase	$\widehat{\mathrm{Var}}(\hat{\mu}_{xe})$	$\widehat{\mathrm{Var}}(\hat{\sigma}_{xe})$	$\widehat{\mathrm{Cov}}(\hat{\mu}_{xe}, \hat{\sigma}_{xe})$
0	0.0002020	0.0005184	0.0000347
0.4	0.0001941	0.0005120	0.0000539
0.8	0.0001893	0.0005014	0.0000702

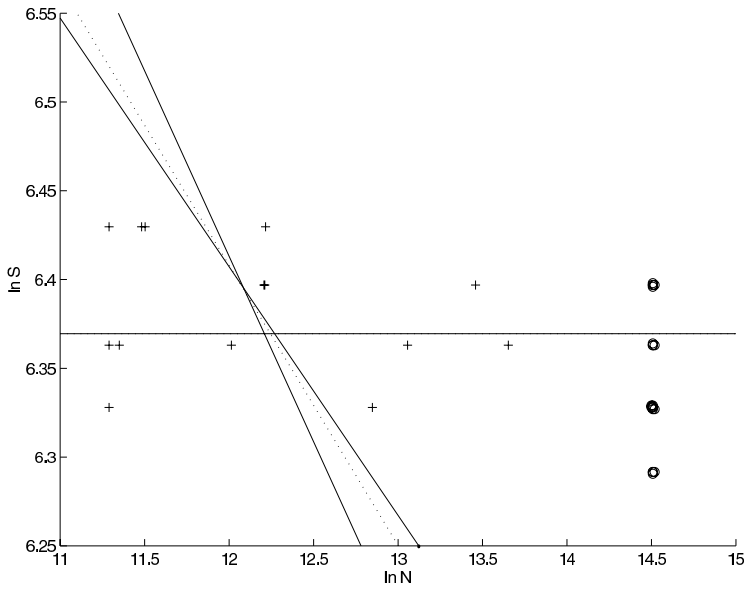

Fig. 1 Example 1.

estimated parameters for the likelihood function (1) are given in Table 3. The estimates for the endurance limit from Model 2 are $\hat{\sigma}_{xel} = 0.08286$ and $\hat{\mu}_{xel} = 6.4219$.

The approximative variance estimates in Model 1 are given in Table 4, and for Model 2 the corresponding estimates are $\widehat{\mathrm{Var}}(\hat{\mu}_{xel}) = 0.0005272$ and $\widehat{\mathrm{Var}}(\hat{\sigma}_{xel}) = 0.002192$, $\widehat{\mathrm{Cov}}(\hat{\mu}_{xel}, \hat{\sigma}_{xel}) = 0.0002935$.

Table 3 ML results for Example 2 with Model 1

Weight increase	$\hat{\sigma}_v$	$\hat{v}(x)$	$\hat{\sigma}_{xe}$	$\hat{\mu}_{xe}$
0	1.3905	$53.9795 - 6.2239\,x$	0.0100	6.3794
0.4	1.9162	$58.9594 - 6.9925\,x$	0.0094	6.3788
0.8	2.3231	$60.3158 - 7.2016\,x$	0.0092	6.3786

Table 4 Approximative variance estimates for Example 2

Weight increase	$\widehat{\mathrm{Var}}(\hat{\mu}_{xe})$	$\widehat{\mathrm{Var}}(\hat{\sigma}_{xe})$	$\widehat{\mathrm{Cov}}(\hat{\mu}_{xe}, \hat{\sigma}_{xe})$
0	0.0012	0.2016	−0.0141
0.4	0.0200	1.0096	−0.1412
0.8	0.0363	1.4048	−0.2249

In this example, the median of the endurance limit is approximately $e^{6.422} = 615$ (MPa) and that of the fatigue limit is approximately $e^{6.379} = 589$ (MPa), so they do not coincide as in the previous example. But the two limits are not the same and in this example the endurance limit is really the limit at n_0 and not at infinity as the fatigue limit is. Here the $\Phi(\frac{y_0 - v(x_i)}{\sigma_v/w_i})$ factor in (1) is not approximately one. Therefore, the $\Phi(\frac{y_i - \mu_{xe}}{\sigma_{xe}})$ factor in (1) increases and consequently μ_{xe} decreases and there is a difference between the two models. This occurs when some of the finite lives are close to the endurance life.

The result of this example is shown in Fig. 2, with the same notation as in the previous figure. There is one exception: since the estimated fatigue limit and endurance limit do not coincide there are two horizontal lines. The solid line is the fatigue limit and the other is the endurance limit.

SIMULATIONS

To see how good the estimated parameters are, one can simulate a large number of staircase tests and see how much the estimated parameters deviate from the true ones.

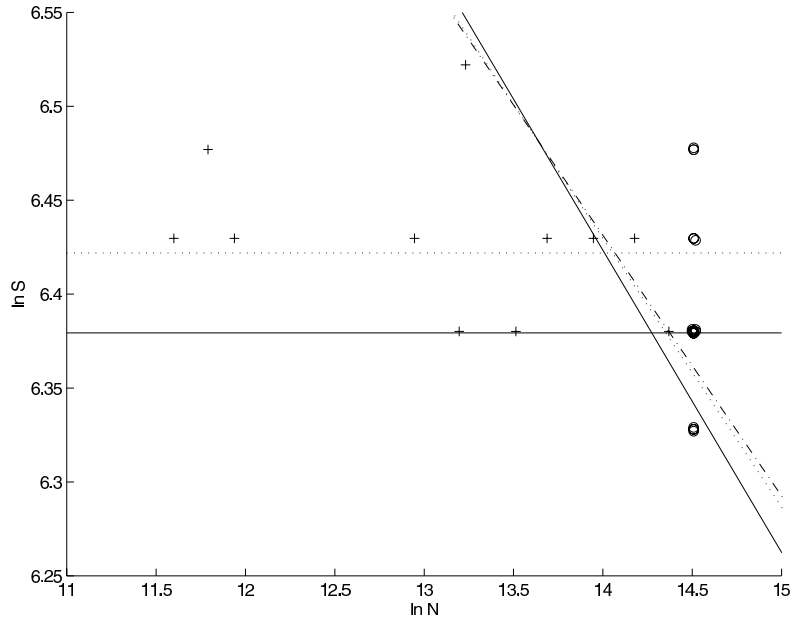

Fig. 2 Example 2.

The procedure which was used to simulate the staircase tests was as follows:

1 Choose a stress start level s_1, step length s_{step} and a stress value n_0 where the test is interrupted.
2 Choose $\ln(S_e)$ randomly from a normal distribution with the expectation μ_{xe} and the variance σ_{xe}^2.
3 If $s_i < S_e$ then the specimen is censored and the logarithm of the lives length is put to y_o. Otherwise choose the logarithm of the lives length, y_i, from a normal distribution with the expectation $\alpha + \beta x_i$ and the variance σ_v^2/w_i^2.
4 If $y_i > y_o$, then set $y_i = y_o$.
5 Let the next s_{i+1} be equal to $s_i + s_{step}$ if the y_i is censored, that is, equal to y_o. Otherwise let the next s_{i+1} be equal to $s_i - s_{step}$.
6 Restart the procedure from step 2.

Using the above method, 1000 tests with 30 specimens in each test were simulated as the procedure described above. The parameters were chosen so that the simulated staircase test corresponded to the staircase test in Example 1. The parameter values were chosen as $s_1 = 540$, $s_{step} = 20$, $n_o = 2 \times 10^6$, $\mu_{xe} = 6.37$, $\sigma_{xe} = 0.057$, $v(x) = 43 - 5x$, $\sigma_v = 0.8$ and all $w_i = 1$.

In the 1000 simulated tests there were 12 tests which are not included in the following results. This is because they are not a realistic staircase test in the sense that they do not overlap. Overlapping does not occur if there is a stress level which separates the failures from the run-outs.

The estimates in the first model are $\hat{\sigma}_v = 0.73188$, $\hat{v}(x) = 49.7329 - 6.05366x$, $\hat{\sigma}_{xe} = 0.04971$ and $\hat{\mu}_{xe} = 6.37050$. In the second model they are $\hat{\sigma}_{xel} = 0.049697$ and $\hat{\mu}_{xel} = 6.37051$.

The mean of the approximative variance estimators in the two models is the same:

$$\widehat{\mathrm{Var}}\begin{pmatrix}\hat{\mu}_{xe}\\ \hat{\sigma}_{xe}\end{pmatrix} = \widehat{\mathrm{Var}}\begin{pmatrix}\hat{\mu}_{xel}\\ \hat{\sigma}_{xel}\end{pmatrix} = 10^{-4}\begin{pmatrix}1.83 & 0.41\\ 0.41 & 5.39\end{pmatrix},$$

where the diagonal elements are variances and off-diagonal covariances. The approximative variance estimators are the same since the endurance and fatigue limit are almost the same.

One way of controlling the approximative variance estimators is to compare them with the empirically calculated variances. The empirically calculated variances for $\hat{\mu}_{xe}$ and $\hat{\sigma}_{xe}$ are 0.000214 and 0.000403, respectively, and the covariance is 0.000027. The corresponding results for the endurance limit are the same. The empirically calculated variances and the approximate calculated variances are of the same size and thus the approximate variance estimates seem to be useful.

CONFIDENCE INTERVAL FOR LOWER PERCENTILE OF THE FATIGUE LIMIT AND THE ENDURANCE LIMIT

One is interested in estimating the lower percentile of the fatigue limit which will be denoted by, S_{e_p}, where p is the probability of the lower tail. Write

$$X_{e_p} = \ln(S_{e_p}) = \mu_{xe} - \lambda_p \sigma_{xe} \quad \text{and} \quad P(S_e \le S_{e_p}) = p.$$

The asymptotic distribution of

$$\frac{\hat{\mu}_{xe} - \lambda_p\hat{\sigma}_{xe} - (\mu_{xe} - \lambda_p\sigma_{xe})}{\sqrt{\widehat{\mathrm{Var}}(\hat{\mu}_{xe} - \lambda_p\hat{\sigma}_{xe})}}$$

is standard normal distribution. By using this a one-sided $(1-\alpha)100\%$ confidence interval for X_{e_p} is obtained as

$$I = [\hat{\mu}_{xe} - \lambda_p\hat{\sigma}_{xe} - \lambda_\alpha\sqrt{\widehat{\mathrm{Var}}(\hat{\mu}_{xe} - \lambda_p\hat{\sigma}_{xe})}; \infty],$$

where

$$\widehat{\mathrm{Var}}(\hat{\mu}_{xe} - \lambda_p\hat{\sigma}_{xe}) = \widehat{\mathrm{Var}}(\hat{\mu}_{xe}) + \lambda_p^2\widehat{\mathrm{Var}}(\hat{\sigma}_{xe}) - 2\lambda_p\widehat{\mathrm{Cov}}(\hat{\mu}_{xe}, \hat{\sigma}_{xe}).$$

The lower confidence bound of S_{e_p} is denoted by $\underline{S}_{e_p}$ and

$$\underline{X}_{e_p} = \ln(\underline{S}_{e_p}) = \hat{\mu}_{xe} - \lambda_p\hat{\sigma}_{xe} - \lambda_\alpha\sqrt{\widehat{\mathrm{Var}}(\hat{\mu}_{xe} - \lambda_p\hat{\sigma}_{xe})}.$$

The lower percentile for the endurance limit, S_{el_p}, is calculated in the same way but with μ_{ex} and σ_{xe} replaced by μ_{exl} and σ_{xel}.

In Tables 5 and 6 the above calculations are done with different values of p and α in both examples and both models. Table 5 shows the result of Example 1 and Table 6 corresponds to Example 2. In Example 1, S_{e_p} and S_{el_p} are the same, but $\underline{S}_{e_p}$ and $\underline{S}_{el_p}$ are different. The reason for

Table 5 Confidence interval from Example 1

p	α	$\underline{S}_{el_p}$	S_{el_p}	$\underline{S}_{e_p}$	S_{e_p}	$1-q$	$\underline{S}^*_{el_q}$	$\underline{S}^*_{e_q}$
0.01	0.01	444	511	451	511	0.98	424	476
0.01	0.05	463	511	468	511	0.94	515	521
0.05	0.01	480	531	486	531	–	–	–
0.05	0.05	495	531	499	531	0.90	533	535

Table 6 Confidence interval from Example 2

p	α	$\underline{S}_{el_p}$	S_{el_p}	$\underline{S}_{e_p}$	S_{e_p}	$1-q$	$\underline{S}^*_{el_q}$
0.01	0.01	397	507	47	576	0.98	–
0.01	0.05	427	507	97	576	0.94	482
0.05	0.01	452	537	98	580	0.94	–
0.05	0.05	475	537	163	580	0.90	531

this is that $\hat{\mu}_{xe}$, $\hat{\mu}_{xel}$ and $\hat{\sigma}_{xe}$, $\hat{\sigma}_{xel}$ are almost the same but the approximative variance estimations are different. One can also see that p and α influence each other. A higher value on p can be compensated by a lower value on α to get the same value on $\underline{S}_{e_p}$. In Example 2, there is a large difference in S_{e_p} and S_{el_p} and an even larger difference in $\underline{S}_{e_p}$ and $\underline{S}_{el_p}$. The value on $\underline{S}_{e_p}$ is very low. The reason for this low value is the large approximate variance estimators.

One can get a lower bound $\underline{X}^*_{e_q} = \hat{\mu}_{xe} - a_e\hat{\sigma}_{xe}$ of X_e which will be exceeded with probability $1-q$, by calculating $P(S_e > \underline{S}^*_{e_q})$ and let this probability be equal to $1-q$. This is the same thing as

$$P(X_e - \underline{X}^*_{e_q} > 0) \approx \Phi\left(\frac{E[X_e - \underline{X}^*_{e_q}]}{\sqrt{\mathrm{Var}(X_e - \underline{X}^*_{e_q})}}\right) = 1-q,$$

where $E[X_e - \underline{X}^*_{e_q}] \approx a_e\sigma_{xe}$ and

$$\mathrm{Var}(X_e - \underline{X}^*_{e_q}) \approx \sigma_{xe}^2 + I^{-1}(1,1) + a_e^2 I^{-1}(2,2) - 2a_e I^{-1}(2,1).$$

One can decide which probability $1-q$ one is interested in and solve Eq (3) with respect to a_e and calculate $\underline{X}^*_{e_q}$:

$$\Phi\left(\frac{a_e\hat{\sigma}_{xe}}{\sqrt{\sigma_{xe}^2 + I^{-1}(1,1) + a_e^2 I^{-1}(2,2) - 2a_e I^{-1}(2,1)}}\right) = 1-q. \quad (3)$$

The above calculations and the corresponding for $\underline{S}^*_{el_q}$ are also shown in Tables 5 and 6 for different values of q. For Example 1, the two models give almost the same result. In the case when $\hat{\sigma}_{xe}$ is smaller than $I^{-1}(2,2)$ Eq. (3) can only be solved for small values on $1-q$. This occurs for Example 2 with the estimates of Model 1.

To check the confidence interval, 1000 staircase tests were simulated similar to the procedure described before with the parameters chosen as $s_1 = 540$, $s_{step} = 20$, $n_o = 2 \times 10^6$, $\mu_{xe} = 6.3693$, $\sigma_{xe} = 0.0571$, $v(x) = 42.6965 - 4.7865x$, $\sigma_v = 0.7920$ and all $w_i = 1$. In the 1000 simulated tests there were six tests which are not included in the result because they do not overlap.

The number of times the confidence interval contains X_{e_p} is counted. This number is denoted by l_e. This calculation and the corresponding for X_{el_p} are shown in Table 7. The simulations show that the coverage for X_{e_p} is too low, i.e., the estimate of X_{e_p} is too high. This can be explained by the lack of normality of $\hat{\sigma}_{xe}$ and $\hat{\sigma}_{xel}$; compare Fig. 3 in which the distribution for $\hat{\sigma}_{xel}$ is shown. The distribution for $\hat{\sigma}_{xe}$ has the same shape.

The theoretical probability of the lower tail $P(S_e \le S_{e_p}) = p$ was compared to the mean of the simulated $\Phi(\frac{\underline{X}_{e_p} - \mu_{xe}}{\sigma_{xe}})$. This and the corresponding estimates for the endurance limit are shown in Table 7. This shows that the estimated X_{e_p} and X_{el_p} are too high.

Table 7 Control of different confidence interval

$1-\alpha$	$\frac{l_e}{\sqrt{n_0}}$	$\frac{l_{el}}{\sqrt{n_0}}$	p	Mean of $\Phi\left(\frac{\underline{X}_{e_p} - \mu_{xe}}{\sigma_{xe}}\right)$	Mean of $\Phi\left(\frac{\underline{X}_{el_p} - \mu_{xel}}{\sigma_{xel}}\right)$
0.99	0.8340	0.8330	0.01	0.0535	0.0537
0.95	0.7425	0.7425	0.01	0.0535	0.0537
0.99	0.8471	0.8461	0.05	0.1092	0.1100
0.95	0.7374	0.7334	0.05	0.1092	0.1100

The mean of the simulated $1 - \Phi(\frac{\underline{X}^*_{e_q} - \mu_{xe}}{\sigma_{xe}})$ is compared to $P(S_e > \underline{S}^*_{e_q}) \approx 1-q$ and $P(S_e > \underline{S}^*_{el_q}) \approx 1-q$ are checked in a similar way. The results of these calculations are shown in Table 8. These calculations show that the lower bound of the fatigue limit $\underline{X}^*_{e_q}$ and $\underline{X}^*_{el_q}$ is overestimated.

LOGISTIC DISTRIBUTION FOR THE LOGARITHM OF THE FATIGUE LIMIT AND THE ENDURANCE LIMIT

The estimates for μ_{xe}, μ_{xel}, σ_{xe} and σ_{xel} almost seem to be the same if the logarithm of the fatigue and of the endurance limit follows the logistic distribution instead of the normal distribution.

CONCLUSIONS

As we have shown, it is possible to estimate the distribution of the fatigue limit although the fatigue limit itself is impossible to observe. One never knows if a failure will occur eventually. In order to make the estimate possible finite lives have to be considered.

The finite lives seem to give additional information in some cases and in others just the information that they are finite. According to this it is useful to use the finite lives in some cases and in others one does not seem to gain anything. When there is no information in the finite lives, the estimated limits coincide. In this case Model 2 is preferred since the likelihood is easier to maximize. But the two models do not estimate the same limit; Model 1 estimates the fatigue limit and not the endurance limit.

The finite lives give information when some specimens break near the endurance life n_o. This is because the $\Phi(\frac{y_o - v(x)}{\sigma_v/\sqrt{w_i}})$ factor in (1) is not approximately equal to one, and accordingly, functions (1) and (2) do not coincide. In this case the fatigue limit estimate is lower than the endurance limit estimate. The reason for this is that in this case the endurance limit is really the limit at n_o cycles and not at infinity. The fatigue limit, on the other hand,

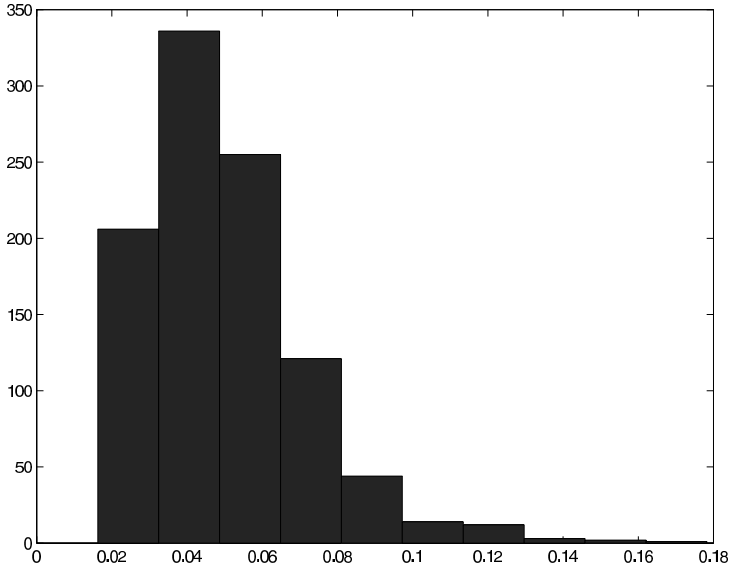

Fig. 3 Distribution for $\hat{\sigma}_{xe1}$.

Table 8 Control of the lower limits $\underline{X}^*_{e_q}$ and $\underline{X}^*_{el_q}$

a_e	$1-q$	Mean of $1-\Phi\left(\frac{\underline{\hat{S}}^*_{eq}-\mu_{xe}}{\sigma_{xe}}\right)$	Mean of $1-\Phi\left(\frac{\underline{\hat{S}}^*_{elq}-\mu_{xel}}{\sigma_{xel}}\right)$
2.0537	0.98	0.9289	0.9284
1.5548	0.94	0.8800	0.8791
1.2816	0.90	0.8408	0.8396

corresponds to infinite life. As the interest was in the fatigue limit rather than in the endurance limit at n_o cycles, the staircase test was carried out at too low a run-out level.

The estimate of the lower percentile for the endurance limit and the fatigue limit seems to be too high according to the simulated confidence intervals.

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APPENDIX

Table A1 Staircase data for Example 1, $n_o = 2 \times 10^6$

Specimen no	S	N $\times 10^3$	Specimen no	S	$N \times 10^3$
1	540	n_o	16	600	n_o
2	560	80	17	620	202
3	540	n_o	18	600	n_o
4	560	380	19	620	80
5	540	n_o	20	600	n_o
6	560	n_o	21	620	99
7	580	n_o	22	600	201
8	600	700	23	580	n_o
9	580	80	24	600	n_o
10	560	n_o	25	620	97
11	580	468	26	600	200
12	560	n_o	27	580	85
13	580	165	28	560	n_o
14	560	n_o	29	580	850
15	580	n_o	30	560	n_o

Table A2 Staircase data for Example 2, $n_o = 2 \times 10^6$

Specimen no	S	N $\times 10^3$	Specimen no	S	$N \times 10^3$
1	590	n_o	14	560	n_o
2	620	1140	15	590	n_o
3	590	1740	16	620	n_o
4	560	n_o	17	650	132
5	590	n_o	18	620	109
6	620	419	19	590	n_o
7	590	n_o	20	620	1436
8	620	880	21	590	n_o
9	590	538	22	620	n_o
10	560	n_o	23	650	n_o
11	590	n_o	24	680	538
12	620	153	25	650	n_o
13	590	739	–	–	–

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Estimating inclusion distributions of hard metal using fatigue tests

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Abstract

The maximum inclusion size in clean steels influences fatigue behaviour and other mechanical properties. In this paper three different approaches to estimating the distribution of the inclusion sizes which cause the failure are compared. The data used are from a staircase test. The load levels on the broken and unbroken specimens as well as the size of the inclusions which cause the failures are recorded. One model uses only the load levels and the other two models use the recorded inclusion size on the broken specimens and the load level for the unbroken specimens. The difference between the models using the inclusion size is that one is based on the assumption that the specimen breaks in the largest inclusion, and the other one on the assumption that the specimen breaks in some inclusion over a critical size, not necessarily the largest one. It is shown that from a staircase test with measured inclusion size on the broken specimens it is better to use both load levels and inclusion sizes to estimate the inclusion size distribution instead of using just the different load levels. The two models using the inclusion sizes almost coincide for big inclusions. These models are also used to verify or reject possible fatigue limit models.
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Keywords: Maximum likelihood; Extreme value; Staircase test

1. Introduction

The mechanical properties of hard and clean metals are in part governed by the inclusion size distribution, e.g. the fatigue limit is determined by the size of the maximum projected area on the plane perpendicular to the load [1].

Direct observations of inclusions sizes are difficult since the inclusions are rare and often occur in the interior of the specimen. A usual way of observing is the following, a number of areas of the same size are observed on polished cross-sections by optical microscopy. The largest inclusion in each area is identified and measured and then the extreme value theory is used to extrapolate the maximum inclusion size distribution of the specimen of interest. In [2] the method of estimating the inclusion size distribution is based on measuring the maximum size in randomly chosen areas or volumes. In [3] the method is based on measuring the size of inclusions larger than a chosen size in a small volume.

In a staircase test a number of specimens are tested at different load levels. The result from a staircase test is also the number of failures and survivors of different load levels. In [4] and [5] the distribution of the inclusions was estimated using an estimate of the endurance limit and the relation between the endurance limit and the inclusion size [1].

The inclusion size distribution can also be estimated by direct observations on the maximal inclusion size in the specimens or by the load levels from a staircase test. Here the inclusion size and the load level are combined through a staircase test. The load levels on the unbroken specimens are used. For the broken specimens the inclusion which caused the failure is used. This is possible since if the fatigue failure starts at the maximal projected area perpendicular to the load [1], direct observations of this inclusion are possible to carry out on the failed specimens by measuring the inclusion where the specimen breaks. The inclusion sizes are used under two different assumptions, the specimens breaks in the largest inclusion or in some of the inclusions over some critical size.

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According to general statistical theory maximal sizes are approximately distributed according to one extreme value distribution. Here the Gumbel and the Frechet extreme value distribution are applied. As a comparison the consequence of assuming lognormal fatigue strength is also calculated. This method is often used in practice [6].

2. Inclusion size distribution estimate based on fatigue test data

According to [1] one can use relation (1) below, the so called ($\sqrt{\text{area}_{\max}}$−model) to determine the fatigue limit, where HV is the Vickers hardness in Hv, R is the stress ratio $S_{\min}/S_{\max}$ and $\text{area}_{\max}$ is the largest inclusion area projected onto a plane perpendicular to the applied load in μm² in the volume of the material that is subjected to the maximum stress. The constant K is depending on the location of the inclusion, K=1.43 for surface inclusion and K=1.56 for internal inclusion. The exponent α is given as $\alpha = (HV\cdot 10^{-4} + 0.226)$. The fatigue limit S_e is given by

$$S_e = K(HV + 120)\sqrt{\text{area}_{\max}}^{-1/6}\left(\frac{1-R}{2}\right)^{\alpha}. \quad (1)$$

For $q = K(Hv + 120)\left(\frac{1-R}{2}\right)^{\alpha}$, Eq. (1) can be written as $S_e = h(A) = qA^{-1/6}$ where A is the square root of the largest projected inclusion area perpendicular to the load, $\sqrt{\text{area}_{\max}}$.

The aim is to estimate the inclusion size distribution from observations on the load levels of the specimens and Eq. (1), using the information that if the specimen is broken the load level is above the fatigue limit and vice versa for the unbroken specimens. This is the same as $h^{-1}(s)$ is less than the inclusion sizes for the broken specimens and bigger than the inclusion sizes for the unbroken specimens on the stress level s. Hence the relation between the cumulative probability distribution function of S_e and A is

$$F_{S_e}(s) = \mathrm{P}(qA^{-1/6} \leq s) = 1 - F_A[(s/q)^{-6}] = 1 - F_A[h^{-1}(s)]. \quad (2)$$

Note that for each load level there is a critical inclusion size for which inclusion sizes below this size will not cause failure, $h^{-1}(s)$. This size will also be denoted a_c.

From the staircase test one has k different load levels and the number of survivors and failures at each level. Let F_A(a) denote the cumulative distribution for the maximum inclusion size in the specimen, n_i the number of failures and m_i the number of survivors at load level s_i. The likelihood of the experiment in terms of the inclusion size distribution is therefore

$$\text{lik} = \prod_{i=1}^{k} (1 - F_A[h^{-1}(s_i)])^{n_i}(F_A[h^{-1}(s_i)])^{m_i}. \quad (3)$$

Here two different distributions for the inclusion size are considered, the Gumbel and the Frechet distributions with threshold zero. The likelihood function when using the Gumbel assumption is

$$\text{lik}(\mu,\sigma) = \prod_{i=1}^{k}\left(1 - \mathrm{e}^{-\mathrm{e}^{-\frac{h^{-1}(s_i)-\mu}{\sigma}}}\right)^{n_i}\left(\mathrm{e}^{-\mathrm{e}^{-\frac{h^{-1}(s_i)-\mu}{\sigma}}}\right)^{m_i} \quad (4)$$

and the likelihood of the Frechet assumption is

$$\text{lik}(\beta,\lambda) = \prod_{i=1}^{k}\left(1 - \mathrm{e}^{-\left(\frac{h^{-1}(s_i)}{\beta}\right)^{-\lambda}}\right)^{n_i}\left(\mathrm{e}^{-\left(\frac{h^{-1}(s_i)}{\beta}\right)^{-\lambda}}\right)^{m_i}. \quad (5)$$

As a comparison one can estimate the fatigue limit distribution and then use relation (1) to estimate the inclusion size distribution.

One way of estimating the fatigue limit distribution is to assume that the logarithm of the fatigue strength follows a normal distribution X_e=ln(S_e)~$N(\mu_e, \sigma_e)$ and that the specimen breaks if the load level is above the fatigue limit [6]. The likelihood for estimating the fatigue limit distribution is then

$$lik(\mu_e,\sigma_e) = \prod_{i=1}^{k} (F_{X_e}[ln(s_i)])^{n_i}(1 - F_{X_e}[ln(s_i)])^{m_i} \quad (6)$$

The difference between this way and the former ways of estimating the inclusion size distribution is that the former are based on the assumption that the inclusion sizes have some specified distribution and the latter method is based on the assumption that the fatigue limit has some specified distribution.

3. Inclusion size distribution estimate based on fatigue test data augmented with inclusion size measurements

These models estimate the inclusions size distribution from the load levels of the unbroken specimens, inclusion sizes of the broken specimens and eq. (1) under different assumptions.

The size of the inclusion where the specimen breaks is measured and is assumed to be the largest inclusion in that specimen. The test data used in this model are also k different load levels and the number of survivors and failures at each load level. For the broken specimens there are also the inclusions sizes $a_{i1},\ldots,a_{in_i}$. The inclusion sizes for the unbroken specimens are all assumed to be below $h^{-1}(s_i)$, the critical size i=1,…,k.

Let again $F_A(a)$ be the cumulative distribution function for the inclusion size and $f_A(a)$ is the corresponding density function. The likelihood for the experiment under these assumptions is

$$\text{lik} = \prod_{i=1}^{k}\left(\prod_{j=1}^{n_i} f_A(a_{ij})\right)(F_A[h^{-1}(s_i)])^{m_i}. \quad (7)$$

Assuming that the inclusion size follows a Gumbel distribution one gets

$$\text{lik} = \prod_{i=1}^{k}\left(\prod_{j=1}^{n_i}\frac{1}{\sigma}e^{-\frac{a_{ij}-\mu}{\sigma}}e^{-e^{-\frac{a_{ij}-\mu}{\sigma}}}\right)\left(e^{-e^{-\frac{h^{-1}(s_i)-\mu}{\sigma}}}\right)^{m_i}. \quad (8)$$

If instead one assumes that the inclusion size follows a Frechet distribution with threshold zero, one gets

$$\text{lik} = \prod_{i=1}^{k}\left(\prod_{j=1}^{n_i}\frac{\lambda}{\beta}\left(\frac{a_{ij}}{\beta}\right)^{-\lambda-1}e^{-\left(\frac{a_{ij}}{\beta}\right)^{-\lambda}}\right)\left(e^{-\left(\frac{h^{-1}(s_i)}{\beta}\right)^{-\lambda}}\right)^{m_i}. \quad (9)$$

In this model the specimens are assumed to break in the largest inclusion. Assume now that there is more than one inclusion over the critical size a_c and that the specimen breaks in one of these with the same probability for each of them. This assumption is based on that the crack will begin in the largest inclusion, but how fast the crack will grow depends on the shape of the inclusion. To model this the following assumptions are reasonable (see [3]). Assume that the number of inclusions in the specimen that are greater than a_c are Poisson distributed with the parameter v. The conditional probability that an inclusion is not greater than a, given that it is at least a_c, follows a Generalized Pareto distribution with the cumulative distribution function

$$F_{A|A\geq a_c}(a) = 1-\left(1-\frac{\xi(a-a_c)}{\sigma}\right)^{1/\xi}$$

where $\sigma > 0$, $0 < a-a_c < \sigma/\xi$ if $\xi > 0$ and $0 < a-a_c < \infty$ if $\xi \leq 0$..

Then the inclusion size over some threshold is Generalized Pareto distributed and the number of inclusions N over that threshold is Poisson distributed. When $\xi=0$, the maximum inclusion size follows a Gumbel distribution [7]. For $\xi<0$, the maximum inclusion size follows a Frechet distribution and for $\xi>0$ it has a reversed Weibull distribution. Therefore the assumption that the specimen break in some inclusion over the critical size does not contradict to the assumption that the specimen break in the largest inclusion.

Using the law of total probability the cumulative distribution function of A can be rewritten as

$$\begin{aligned}P(A\leq a) &= \sum_{n=0}^{\infty}P(A\leq a|N=n)P(N=n) = 1\cdot P(N=0)\\ &+\sum_{n=1}^{\infty}P(A\leq a|N=n)P(N=n) = P(N=0)+\sum_{n=1}^{\infty}P(A_1\\ &\leq a)P(N=n) = P(N=0)+P(A_1\leq a)[1-P(N=0)]\\ &= e^{-v}+\left[1-\left(1-\frac{\xi(a-a_c)}{\sigma}\right)^{1/\xi}\right](1-e^{-v}) = 1-(1\\ &-e^{-v})\left[1-\frac{\xi(a-a_c)}{\sigma}\right]^{1/\xi}, \quad a > a_c.\end{aligned}$$

Note that $P(A_1\leq a_c)=e^{-v}>0$. Here A_1 follows conditionally a Generalized Pareto distribution and denotes the size of an arbitrary inclusion above the threshold.

For $a>a_c$ the density function of the inclusion size is

$$f_A(a) = \frac{1}{\sigma}(1-e^{-v})\left(1-\frac{\xi(a-a_c)}{\sigma}\right)^{1/\xi-1}$$

and for $\xi=0$ the previous density function becomes exponential above the threshold, i.e $f_A(a) = \frac{1}{\sigma}(1-e^{-v})\left(e^{-\frac{(a-a_c)}{\sigma}}\right), \quad a > a_c.$

The critical size a_{ci}, the expected number of inclusions v_i over that size and the scale parameter σ_i, for the inclusion size distribution depend on the load level s_i. The density function corresponding to the load level s_i is

$$f_A(a) = \frac{1}{\sigma_i}(1-e^{-v_i})\left(1-\frac{\xi(a-a_{ci})}{\sigma_i}\right)^{1/\xi-1}, \quad a > a_{ci}, \quad i = 1,\ldots,k.$$

The critical size for each level is determined by the load level, $a_{ci}=h^{-1}(s_i)$. Denote the smallest critical size as a_{c0} corresponding to the highest load level. The different scale parameters can be related to the scale parameter which corresponds to the lowest critical size in the following way $\sigma_i=\sigma_0-\xi(a_{ci}-a_{c0})$ for $i=1,\ldots,k$, where σ_0 is the scale parameter corresponding to the lowest critical size, i.e. the largest load level. This is true since

$$P(A > a_{ci}) = P(A > a_{ci}|A > a_{ci-1})P(A > a_{c0}), \quad a_{ci} > a_{ci-1} > a_{c0}$$

using the Pareto distribution this equation can be expressed as

$$\left(1-\frac{\xi(a_{ci}-a_{c0})}{\sigma_0}\right)^{1/\xi} = \left(1-\frac{\xi(a_{ci}-a_{ci-1})}{\sigma_{i-1}}\right)^{1/\xi}\left(1-\frac{\xi(a_{ci-1}-a_{c0})}{\sigma_0}\right)^{1/\xi}$$

evaluating this expression gives $\sigma_i = \sigma_0-\xi(a_{ci}-a_{c0})$.

There is a relation between the different v_i values $i=1,\ldots,k$. Each v_i value can be related to v_0, the expected number of inclusions over a_{c0}. The expected number of inclusions over a_{ci} is $v_i = v_0P(A > a_{ci}|A > a_{c0})$ for $i=1,\ldots,k$ [7]. Hence $v_i = v_0\left(1-\frac{\xi(h^{-1}(s_i)-a_{c0})}{\sigma_0}\right)^{1/\xi}, \quad h^{-1}(s_i)\geq a_{c0} \quad i = 1,\ldots,k..$

The likelihood for this model is

$$\prod_{i=1}^{k}\left(\prod_{j=1}^{n_i}f_A(a_{ij})\right)(P(N_i=0))^{m_i} = \prod_{i=1}^{k}\left(\prod_{j=1}^{n_i}\frac{1}{\sigma_i}(1-e^{-v_i})\left(1-\frac{\xi(a_{ij}-h^{-1}(s_i))}{\sigma_i}\right)^{1/\xi-1}\right)(e^{-v_i})^{m_i}. \quad (10)$$

The likelihood in (10) holds when all inclusion sizes are bigger than their corresponding critical value.

For $\xi=0$ all scale parameters are the same and the expected number of inclusions over the smallest critical sizes a_0 is

$$v_i = v_0e^{-\frac{(h^{-1}(s_i)-a_{c0})}{\sigma}}, \quad h^{-1}(s_i)\geq a_{c0} \quad i = 1,\ldots,k.$$

The likelihood when ϵ is zero is therefore

$$\prod_{i=1}^{k}\left(\prod_{j=1}^{n_i}\frac{1}{\sigma}\left(1-e^{-v_0e^{-\left(\frac{h^{-1}(s_i)-a_{c0}}{\sigma}\right)}}e^{-\frac{(a_{ij}-h^{-1}(s_i))}{\sigma}}\right)\right)\left(e^{-v_0e^{-\frac{(h^{-1}(s_i)-a_{c0})}{\sigma}}}\right)^{m_i}. \quad (11)$$

Note that the likelihood in (10) uses one more parameter to describe the inclusion size distribution than the other likelihoods. But later in the estimation procedure we will fix $\xi=0$ and just fit two parameters, v_0 and σ. Otherwise the model should be questioned.

Note that in order to apply the described likelihoods, the observed inclusion size a_{ij} of the failure number j on the load level s_i has to satisfy the consistency condition that a_{ij} is bigger than the critical size $a_{ci} = h^{-1}(s_i)$.

4. Simulation

To study the systematic and random error when estimating the 95% quantile of the distribution of the inclusion size, a simulation study of a staircase test with 30 specimens is carried out. The staircase test was simulated under four different assumptions:

1. Size distribution is assumed to be Gumbel.
2. Size distribution is assumed to be Frechet with $\lambda=3$.
3. Size distribution is assumed to be exponential above the threshold, i.e. Generalized Pareto with $\xi = 0$.
4. Size distribution is assumed to be Generalized Pareto with $\xi=-0.1$.

For each of the four size distributions the following models are fitted:

1. Size distribution is assumed to be Gumbel with and without measured inclusion sizes, i.e. likelihood (8) and (4) respectively.
2. Size distribution is assumed to be Frechet with and without measured inclusion sizes, i.e. likelihood (9) and (5), respectively.
3. Size distribution is assumed to be Generalized Pareto, with $\xi = 0$, i.e. likelihood (11).
4. The Logarithm of the fatigue limit follows a normal distribution, i.e. likelihood (6).

The procedure which was used to simulate the staircase test when the inclusion sizes are assumed to be Gumbel or Frechet distribution was as follows:

1. Choose a start level s_1, step length s_{step} and a constant q.
2. Choose A randomly from a Frechet or a Gumbel distribution.
3. If $s_i < qA^{-1/6}$ the specimen is unbroken and s_{i+1} is equal to s_i+s_{step}.
4. If $s_i > qA^{-1/6}$ the specimen is broken and s_{i+1} is equal to $s_i - s_{step}$ and save the inclusion size A.
5. Restart the procedure from number 2.

Using the method above, 1000 staircase tests were simulated according to the procedure described above with Frechet distribution and 1000 staircase tests with Gumbel distribution. The parameters were chosen so that the simulated data in some sense correspond to two of the data sets B and C in the following examples. The parameter values were chosen as $s_1=800$, $s_{step}=20$ and $q=1350$. For the Gumbel distribution the scale parameter $\sigma=7$ and the location parameter $\mu=20$. For the Frechet distribution the location parameter $\alpha=0$, the scale parameter $\beta=15$ and the shape parameter $\lambda=3$. Under the Gumbel assumption the expectation and the variance for the inclusion size are 24 and 81, respectively, and for the Frechet assumption the corresponding are 20 and 190, respectively.

To simulate the staircase test when the inclusion size over some threshold follows a Generalized Pareto distribution, the following procedure was used:

1. Choose a start level s_1, step length s_{step}, a constant q, expected number of inclusions over a threshold v_0, and the parameters in the Generalized Pareto distribution.
2. Calculate v_i, the expected number of inclusions over a_{ci}.
3. Choose the number of inclusions over the critical size N randomly from a Poisson distribution with parameter v_i.
4. If $N=0$ the specimen is unbroken and s_{i+1} is equal to s_i+s_{step}.
5. If $N > 0$ the specimen is broken and s_{i+1} is equal to s_i-s_{step} and choose A randomly from a Generalized Pareto distribution with threshold a_{ci}.
6. Restart the procedure from number 2.

Using the method above, 1000 staircase tests were simulated according to the procedure described above. The parameter values were chosen as $s_1=800$, $s_{step}=20$, $q=1350$, $\sigma=8$, $\xi=0$ and $v_0=10$. The expected number of

Table 1
Mean and standard deviation for the estimated 95% quantile

	Gumbel dist correct 95% quantile is 40.79		Frechet dist correct 95% quantile is 40.37	
Dist assumption	Mean	SD	Mean	SD
A ~ Gumbel with	40.47	4.24	38.86	9.03
A ~ Gumbel without	37.81	9.17	29.55	7.99
A ~ Prechet with	43.25	5.48	40.03	8.53
A ~ Frechet without	48.70	47.75	37.25	39.23
A ~ Exponential	38.86	4.40	37.79	9.23
log(*S*) ~ Normal	38.83	19.50	30.55	16.68

inclusions over the threshold a_0 corresponds to a load level at 1000 MPa. A simulation was also carried out with the same parameters value except that ξ was -0.1.

In Tables 1 and 2 the sample mean and standard deviation for the 95% quantile of the estimated inclusion size distribution are shown. In Table 1 the simulated staircase test is from the method of assuming that the inclusion size follows a Gumbel distribution or a Frechet distribution, Table 2 the inclusion size follows a Generalized Pareto distribution with ξ=0 or a Generalized Pareto distribution with ξ=-0.1. The model which uses the direct measured inclusion sizes likelihood (7) is denoted by 'with' in the tables. The model which just uses the load levels likelihood (3) is denoted by 'without'. Frechet and Gumbel in the tables stand for which assumption one has on the inclusion size. The model which uses the Generalized Pareto distribution likelihood (11) is denoted by 'exponential'. In the calculations ξ is assumed to be zero all the time the distribution is also exponential. The model which assumes that the logarithm of the fatigue limit is normally distributed, i.e. likelihood (6), is denoted by 'normal'.

According to Tables 1 and 2 the estimated inclusion size distribution seems to be better estimated using the models which use the measured inclusion size. Whether it is better to use the exponential model or the model which assumes that the specimens break in the largest inclusion depends on the distribution on the inclusion size. The exponential model is not always the best one but it seems as if the estimate does not get too wrong even if ξ is always assumed to be zero. The model which uses the inclusion size under Gumbel and Frechet distribution is more dependent on which inclusion size distribution which is the correct distribution.

In the simulation in Table 2 the models using the inclusion sizes are much better than the models that just use the different load levels. In the simulation in Table 1 the difference is not that large.

Table 2
Mean and standard deviation for the estimated 95% quantile

	Exponential dist correct 95% quantile is 40.79		Pareto dist correct 95% quantile is 40.37	
Dist assumption	Mean	SD	Mean	SD
A ~ Gumbel with	44.59	4.98	50.15	6.83
A ~ Gumbel without	53.28	15.03	53.38	16.38
A ~ Frechet with	47.83	6.30	54.00	8.45
A ~ Frechet without	128.66	1005.40	121.33	628.57
A ~ Exponential	42.69	5.30	48.61	7.28
log(*S*) ~ Normal	62.24	93.01	64.04	107.88

5. Example

The different ways of estimating the inclusion size distributions are used to estimate the inclusion size distribution on three different data sets. The data sets are from two different steels and will be denoted by data set A, B and C. For data set A the hardness is 561 Hv and R is -0.5 for data set B and C the corresponding parameters are 745 Hv and -1. The parameter K is 1.43 for all data sets. In Table 3 the load levels and the square root of the inclusion size for the broken specimen are given. They are denoted by s and $\sqrt{\text{area}}$, respectively. In Table 4 the number of unbroken specimens denote by n at each load level s are presented. One problem with these data sets is that only for data set A the inclusion sizes are greater than the critical size a_c, for all inclusions. To qualitively judge if the model fit is good, a

Table 3
Data set for the broken specimens

Data set A		Data set B		Data set C	
s	$\sqrt{\text{area}}$	*s*	$\sqrt{\text{area}}$	*s*	$\sqrt{\text{area}}$
620	20	830	35.0	1200	11.5
640	17	830	26.5	1200	38.1
640	17	830	39.5	1200	24.6
640	23	830	43.3	1200	8.0
660	13	830	25.4	1000	23.6
680	15	830	27.8	1000	26.6
640	19	800	24.8	1000	21.3
620	30	800	33.2	1000	22.9
640	29	800	44.9	1000	17.2
620	14	800	30.0	830	28.7
600	25	800	30.6	830	27.4
600	33	800	28.9	830	18.7
600	15	780	40.5	800	18.2
600	19	780	44.2	800	28.1
600	20	780	26.9	800	12.9
		780	22.8	780	39.9
		750	37.5	780	24.9

Table 4
Data set for the unbroken specimens

Data set A		Data set B		Data set C	
s	*n*	*s*	*n*	*s*	*n*
600	18	830	3	830	2
620	10	800	3	800	3
640	8	780	5	780	5
660	3	750	5		

usual approach is to do a probability plot of the empirical distribution function, F_n. However, it is not straight forward to define the empirical distribution function when part of the sample is censored. To overcome this difficult the approach by Kaplan–Meier is adopted. Their estimate can be derived as a nonparametric estimate which is consistent and asymptotically normal distributed, [8].

To estimate $F_n(i)$ the following method is used. Since the data are left censored instead of right censored, the order number is changed to fit to the Kaplan–Meier method. Multiply all observations by -1 and order them $a_{(1)} < a_{(2)} < \ldots < a_{(n)}$. Let J_a denote the set of all indices j where $a_{(j)} \leq -a$ and $a_{(j)}$ represent an observed inclusion size. Let $n_{(j)}$ denote the number of specimens, with observed or possible inclusion sizes less than $a_{(j)}, j = 1,\ldots,n$. Then the Kaplan–Meier estimate of $F(a)$ is defined as $\hat{F}(a) = \Pi_{j \in J_a} \frac{n_{j-1}}{n_j}$.

In Figs. 1 and 2 the inclusion size distribution is estimated without the observed inclusion size (3). The solid line is under Gumbel assumption on the inclusion size. The dotted line is under Frechet assumption on the inclusion size and the dash-dotted line is under the assumption that the logarithm of the fatigue limit is normally distributed. The estimated empirical distribution is denoted by crosses. For data set B and C the assumption that all inclusions are greater then the critical size a_c is used even if it is not.

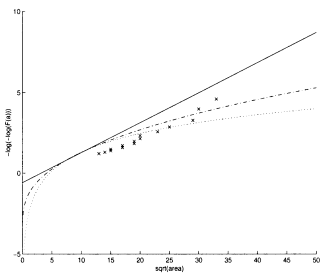

Fig. 1. Estimated probability plot from data set A inclusion sizes unobserved. Estimated Gumbel, —; Frechet, …; lognormal, –•–.

The estimated distribution function for the inclusion size distribution does not fit the estimated empirical distribution too well. When the probability for negative inclusion sizes is small in the estimated Gumbel distribution, the estimated inclusion size distribution under the Gumbel and the lognormal assumption is essentially the same. For the data set in data set B there is one specimen which has broken although it should not have done that and for data set C there are three specimens. One possible explanation for this is a measurement error on the inclusion size or the load level.

In Table 5 the 95% quantile, the expected value and the standard deviation for the inclusion size under the different distribution assumptions on the inclusion size are shown. Under the Frechet assumption on the inclusion size the variance is $\beta^2(\Gamma(1-2/\lambda)-\Gamma(1-1/\lambda)^2)$ if $\lambda>2$ and else infinite. Since $\hat{\lambda} < 2$ when using the data set in data set A the standard deviation is not computed for this data set. In Figs. 3 and 4 the dash-dotted lines are under the assumption that the logarithm of the fatigue limit is normally distributed, the same as before. The other two estimated distribution functions are from using the inclusion size of the broken specimens, (7). The solid line is under the Gumbel assumption on the inclusion size, dotted line is under Frechet assumption on the inclusion size. The crosses are the estimated distribution function with the Kaplan–Meier method. For data set B and C the load level for the broken specimens are all assumed to be $h^{-1}(s) < a_c$.

The estimated distributions seem to fit quite well the empirical distribution under the Gumbel and Frechet assumption. For the estimate from data set A under the Gumbel assumption the distribution is somewhat strange, since there is quite a big probability that the inclusion size is less than zero. This is a problem with the Gumbel distribution since it is defined for negative sizes as well. If the inclusions are big this is no problem. This material is a very clean steel with few and small inclusions so this can be the reason why the probability for negative inclusions is quite big.

Table 6 correspond to Table 5 but with the likelihood which uses the inclusions size on the broken specimens (7). The standard deviation under Frechet assumption for data set A is not computed for the same reason as before.

The model which uses the Generalized Pareto distribution is only tested on data set A, because this is the only data set which is consistent with the assumptions of the model.

The estimates for the Generalized Pareto distribution using data set A are made when ξ is assumed to be zero and are obtained as $\hat{\sigma} = 11.3727$ and $\hat{v}_0 = 0.4654$. The lowest critical inclusion size is then a_{c0}=5.3040. The expected value of the inclusion size is 6.21 and variance 113.11.

In Fig. 5 the dash-dotted line shows the estimated

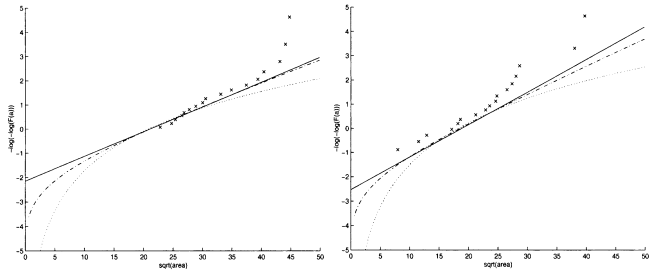

Fig. 2. Estimated probability plot from data set B and C inclusion sizes unobserved. Estimated Gumbel, —; Frechet, …; lognormal, –•–.

Table 5
The 95% quantile, the expected value arid the standard deviation for the estimated inclusion size

Distribution assumption		Data set A	Data set B	Data set C
A ~ Gumbel	95% quaritile	19.16	50.37	41.27
A ~ Gumbel	Expected value	0.08	15.48	14.66
A ~ Gumbel	Standard deviation	6.90	12.60	9.60
A ~ Frechet	95% quantile	27.09	73.23	60.39
A ~ Frechet	Expected value	10.13	32.17	27.38
A ~ Frechet	Standard deviation		40.10	28.50
log(S) ~ Normal	95% quantile	22.50	51,55	43.86
log(S) ~ Normal	Expected value	7.94	27.21	23.66
log(S) ~ Normal	Standard deviation	8.30	12.90	10.70

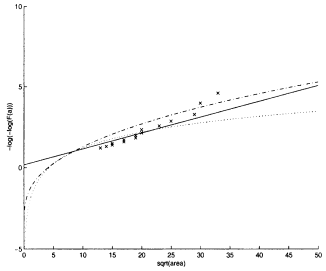

Fig. 3. Estimated probability plot from data set A inclusion sizes observed. Estimated Gumbel, —; Frechet, …; lognormal, –•–.

inclusion size distribution using (11) and the solid line shows the corresponding inclusion size distribution using (8). The crosses are again the estimated distribution function from the Kaplan-Meier method. The distribution function from (11) and that from (8) almost seem to be the same for large inclusion sizes. On the basis of the estimate from (11) there seem to be few inclusions over the critical size. Therefore the assumption for the previous model that the specimen breaks in the largest inclusions seems reasonable.

5.1. Comparison for estimation with and without direct size measurements

The inclusion size distribution has been estimated by different approaches with and without direct size measurement of the inclusion sizes. All estimated distribution functions of the data from data set A are shown in the left part of Fig. 6 except the one which uses the assumption that the specimen does not necessarily break in the largest inclusion. In Fig. 7 the estimates are from data set B and C.

When the specimens have broken the fatigue limit is below the load level, s, using (2) one gets that the inclusion sizes is above $h^{-1}(s)$. The only data that this condition holds for are those in data set A. In the data set B and C there are some specimens which broke at a too low load. Either the model is not applicable in these cases or the deviations are due to measurements errors.

Looking at Figs. 6 and 7 the model using the inclusion sizes seems to fit the estimated empirical distribution better. For most of the data sets according to Tables 5 and 6 the model which uses only the different load levels has a higher value for the 95% quantile, the variance and the expected value of the inclusion size.

6. Conclusions

The distribution of the inclusion size where the failure occurs has been estimated using different approaches. It seems as if, according to the simulations, the models which use the inclusion sizes for the broken specimens better estimate the wanted distribution. The added information obtained by observing also the actual inclusion sizes reduces the standard deviation of estimates drastically e.g. the standard deviation of the estimate of the 95% quantile is reduced by a factor 2 in the Gumbel case and a factor 4 in the Frechet case according to Table 1. When estimating the size distribution and using the model which uses the inclusion sizes, there is no information in the load level for the broken specimens when the actual size is observed.

As mentioned above, it is better to use the models which use the measured inclusions sizes. Which is the best model depends on the distribution assumption on the inclusion sizes. The model which uses the assumption that the specimen break in some inclusion over the threshold seems to be better than the model which uses Generalized Pareto distribution. The Generalized Pareto model seems to be least sensitive to a wrong distribution assumption on the inclusion sizes. This model also gives an estimate of how many inclusions there are over the critical size. When there are few inclusions over that size the difference between the models which use the inclusion size is small. If there is only one inclusion over the critical size and the specimen breaks it has to break in that inclusion.

The sizes of the inclusions are hard to measure as mentioned above. From a staircase test the size of the inclusion which caused the failure can be measured. Hence the staircase method can be used to find big inclusions.

From the measured inclusion size on the broken specimens it can also be concluded whether or not the $\sqrt{\text{area}}_{\max}$-model is valid. This can be done by comparing the size of the inclusion causing the failure to the critical inclusion size on the load level in question. The data set B and C possibly violate this condition.

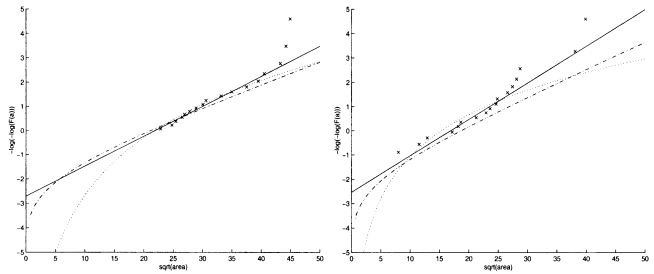

Fig. 4. Estimated probability plot from data set B and C inclusion sizes observed. Estimated Gumbel, —; Frechet, …; lognormal, –•–.

Table 6
The 95% quantile the expected vale and the standard deviation for the estimated inclusion size

Distribution assumption		Data set A	Data set B	Data set C
A ~ Gumbel	95% quantile	28.55	45.90	36.60
A ~ Gumbel	Expected value	−7.55	17.24	13.03
A ~ Gumbel	Standard deviation	13.10	10.40	8.50
A ~ Frechet	95% quantile	34.96	51.91	49.95
A ~ Frechet	Expected value	13.14	28.03	22.72
A ~ Frechet	Standard deviation	189.70	23.30	
log(S) ~ Normal	95% quantile	22.50	51.55	43.86
log(S) ~ Normal	Expected value	7.94	27.21	23.66
log(S) ~ Normal	Standard deviation	8.30	12.90	10.70

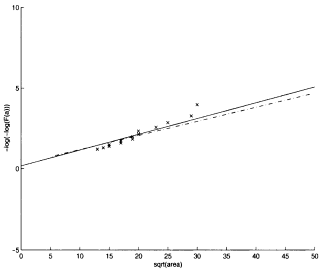

Fig. 5. Estimated probability plot for the inclusion sizes data set A. Estimated Gumbel, —; Exponential, –•–.

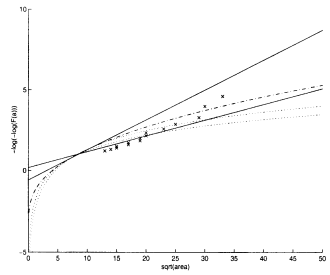

Fig. 6. Estimated probability plot for the inclusion size.. This figure contains Fig. 1 together with 3, data set A.

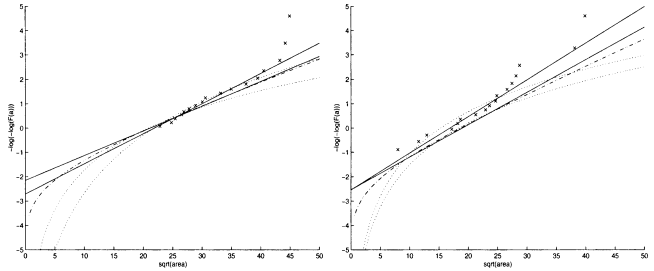

Fig. 7. Estimated probability plot for the inclusion size.. This figure contains Fig. 2 together with 4, data set B and C.

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Estimating fatigue limit distributions under inhomogeneous stress conditions

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Abstract

A model for calculating the fatigue limit distribution based on the inclusion size is presented in this paper. The model uses the $\sqrt{area_{max}}$-model as a relation between the inclusion sizes and the fatigue limit and assuming that the inclusion size above a fixed threshold follows a Generalized Pareto distribution. The stress is considered to be inhomogeneous in the component. The data that is used is from two different fatigue tests, rotating bending and uniaxial test. The fatigue limit is compared for these two tests and for two different components. Confidence intervals for the fatigue limit distribution, inclusion size and the inclusion intensity are calculated.

Keywords: Inhomogeneous stress; Fatigue limit; Inclusion size

1 Introduction

The mechanical properties of hard and clean metals are in part governed by the maximum size of the nonmetallic inclusion, e.g. the fatigue limit is determined by the size of the maximum projected area on the plane perpendicular to the load, $\sqrt{area_{max}}$-model, Murakami and Beretta [1].

A test to estimate the fatigue limit is the stair case test. The test data from a stair case test are the number of broken and unbroken components at each tested stress level.

When the stress distribution in the component is homogeneous all inclusions are exposed to the same stress, but when there is an inhomogeneous stress distribution a large inclusion may not be dangerous since the local stress may be small.

The fatigue limit is estimated under inhomogeneous stress conditions in Jablonski and Kienzler [2] and in Bomas [3]. This is done by using Weibull's weakest-link concept, provided the fatigue limit from a reference volume is known. The fatigue limit is assumed to follow an extreme value distribution. In Yates et al [4] the fatigue limit is also estimated under inhomogeneous stress conditions but the inclusion size above a fixed threshold is assumed to follow a Generalized Pareto distribution.

In this paper, the nonmetallic inclusion size is used to estimate the fatigue limit distribution under inhomogeneous stress conditions. The fatigue limit distributions are compared for different geometry and test conditions. This is done by combining the $\sqrt{area_{max}}$-model and the assumption that the inclusion size above a fixed threshold follows a Generalized Pareto distribution (Anderson et al [5]). The model is described in Section 2. Two different test procedures are considered: rotating bending and uniaxial tension-compression. In Section 6 and 7 fatigue limit distributions are estimated using data from these two different test procedures. Although only data on failures and run-outs on different stress levels are given, confidence intervals for the inclusion size and the inclusion intensity are possible to calculate for the two different examples. In Section 8 and 10 the difference or equality between the different estimated distribution functions are discussed. This is done by using a confidence interval produced by the profile likelihood method. The proposed model allows a prediction of the fatigue limit in rotating bending based on data from uniaxial tests and vice versa.

2 Model

According to Murakami and Beretta [1], the following relation can be used for hard steels, the so called $\sqrt{area_{max}}$-model, to determine the fatigue limit, $S_e = C(HV + 120)\sqrt{area_{max}}^{-1/6}\left(\frac{1-R}{2}\right)^{\alpha}$ where HV is the Vickers hardness, R is the stress ratio S_{min}/S_{max} and $area_{max}$ is the largest inclusion area projected onto a plane perpendicular to the applied load in the volume of the material that is subjected to the maximum stress. The constant C depends on the location of the inclusions, $C = 1.43$ for surface inclusions and $C = 1.56$ for internal inclusions. The exponent α is given as $\alpha = (HV \cdot 10^{-4} + 0.226)$. The maximum inclusion sizes are considered since homogeneous stress is assumed in the $\sqrt{area_{max}}$-model.

Rewrite S_e as

$$S_e = h(D_{max}) = qD_{max}^{-1/6} \tag{1}$$

where D_{max} is the square root of the maximum inclusion size. Since a component will fail if the stress level is above the fatigue limit there will be a failure if the stress level is above $h(D_{max})$. Assume that the stress level is not homogeneous in the component. Then one has to look at each point $\boldsymbol{x}$ in the component and see if there is an inclusion there and if the stress level $s(\boldsymbol{x})$ at that point is above the critical value $h(D)$ for that inclusion, where D is the square root of the inclusion size at the point $\boldsymbol{x}$. Here the fact is neglected that the inclusion affects the stress field and that the stress field varies along the inclusion.

Denote the volume of the component by V. From Yates et al [4] one

can get that the probability that a component will fail is

$$\mathrm{P}(\text{The component fails}) = 1 - exp\Big(- \int_V \lambda(\boldsymbol{x}) d\boldsymbol{x} \Big).$$

The assumption used is that inclusions with a size greater than a_o appear as the Poisson process with the intensity λ_o, and the sizes of inclusions are random and independent of each other and their locations. The intensity of inclusions that can cause failure at the location $\boldsymbol{x}$ in the component is $\lambda(\boldsymbol{x})$. The failure intensity $\lambda(\boldsymbol{x})$ satisfies

$$\lambda(\boldsymbol{x}) = \lambda_o \cdot \mathrm{P}(s(\boldsymbol{x}) > h(D) | D > a_o).$$

Assume that the inclusion size above a fixed threshold follows a Generalized Pareto distribution (Anderson et al [5]). The probability that there is a failure in the point $\boldsymbol{x}$ given that there is an inclusion bigger than a_o is then the following

$$\mathrm{P}\Big(s(\boldsymbol{x}) > h(D) | D > a_o\Big) = \mathrm{P}\Big(h^{-1}(s(\boldsymbol{x})) < D | D > a_o\Big) =$$

$$\mathrm{P}\Big(D > q^6 s(\boldsymbol{x})^{-6} | D > a_o\Big) = \Big(1 + \frac{\xi(q^6 s(\boldsymbol{x})^{-6} - a_o)}{\sigma_o}\Big)^{-1/\xi}$$

where σ_o is a scale parameter, ξ is a shape parameter and a_o is a threshold. If $\xi = 0$ the exponential distribution is obtained,

$$\mathrm{P}\Big(D > q^6 s(\boldsymbol{x})^{-6} | D > a_o\Big) = exp\Big(- \frac{(q^6 s(\boldsymbol{x})^{-6} - a_o)}{\sigma_o}\Big).$$

The probability that the component will fail is then

$$\mathrm{P}(\text{The component fails}) = 1 - exp\Big(- \int_V \lambda(\boldsymbol{x}) d\boldsymbol{x} \Big) =$$

$$1 - exp\Big(- \lambda_o \int_V exp\Big\{ - \frac{(q^6 s(\boldsymbol{x})^{-6} - a_o)}{\sigma_o} \Big\} d\boldsymbol{x} \Big) =$$

$$1 - exp\Big(- \mu \int_V exp\Big\{ - \frac{q^6 s(\boldsymbol{x})^{-6}}{\sigma_o} \Big\} d\boldsymbol{x} \Big) \qquad (2)$$

where $\mu = \lambda_o e^{\frac{a_o}{\sigma_o}}$. The intensity μ can be interpreted as the intensity of the inclusions of any size.

3 Component

Two different types of components will be considered (see Figures 1 and 2). The component in Figure 1 will be denoted as the hourglass component and the other component as the cylindrical component.

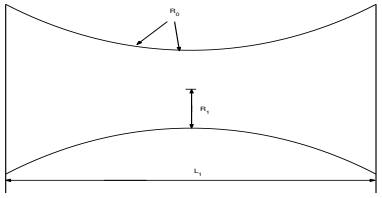

Figure 1: *Shape of fatigue specimen*

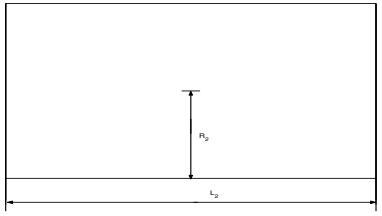

Figure 2: *Shape of fatigue specimen*

4 The stress level at point $\boldsymbol{x}$

To calculate the probability that a component will fail one needs to know the stress at each point $\boldsymbol{x}$. The stress will depend on the test procedure and the shape of the component.

4.1 The uniaxial test

For the uniaxial test considered, the stress is different in the different cross sections of the component, but the same within the same cross section. The maximum stress occurs at the smallest cross section.

Let L be the length of the component and $A(l)$ the area of the cross section at position l measured from the end of the component and hence

$0 \leq l \leq L$. The stress at point $\boldsymbol{x}$ depends on l and thus

$$s(\boldsymbol{x}) = s(l) = \frac{A_{min}}{A(l)} \cdot s, 0 < l < L$$

where A_{min} is the smallest cross section of the component and s is the maximal stress along the component. The stress concentrations are neglected since the component is assumed to have large radius.

For the component in Figure 1 the stress at point $\boldsymbol{x}$ is $s(\boldsymbol{x}) = s(l) = \frac{R_1^2}{r(l)^2} \cdot s$, $0 < l < L_1$, where $r(l)$ is the radius at position l and $r(l) = R_1 + R_0 - \sqrt{R_0^2 - (L_1/2 - l)^2}$, $0 < l < L_1$. For the component in Figure 2 the stress is the same for all points $\boldsymbol{x}$, $s(\boldsymbol{x}) = s$.

4.2 Rotating bending test

In the rotating bending test considered, the type of the test machine is a 4-point loading type. The stress on the surface is the same for the whole component. The only shape of the component that is considered for rotating bending test is the cylindrical shape.

For the component in Figure 2 the maximum stress occurs at the surface and is linearly decreasing to zero in the centre of the component. The stress at a point $\boldsymbol{x}$ depends on its distance from the centre r and satisfies

$$s(\boldsymbol{x}) = s(r) = \frac{r}{R_2} s \ , \ \text{and } 0 \leq r \leq R_2.$$

5 Estimating parameters

The test data which are used are from fatigue tests and consist of the number of broken and unbroken components on each stress level $s_i, i = 1, \cdots, k$. One method to estimate the parameters is the maximum likelihood method. This is done by splitting the observations into two disjoint sets, B corresponding to broken observations and U to the unbroken ones. The likelihood function is then

$$L = \prod_{i \in B} \text{P(Component nr } i \text{ is broken)} \prod_{j \in U} \text{P(Component nr } j \text{ is unbroken)}.$$

5.1 Likelihood function for the uniaxial test

Assume that the component which is exposed to the uniaxial test looks like the component in Figure 1.

The probability that the component number i will break depends on the tension s_i on which it is tested. The tension $s_i(\boldsymbol{x})$ is the local tension stress which gives rise to the failure intensity $\lambda_i(\boldsymbol{x})$ and

$$\text{P(The component number } i \text{ is broken)} = 1 - exp\Big(- \int_V \lambda_i(\boldsymbol{x}) d\boldsymbol{x} \Big) =$$

$$1 - exp\Big(- \lambda_o \int_V exp\Big\{ - \frac{(q^6 s_i(\boldsymbol{x})^{-6} - a_o)}{\sigma_o} \Big\} d\boldsymbol{x} \Big) =$$

$$1 - exp\Big(- \pi\mu \int_{l=0}^{L_1} r(l)^2 exp\Big\{ - \frac{q^6 r(l)^{12}}{R_1^{12} s_i^6 \sigma_o} \Big\} dl \Big) \qquad (3)$$

where $\mu = \lambda_o e^{\frac{a_o}{\sigma_o}}$ as before.

The likelihood function is

$$L(\mu, \sigma_o) = \prod_{i \in B} \Bigg[1 - exp\Big(- \pi\mu \int_{l=0}^{L_1} r(l)^2 exp\Big\{ - \frac{q^6 r(l)^{12}}{R_1^{12} s_i^6 \sigma_o} \Big\} dl \Big) \Bigg] \cdot$$

$$\prod_{j \in U} exp\Big(- \pi\mu \int_{l=0}^{L_1} r(l)^2 exp\Big\{ - \frac{q^6 r(l)^{12}}{R_1^{12} s_j^6 \sigma_o} \Big\} dl \Big). \qquad (4)$$

For the component in Figure 2 the probability that the component number i will break is

$$\text{P(The component number } i \text{ is broken)} = 1 - exp\Big(- \int_V \lambda_i(\boldsymbol{x}) d\boldsymbol{x} \Big) =$$

$$1 - exp\Big(- \pi\mu L_2 R_2^2 exp\Big\{ - \frac{q^6}{s_i^6 \sigma_o} \Big\} \Big).$$

The likelihood function is

$$L(\mu, \sigma_o) = \prod_{i \in B} \Bigg[1 - exp\Big(- \pi\mu L_2 R_2^2 exp\Big\{ - \frac{q^6}{s_i^6 \sigma_o} \Big\} \Big) \Bigg] \cdot$$

$$\prod_{j \in U} exp\Big(- \pi\mu L_2 R_2^2 exp\Big\{ - \frac{q^6}{s_j^6 \sigma_o} \Big\} \Big).$$

5.2 Likelihood function for the rotating bending test

The only component which is exposed to the rotating bending test in our study is the cylindrical one. The probability that the component will break is

$$\text{P(The component number } i \text{ is broken)} = 1 - exp\Big(- \int_V \lambda_i(\boldsymbol{x}) d\boldsymbol{x} \Big) =$$

$$1 - exp\Big(- \mu \int_V exp\Big\{ - \frac{q^6}{s_i(\boldsymbol{x})^6 \sigma_o} \Big\} d\boldsymbol{x} \Big) =$$

$$1 - exp\Big(- \mu 2\pi L_2 \int_{r=0}^{R_2} r \cdot exp\Big\{ - \frac{q^6 R_2^6}{(r s_i)^6 \sigma_o} \Big\} dr \Big) = \{\rho = r/R_2\} =$$

$$1 - exp\Big(- \mu 2\pi L_2 R_2^2 \int_{\rho=0}^{1} \rho \cdot exp\Big\{ - \frac{q^6}{(\rho s_i)^6 \sigma_o} \Big\} d\rho \Big).$$

The likelihood function is

$$L(\mu, \sigma_o) = \prod_{i \in B} \left[1 - exp\Big(- \mu 2\pi L_2 R_2^2 \int_{\rho=0}^{1} \rho \cdot exp\Big\{ - \frac{q^6}{(\rho s_i)^6 \sigma_o} \Big\} d\rho \Big) \right] \cdot$$

$$\prod_{j \in U} exp\Big(- \mu 2\pi L_2 R_2^2 \int_{\rho=0}^{1} \rho \cdot exp\Big\{ - \frac{q^6}{(\rho s_j)^6 \sigma_o} \Big\} d\rho \Big). \qquad (5)$$

6 Example of rotating bending test

The constant q in equation (1) depends on the material parameters and whether the inclusion is a surface inclusion or an interior inclusion, i.e. C=1.43 or C=1.56 respectively. Therefore the integral in equation (2) should really be written as two integrals, one for the surface and one for the interior of the component with two different q's. In the following examples this is neglected and only one q is used. Some calculations on using two different q's have been done and there is no great difference for the probability that the component fails.

This example is from a rotating bending test on a cylindric component, the test data are shown in Table 3 of the Appendix, where L_2 is 20 mm,

R_2 = 4 mm and the hardness is 261 Hv. The constant C is assumed to be 1.43 and R=-1, therefore q is 544.83. The unknown parameters μ and σ_o are estimated from equation (5). The estimate for μ and σ_o is denoted $\hat{\mu}_{rot}$ and $\hat{\sigma}_{rot}$ respectively, $\hat{\theta}_{rot} = (\hat{\mu}_{rot}, \hat{\sigma}_{rot})$. The estimated parameters are $\hat{\theta}_{rot}$=(0.1, 1.5). The units are mm^{-3} for $\hat{\mu}_{rot}$ and μm for $\hat{\sigma}_{rot}$. The probability that the component will fail equation (2) will be denoted as $F_{A,B}(s)$ where A depends on the distribution of $s(x)$ in V and B shows which test procedure that has been used for estimate θ. In Figure 3 the probability $F_{rot,\hat{\theta}_{rot}}$ that the component will fail is shown as a function of the applied maximal stress. On the x-axis is the stress level, s. The accuracy of these estimates will be discussed in the next section.

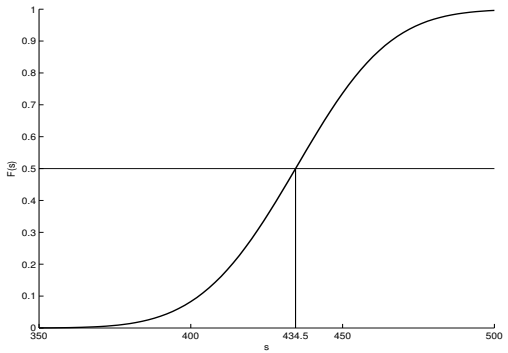

Figure 3: *Distribution function for the fatigue limit,* $F_{rot,\hat{\theta}_{rot}}(s)$.

The probability that the component will fail using a uniaxial test procedure on a cylindrical component is denoted as $F_{uniC,\theta}(s)$. In Figure 4 two different estimated probabilities that the component will fail are given, one using a rotating bending test procedure $F_{rot,\hat{\theta}_{rot}}(s)$, the solid line, and one using a uniaxial test procedure $F_{uniC,\hat{\theta}_{rot}}(s)$, the dotted line. For both these distribution functions the same estimated parameters are used, $\hat{\mu}_{rot}$ and $\hat{\sigma}_{rot}$. As expected, the estimated fatigue limit for the rotating bending test is much higher than for the uniaxial test. The solid line is the same as in Figure 3.

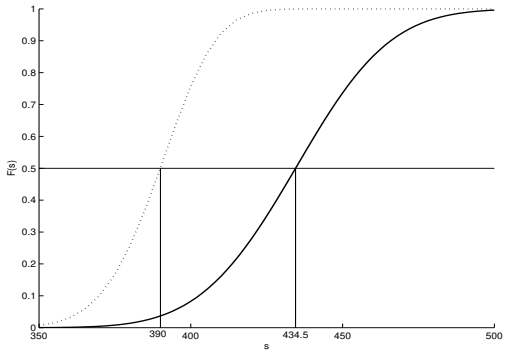

Figure 4: *Distribution function for the fatigue limit,* $F_{rot,\hat{\theta}_{rot}}(s)$*, solid line, and* $F_{uniC,\hat{\theta}_{rot}}(s)$*, dotted line.*

In Figure 5 the estimated parameters from the rotating bending test above are used but the probability that the component fails is calculated for a uniaxial test on an hourglass shape of the component $F_{uniH,\hat{\theta}_{rot}}(s)$, where L_1=35 mm, R_1=3 mm and R_0 is 45 mm. The dotted line is $F_{uniH,\hat{\theta}_{rot}}(s)$ and the solid line is $F_{rot,\hat{\theta}_{rot}}(s)$, the same as in Figure 3. In this case the difference is smaller since the highly loaded volume of the hourglass component is smaller than the volume of the cylindrical one.

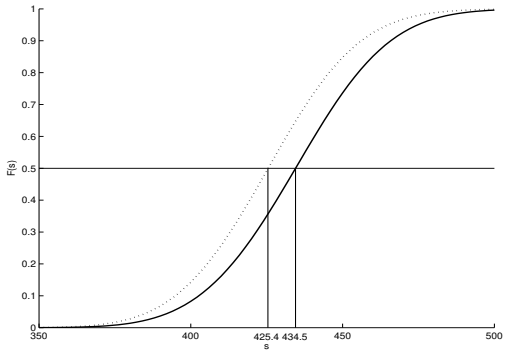

Figure 5: *Distribution function for the fatigue limit,* $F_{rot,\hat{\theta}_{rot}}(s)$*, solid line, and* $F_{uniH,\hat{\theta}_{rot}}(s)$*, dotted line.*

In Figure 3-5 three different medians of the fatigue limit are estimated. The median of $F_{rot,\hat{\theta}_{rot}}(s)$ is 434.5 MPa, $F_{uniC,\hat{\theta}_{rot}}(s)$ is 390 MPa and $F_{uniH,\hat{\theta}_{rot}}(s)$ is 425.4 MPa.

6.1 Confidence interval for the estimated parameters

The confidence interval for the parameters is here estimated by the method of profile likelihood (Venzon [6]). The analysis is based on a calculation of the maximum of the likelihood function L corresponding to different fixed values of the parameter of interest. The interval for μ the following is used $L(\mu) = L(\mu; \hat{\sigma}(\mu)) = \max_{\sigma} L(\mu; \sigma)$. In Figure 6 the logarithm of the profile likelihood is plotted versus different values on μ. The 95% confidence interval for the inclusion intensity μ is $(0.007, 6.1)$ and the 90% confidence interval is $(0.01, 2.9)$.

The corresponding intervals for σ_o are shown in Figure 7 and is $(0.6, 6.8)$ for the 95% confidence interval and $(0.7, 4.7)$ for the 90% confidence interval. The parameter σ_o is the mean of that part of the inclusion size which exceeds a_o.

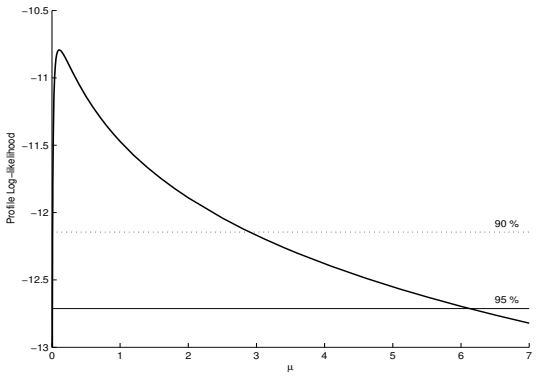

Figure 6: *Profile likelihood for* μ.

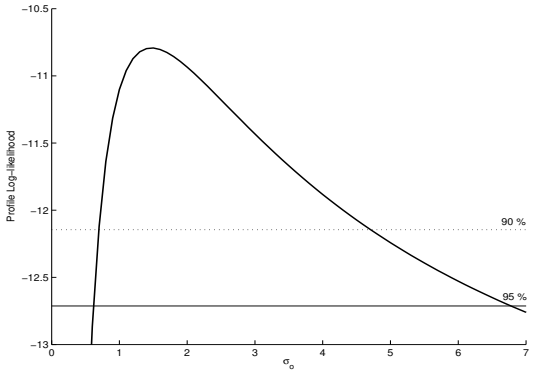

Figure 7: *Profile likelihood for* σ_o.

7 Example of uniaxial test

The test data for this example are shown in Table 4 of the Appendix. The data came from an uniaxial test and the component has an hourglass shape. The material is the same as in the previous example and the parameters for the component shape is L_1=35 mm, R_1=3 mm and R_0=45. The estimated parameters μ and σ_o from equation (4) are 0.8 and 0.9 respectively and are denoted as $\hat{\theta}_{uni} = (\hat{\mu}_{uni}, \hat{\sigma}_{uni})$. In Figure 8 the probability that the component will fail $F_{uniH,\hat{\theta}_{uni}}(s)$ is plotted versus the stress value. The estimated median for $F_{uniH,\hat{\theta}_{uni}}(s)$ is 430 MPa.

Under a rotating bending test procedure on a cylindrical component using $\hat{\theta}_{uni}$, the probability that the component will fail is denoted by $F_{rot,\hat{\theta}_{uni}}(s)$. The parameters for the cylindrical component are L_2=20 mm and R_2=4 mm. In Figure 9 $F_{rot,\hat{\theta}_{uni}}(s)$ is the dotted line and $F_{uniH,\hat{\theta}_{uni}}(s)$ is the solid line, the same as in the previous figure. The estimated median for $F_{rot,\hat{\theta}_{uni}}(s)$ is 437 MPa. Again the estimated fatigue limit for rotating bending test is higher than for uniaxial test as expected.

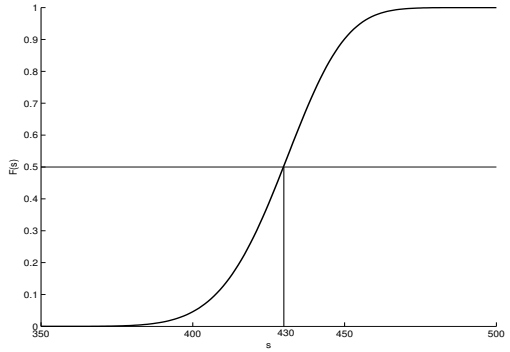

Figure 8: *Distribution function for the fatigue limit,* $F_{uniH,\hat{\theta}_{uni}}(s)$.

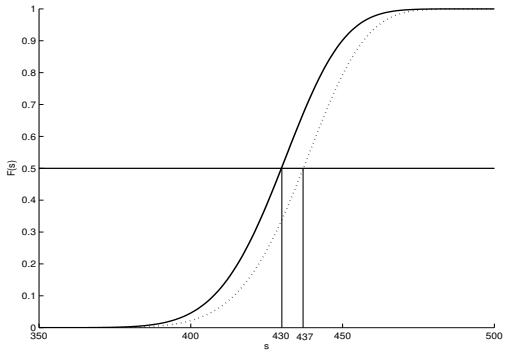

Figure 9: *Distribution function for the fatigue limit* $F_{uniH,\hat{\theta}_{uni}}(s)$ *the solid line and* $F_{rot,\hat{\theta}_{uni}}(s)$ *the dotted line.*

7.1 Confidence intervals for the parameters

The confidence intervals for μ and σ_o are calculated in the same way as in the previous example. In Figure 10 the logarithm of the profile likelihood function is plotted versus σ_o. The 90% confidence interval for σ_o is (0.4, 2.9) and a 95% interval is (0.4, 4.5). The confidence intervals for σ_o from the uniaxial test and the rotating bending test from the previous example have a non-void intersect. For the confidence interval for μ one gets quite large intervals (see Figure 11).

The reason that the μ-intervals are large may be due to the fact that the data sets are small and that only binary data are used. The number of data points for the rotating bending test is 17 (10 broken and 7 unbroken). The corresponding number for the uniaxial test is 24 (15 broken and 9 unbroken).

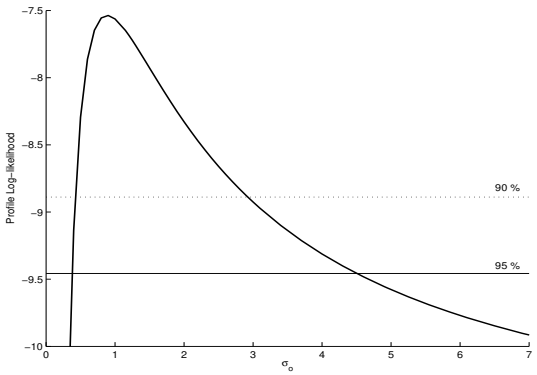

Figure 10: *Profile likelihood for* σ_o.

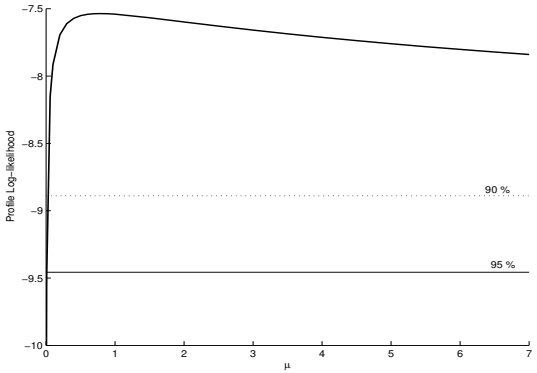

Figure 11: *Profile likelihood for* μ.

8 Difference between the estimated distribution function

8.1 Rotating bending test procedure and cylindrical component

From the examples the parameters (μ, σ_o) have been estimated twice and the estimates were obtained as (0.1, 1.5) for the rotating bending test and (0.8, 0.9) for the uniaxial test. Since the material is the same one should expect similar estimates. Therefore we will compare the different estimates with regard to their life distribution. In Figure 12 these two estimates are used to estimate $F_{rot,\theta}(s)$, the probability that the component will fail. The test procedure that is used is rotating bending for a cylindrical component. The solid line is for using $\hat{\theta}_{rot}$ and the dotted line is for $\hat{\theta}_{uni}$. The parameters for the cylindrical component are L_2=20 mm and R_2=4 mm. The distribution functions $F_{rot,\hat{\theta}_{rot}}(s)$ and $F_{rot,\hat{\theta}_{uni}}(s)$ should be the same. In the next section the confidence interval for $F_{rot,\theta}(s)$ will be discussed.

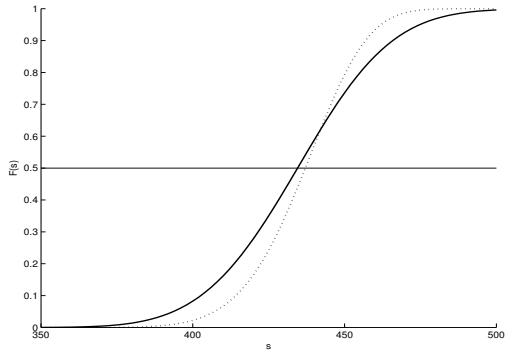

Figure 12: *Distribution function for the fatigue limit, solid line* $F_{rot,\hat{\theta}_{rot}}(s)$, *dotted line* $F_{rot,\hat{\theta}_{uni}}(s)$.

8.2 Uniaxial test procedure and cylindrical component

Figure 13 corresponds to Figure 12 but here the cylindrical component is exposed to a uniaxial test instead of the rotating bending test procedure. Here also the estimated $F_{uniC,\theta}(s)$ should be the same.

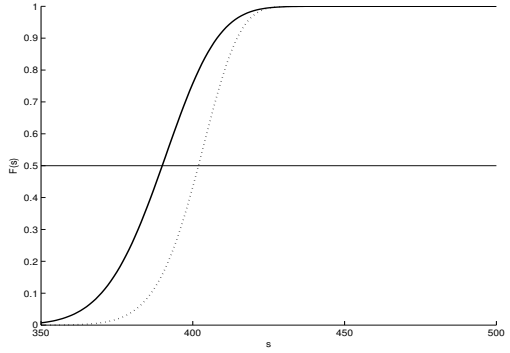

Figure 13: *Distribution function for the fatigue limit, solid line* $F_{uniC,\hat{\theta}_{rot}}(s)$, *dotted line* $F_{uniC,\hat{\theta}_{uni}}(s)$.

8.3 Uniaxial test procedure and hourglass component

The probability that an hourglass component which is exposed to a uniaxial test will fail is plotted versus the stress level in Figure 14. The parameters for the hourglass component are L_1=35 mm, R_1=3 mm and R_0=45 mm. The solid line is obtained when $\hat{\theta}_{rot}$ is used and the dotted line when $\hat{\theta}_{uni}$ is used. Again the estimated $F_{uniH,\hat{\theta}_{rot}}(s)$ and $F_{uniH,\hat{\theta}_{uni}}(s)$ should be the same. Differences: Why? model errors or possible random error.

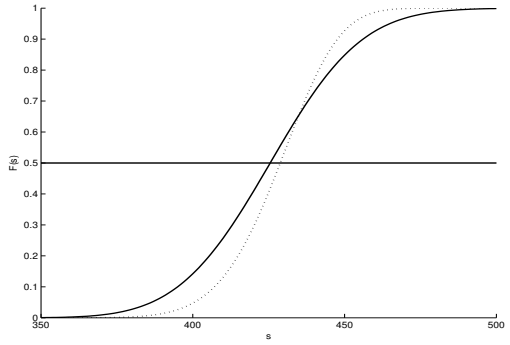

Figure 14: *Distribution function for the fatigue limit, solid line* $F_{uniH,\hat{\theta}_{rot}}(s)$*, dotted line* $F_{uniH,\hat{\theta}_{uni}}(s)$.

9 Overview over Figures 3-14

In Table 1 an overview over Figures 3-14 is shown. The table shows which test procedure, component shape and estimated parameters are used to estimate the distribution function.

Table 1: Overview over Figures 3-14

Figure number	Estimating parameter	Test procedure	Component shape
Figure 3	$\hat{\theta}_{rot}$	Rotating bending	Cylindric
Figure 4, dotted line	$\hat{\theta}_{rot}$	Uniaxial	Cylindric
Figure 5, dotted line	$\hat{\theta}_{rot}$	Uniaxial	Hourglass
Figure 8	$\hat{\theta}_{uni}$	Uniaxial	Hourglass
Figure 9, dotted line	$\hat{\theta}_{uni}$	Rotating bending	Cylindric
Figure 12	$\hat{\theta}_{uni}$ and $\hat{\theta}_{rot}$	Rotating bending	Cylindric
Figure 13	$\hat{\theta}_{uni}$ and $\hat{\theta}_{rot}$	Uniaxial	Cylindric
Figure 14	$\hat{\theta}_{uni}$ and $\hat{\theta}_{rot}$	Uniaxial	Hourglass

10 Confidence interval for the fatigue limit distribution

In Figures 12-14 we can see that the estimated $F(s)$ are not the same as they should be. One can calculate confidence interval for the estimated $F(s)$ and see if the other $F(s)$ is covered by the interval. The confidence interval for $F(s)$ is calculated by using the profile likelihood method for quantiles.

10.1 Rotating bending test procedure and cylindrical component

The α quantile of the distribution for the fatigue limit $F_{rot,\theta}(s)$ is s_α, i.e. $F_{rot,\theta}(s_\alpha) = \alpha$. From the distribution function $F_{rot,\theta}(s)$ one can get an expression for the parameter μ which is the following

$$\mu = \mu(s_\alpha, \sigma_o) = \frac{-\ln(1-\alpha)}{2\pi L_2 R_2^2 \int_{\rho=0}^{1} \rho \cdot exp\Big\{ -\frac{q^6}{(\rho s_\alpha)^6 \sigma_o} \Big\} d\rho}. \qquad (6)$$

Substituting μ with equation (6) in the equation (5) for the likelihood we obtain a likelihood function L_α for the pair s_α and σ_o as $L_\alpha(s_\alpha, \sigma_o) = L(\mu(s_\alpha, \sigma_o); \sigma_o)$. Then the profile likelihood for the quantile s_α is obtained as $L_\alpha(s_\alpha) = \max_{\sigma_o} L_\alpha(s_\alpha, \sigma_o)$. In Figure 15 the confidence interval for $F_{rot,\theta}(s)$ is shown. The solid line is the estimated $F_{rot,\hat{\theta}_{rot}}(s)$ using $\hat{\mu}_{rot} = 0.1$ and $\hat{\sigma}_{rot} = 1.5$, the dotted line is the 90% confidence interval and the dash-dot line is the 95% confidence interval. The dashed line is the estimated $F_{rot,\hat{\theta}_{uni}}(s)$ using $\hat{\mu}_{uni} = 0.8$ and $\hat{\sigma}_{uni} = 0.9$ on a rotating bending test and a cylindrical component. The estimated distribution function $F_{rot,\hat{\theta}_{uni}}(s)$ lies in the confidence interval for $F_{rot,\hat{\theta}_{rot}}(s)$. Even here the confidence interval is quite large.

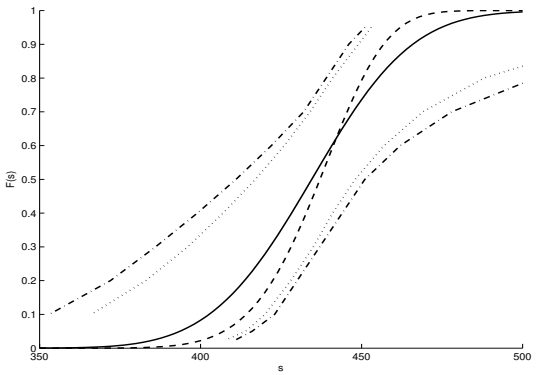

Figure 15: *The solid line is $F_{rot,\hat{\theta}_{rot}}(s)$, the dotted line and the dash-dot line are confidence intervals for $F_{rot,\hat{\theta}_{rot}}(s)$ 90 and 95 %. The dashed line is $F_{rot,\hat{\theta}_{uni}}(s)$.*

10.2 Uniaxial test procedure and hourglass component

To calculate the confidence interval for the distribution function $F_{uniH,\theta}(s)$ estimated from the uniaxial test hourglass component, μ in the equation (4) for the likelihood was substituted with the following

$$\mu = \mu(s_\alpha, \sigma_o) = \frac{-\ln(1-\alpha)}{\pi \int_{l=0}^{L_1} r(l)^2 exp\Big\{ -\frac{g^6 r(l)^{12}}{R_1^{12} s_\alpha^6 \sigma_o} \Big\} dl} \tag{7}$$

In Figure 16 the solid line is the estimated distribution function for the uniaxial test and hourglass component using $\hat{\theta}_{uni}$. The dotted line and the dash-dot line is the 90% respectively 95% confidence interval for the estimated $F_{uniH,\hat{\theta}_{uni}}(s)$. The dashed line is $F_{uniH,\hat{\theta}_{rot}}(s)$. Also in this case $F_{uniH,\hat{\theta}_{rot}}(s)$ is covered by the confidence interval for $F_{uniH,\hat{\theta}_{uni}}(s)$.

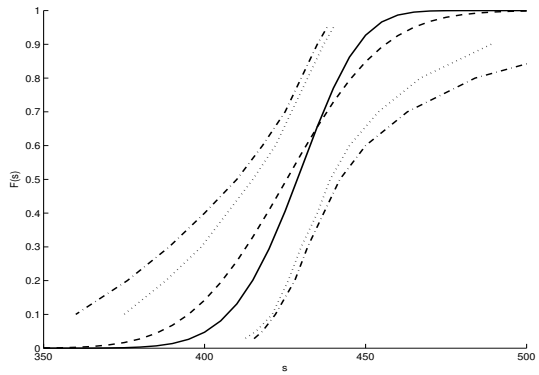

Figure 16: *The solid line is $F_{uniH,\hat{\theta}_{uni}}(s)$, the dotted line and the dash-dot line are confidence intervals for $F_{uniH,\hat{\theta}_{uni}}(s)$ 90 and 95 %. The dashed line is $F_{uniH,\hat{\theta}_{rot}}(s)$.*

11 Calculation on larger dataset

To get a larger data set heat A,B and C tempered at 600°C from [7] are treated as one data set. The same calculation as before where the mean hardness is used as the hardness for the joint data set. The result of this calculation for both the rotating bending and uniaxial test are shown in Table 2. The table shows that the confidence interval for the estimated parameters μ and σ_o from the rotating bending test and the confidence interval for the uniaxial test have a non-void intersection. The model seems also in this case to be consistent with the data. The estimated median of the fatigue limit distribution is 449 MPa for the rotating bending test. The corresponding quantity of the uniaxial test is 441 MPa. When we use the rotating bending data to predict the uniaxial fatigue limit 441 MPa is obtained which coincides with the observed value. And vice versa, when uniaxial data are used to predict the rotating bending fatigue limit, 450 MPa which is close to the observed value 449 MPa.

Table 2: Result from calculation on heat A,B and C

Parameter	Rotating	Uniaxial
σ_o	1.3	2.00
95 % intv	(0.85, 2.9)	(1.1, 4.85)
90 % intv	(0.9, 2.5)	(1.2, 4.00)
μ	0.15	0.04
95 % intv	(0.02, 2.34)	(0.01, 0.23)
90 % intv	(0.03, 1.41)	(0.01 ,0.17)
Median of $F(s)$	449	441
95 % intv	(437, 459)	(423,455)
90 % intv	(440, 457)	(427,452)

12 Conclusion

A model which estimates the probability that a component will fail by combining the $\sqrt{area}_{max}$-model and Generalized Pareto distribution assumptions on the inclusion size has been described. The model can be used to estimate the probability that the component will fail for different test procedures and component shapes.

The model seems to be consistent with the data, the fatigue limit is higher for the rotating bending test than for the uniaxial test procedure, as it should be. The estimated parameters $\hat{\theta}_{rot}$ and $\hat{\theta}_{uni}$ differ somewhat from each other, but there is a large uncertainty in these estimates. Even if the estimated parameters $\hat{\theta}_{rot}$ and $\hat{\theta}_{uni}$ differ somewhat from each other there is a possibility to transfer between different test procedures and component shape. The difference between the estimated distributions functions $F_{rot,\hat{\theta}_{rot}}$ and $F_{rot,\hat{\theta}_{uni}}$ on the one hand and $F_{uniH,\hat{\theta}_{rot}}$ and $F_{uniH,\hat{\theta}_{uni}}$ on the other is not that big. The confidence interval for the estimated parameters and distribution functions are large so these estimates are uncertain. This is due to the small data set.

Note that although the test data only contain the number of broken and unbroken components at each stress level and no inclusion sizes, it is possible to estimate the inclusion intensity and size.

13 Acknowledgements

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A Appendix

Table 3: Rotating bending fatigue test data [7]

Stress level	Number of broken components	Number of unbroken components
410	1	2
420	1	2
430	1	2
440	1	2
450	2	1
460	2	0
470	1	0
480	2	0
500	2	0
520	2	0

Table 4: Uniaxial stress fatigue test data [7]

Stress level	Number of broken components	Number of unbroken components
410	0	3
420	2	1
430	1	2
440	2	1
450	2	0
460	2	0
480	1	0

Estimating curved S-N curves

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Abstract

The fatigue limit distribution is estimated using fatigue data. The stress levels for the broken and unbroken specimens are used. For the broken specimen the number of cycles to failure is also used. By combining the finite life and fatigue limit distribution it is possible to get the probability of not surviving a certain life. This probability is used to estimate a curved S-N curve. The whole S-N curve is estimated at the same time. These curves show the predictive life given a certain stress level. The life and the quantile of the fatigue limit distribution are also predicted by using profile predictive likelihood. In this way the scatter around the S-N curve as well as the uncertainty of the S-N curve are taken into account.

1 Introduction

The S-N curve describes the relation between a given stress amplitude and the number of cycles until a specimen has broken. To obtain this curve each specimen is exposed to a stress amplitude at some given level, and the number of cycles until the specimen breaks is observed. The test is interrupted if the specimen has not broken before some specified number of cycles n_{ro} (e.g. $2 \cdot 10^6$).

Here the fatigue limit means that if the specimen is exposed at a lower stress level than the fatigue limit it will have an infinite life. The fatigue limit here is really the limit at infinity and not at some specified number of cycles n_{ro}.

Several models have been proposed for the S-N curve [3],[4]. Some of the examples are a straight line using $\log(S)$ and N or $\log(N)$, hyperbolas either on a linear or on a logarithmic scale. A curved S-N curve is obtained by combining the distribution for the finite life and the fatigue limit. The whole curve is calculated directly from the probability of not surviving a certain life.

There are two examples which use different test data. One example uses data from a staircase test. The staircase test is the following. Before the test starts the start stress level s_1 and the endurance life n_{ro} must be chosen. The first specimen is exposed to a stress level s_1. If the specimen breaks before n_{ro} cycles, the next specimen is tested at a lower stress level, i.e. $s_2 < s_1$. If the specimen survives n_{ro} cycles, then s_2 is higher than s_1. This procedure is continued until all specimens have been tested. The other example uses data where for one stress level all specimens are broken and one lower stress level where all specimens are unbroken. For the tested levels between there are both broken and unbroken specimens.

In [1] the distribution of the fatigue limit has been estimated on the basis of the stress level and the finite life given by the staircase test. In Section 2 a similar model is used to estimate the fatigue limit distribution. In Section 3 the life given a stress level is predicted by combining the fatigue limit distribution and the finite life distribution. In Sections 4 and 5 the life and the quantile of the fatigue limit distribution are predicted by using profile predictive likelihood. In Sections 6 and 7 there are two examples on real data. In Sections 8 there are calculations on simulated data.

2 Model

In [1] the distribution of the fatigue limit was estimated from fatigue test data. Here a similar model is used to estimate the fatigue limit distribution. The fatigue limit S_∞ is a random variable which is assumed to be normally distributed with the mean μ and the variance σ^2. The logarithm of the finite life N is also a random variable which is assumed to be normally distributed with an expectation $\nu(x)$ which is a function of the stress value S through its logarithm $x = \ln(S)$. The variance of the logarithm of the finite life is τ^2. There are two differences between this model and the model in [1]. The first one is that S_∞ itself is normally distributed rather than the logarithm of S_∞. The second difference is that the variance of the logarithm of the finite life N is constant.

The specimens tested are either censored or uncensored, i.e. unbroken or broken. For a broken specimen the fatigue limit S_∞ is less than the stress s at which it is exposed. The expected logarithmic life can then be expressed by $\nu(x)$. For the unbroken specimen the stress level tested

is either below the fatigue limit, or the life for this specimen is finite but longer than n_{ro}.

To estimate unknown parameters we maximize the likelihood function for our model. This is done by splitting the observations into two disjoint sets, U corresponding to the uncensored observations and C to the censored ones. The likelihood function is then

$$L(\boldsymbol{\theta}) = \prod_{i\in U} f\Big(y_i;(\tau,\nu(x_i))\Big)\Phi\Big(\frac{s_i-\mu}{\sigma}\Big)$$

$$\prod_{i\in C}\Big(1-\Phi(\frac{y_{ro}-\nu(x_i)}{\tau})\Phi(\frac{s_i-\mu}{\sigma})\Big) \tag{1}$$

where $\boldsymbol{\theta} = (\tau,\nu(\cdot),\sigma,\mu)$, $y_{ro} = \ln(n_{ro})$, f the density function for the finite life and Φ the cumulative distribution function for a standard normal variable.

3 Prediction

The likelihood in (1) is based on the fact that the probability that the specimen is censored is

$$P(\text{Censored specimen}) = 1-\Phi\Big(\frac{y_{ro}-\nu(x)}{\tau}\Big)\Phi\Big(\frac{s-\mu}{\sigma}\Big). \tag{2}$$

From the probability in (2), which is the probability that the specimen survives, the relation between life and the stress can be calculated. Given a stress level s, the probability of not surviving a certain life $n_x = e^{y_x}$ is

$$P(Y_x < y_x) = \Phi\Big(\frac{y_x-\nu(x)}{\tau}\Big)\Phi\Big(\frac{s-\mu}{\sigma}\Big) = q. \tag{3}$$

Given a stress level it is also possible to obtain a life that will be survived with a certain probability by solving equation (3) for y_x. Solving this equation for different stress values but the same probability, the curved S-N curve is obtained. We therefore get the curved S-N curve directly from the probability of not surviving a certain life.

4 Predictive profile likelihood

To predict the unobserved value of a future life the profile predictive likelihood is used. First we give a general description of the profile predictive likelihood [2]. Let Y be the data, we want to predict the unobserved value

z of a random variable Z. Let (Y,Z) have the probability density function $f_\theta(y,z;x)$. There are two unknown quantities z and θ, and we want to gain information about z with θ playing the role of a nuisance parameter. The joint likelihood function for z and θ is $l_y(z,\theta) = f_\theta(y,z;x)$. By eliminating θ from l_y we get the predictive likelihood $L(z|y)$ for z. One way of eliminating θ is by maximizing $l_y(z,\theta)$ with respect to θ. Doing this we get the profile predictive likelihood $L_p(z|y) = sup_\theta f_\theta(y,z;x)$. This likelihood can be normalised to be a probability density in z, $\bar{L}_p(z|y)$, and be used to construct prediction intervals for z.

Given a logarithm of the stress x^* the logarithm of the predicted life for x^* is denoted as Z. Since the probability for censored data is known, the likelihood only needs to be maximized for each z-value up to y_{ro}.

The density function for an observation y given x can be written in the following way

$$g(y;x,\theta) = f(y;x,\theta)\Phi(\frac{s-\mu}{\sigma})1_{\{y<y_{ro}\}}$$

$$+\Big(1-\Phi(\frac{y_{ro}-\nu(x)}{\tau})\Phi(\frac{s-\mu}{\sigma})\Big)1_{\{y=y_{ro}\}}. \tag{4}$$

The profile predictive likelihood can then be written as

$$L_p\Big(z|\mathbf{y}\Big) = sup_\theta\Big[g\Big(z;x^*,\theta\Big)\prod_{i=1}^{n} g\Big(y_i;x_i,\theta\Big)\Big] \tag{5}$$

where y_i $i = 1,\cdots,n$ are both the censured and uncensored observations.

For a fixed logarithmic stress value x^*, $L_p\Big(z|\mathbf{y}\Big)$ in equation (5) is calculated for all z-values. Then the profile predictive likelihood is normalised to be a probability density function in z in the following way

$$\bar{L}_p\Big(z|\mathbf{y}\Big) = \frac{L_p\Big(z|\mathbf{y}\Big)}{\int_{u=0}^{y_{ro}} L_p\Big(u|\mathbf{y}\Big)du + L_p\Big(y_{ro}|\mathbf{y}\Big)}. \tag{6}$$

To get a value b that z will not exceed with a certain probability q for a fixed x^*, we can use the following probability

$$P(Z\le b) = \int_{z=0}^{b}\bar{L}_p\Big(z|\mathbf{y}\Big)dz = q \text{ for } b < y_{ro}.$$

When $\theta = \theta_o$ is known, the normalised predictive likelihood is the following

$$\bar{L}_p\Big(z|\mathbf{y}\Big) = g\Big(z;x^*,\theta_o\Big).$$

For $b < y_{ro}$ the probability that z is below b is then the following

$$P(Z \leq b) = \int_{u=0}^{b} g\Big(u; x^*, \theta_o\Big)du = \int_{u=0}^{b} f(u; x^*, \theta_o)\Phi(\frac{s_z - \mu_o}{\sigma_o})du =$$

$$\Phi(\frac{b - \nu_o(x_z)}{\tau_o})\Phi(\frac{s_z - \mu_o}{\sigma_o}).$$

This probability corresponds to the probability in equation (3). But here the uncertainty in θ is not taken into account. If the normalised profile predictive likelihood in (6) is used, the uncertaninty is taken into account.

5 Prediction of the fatigue limit distribution

Above we have predicted the S-N curve, i.e. a quantile in the fatigue life distribution, but it is aslo possible to use a predictive likelihood to predict a quantile in the fatigue limit distribution. Let h be the density function for the fatigue limit distribution. Then the profile predictive likelihood for the stress can be written as

$$L_{p_s}\Big(s|\mathbf{y}\Big) = sup_\theta\Big[h\Big(s;\theta\Big)\prod_{i=1}^{n} g\Big(y_i; x_i, \theta\Big)\Big].$$

$L_{p_s}\Big(s|\mathbf{y}\Big)$ is calculated for all s-values and then normalized to a predictive likelihood $\bar{L}_{p_s}$ in the following way

$$\bar{L}_{p_s}\Big(s|\mathbf{y}\Big) = \frac{L_{p_s}\Big(s|\mathbf{y}\Big)}{\int_{s=-\infty}^{\infty} L_{p_s}\Big(s|\mathbf{y}\Big)ds}.$$

6 Example using staircase data

In this example the data come from a staircase test at Sandvik Materials Technology and are shown in Table 1.

By maximizing the likelihood function in (1) we get estimates for the parameters in the fatigue limit and finite life distribution. These estimates are the following: the expected finite logarithmic life $\hat{\nu} = 42.7 - 4.8x$, the standard deviation of the finite logarithmic life $\hat{\tau} = 0.79$, the median of the fatigue limit $\hat{\mu} = 584.0$ and the standard deviation of the fatigue limit $\hat{\sigma} = 33.2$.

Table 1: Data for example 1, $n_{ro} = 2 \cdot 10^6$ and S is in MPa.

S	N	S	N	S	N	S	N
540	n_{ro}	580	80	620	202	620	97
560	80	560	n_{ro}	600	n_{ro}	600	200
540	n_{ro}	580	468	620	80	580	85
560	380	560	n_{ro}	600	n_{ro}	560	n_{ro}
540	n_{ro}	580	165	620	99	580	850
560	n_{ro}	560	n_{ro}	600	201	560	n_{ro}
580	n_{ro}	580	n_{ro}	580	n_{ro}		
600	700	600	n_{ro}	600	n_{ro}		

Using these estimates we can, from equation (3), calculate the life that the specimen will survive with a certain probability, q given a stress level. In Figure 1 this equation is shown for two different probabilities q=0.1 and q=0.5. The '+' is the broken specimens and 'o' is the unbroken specimens from Table 1. In this prediction the uncertainty in the estimates are not taken into account. The median of the estimated fatigue limit is also shown as the horizontal line.

To take the uncertainty for the estimates into account for the prediction, the profile predictive likelihood described in Section 4 is used. In Figure 2 the life is predicted by using the normalised predictive profile likelihood $\bar{L}_p\Big(z|\mathbf{y}\Big)$ in (6) and is shown as '+'. The integral in $\bar{L}_p\Big(z|\mathbf{y}\Big)$ was calculated numerically. For small quantiles and low stress values the '+' is not close to the curved S-N curve so in this case there is rather a great uncertainty in the estimated curved S-N curve.

If we predict a quantile for the fatigue limit distribution by using $\bar{L}_{p_s}$, then the 50% quantile is 584 MPa and 10% quantile is 525 MPa. These are shown as dotted lines in Figure 2. In Figure 3 the integral of the density function $\bar{L}_{p_s}$ is shown which is used to estimate the quantile. If we predict the median life for the median fatigue limit we obtain n_{ro} in this case. This is an indication that the endurance limit for $n_{ro} = 2 \cdot 10^6$ cycles coincides with the fatigue limit. Here the endurance limit means the following if the specimen is exposed at a lower stress level then the endurance limit it will not break before n_{ro} cycles. In this example it is also true that 90% of the fatigue limits are above 525 MPa and 90 % of the lives exceed $n_{ro} = 2 \cdot 10^6$ cycles on the stress level 525 MPa as indicated by the sequence of '+'.

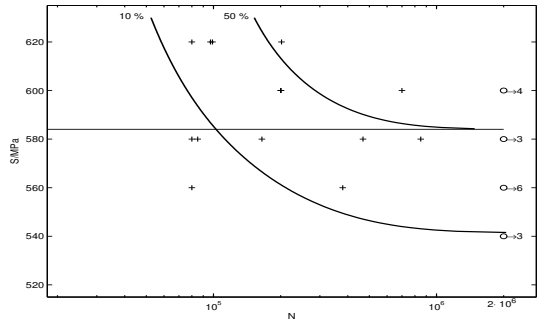

Figure 1: *The horizontal line is the median of the estimated fatigue limit distribution. The other two lines are the estimated curved S-N curve for probabilities 0.1 and 0.5. The censored specimens are denoted as 'o' and the uncensored one as '+'.*

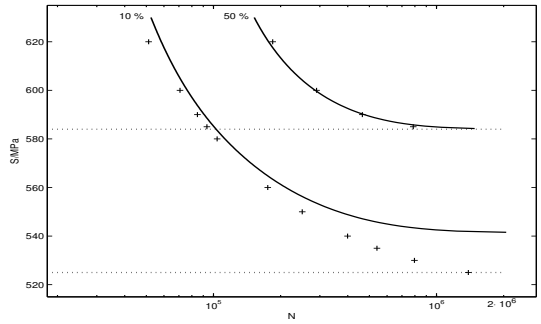

Figure 2: *The predicted life is '+', the solid line is the estimated curved S-N curve, which is the same as in Figure 1. The dotted line is the 50 % and 10 % quantile for the fatigue limit distribution using* $\bar{L}_{p_s}$.

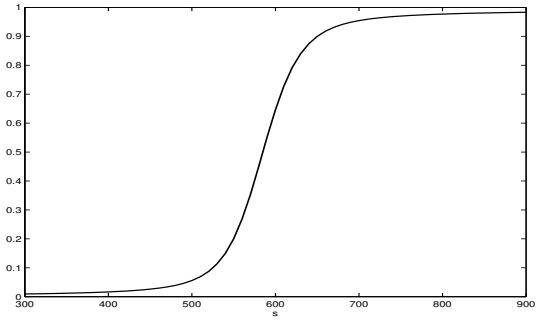

Figure 3: *The integral of* $\bar{L}_{p_s}\left(s|\mathbf{y}\right)$ *which is used to estimate the quantile of the fatigue limit distribution.*

7 Example using S-N data

In this example the data are from the other test procedure described before. These data are also from Sandvik Materials Technology. For one stress level all specimens are broken and for another stress level all are unbroken. The data for this example contain more specimens than the previous example; 59 versus 30 specimens.

The calculations are the same as in the previous example. In Figure 4 the estimated curved S-N curve is shown for probabilities $q = 0.5$ and $q = 0.1$. The median of the estimated fatigue limit is also shown as the horizontal line and is 689 MPa. In Figure 5 the predicted life is shown as the '+'. The dotted line is the predicted 50% and 10% quantile for the fatigue limit distribution by using $\bar{L}_{p_s}$ and is 689 MPa and 635 MPa. Here the predicted life is much closer to the estimated curved S-N curve than in the previous example; the uncertainty in the estimated curve is much smaller. It also looks like the run-out level is to short since the dotted lines don't coincide with the solid lines at the run-out level 10^7 cycles. Which indicates that the endurance limit is well above the fatigue limit.

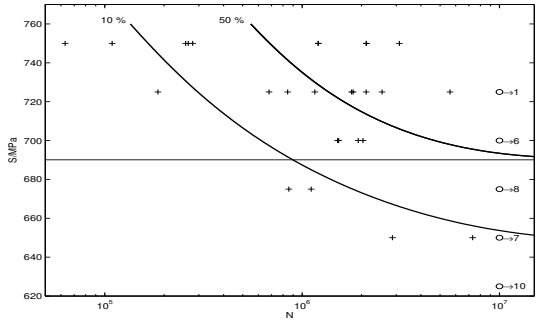

Figure 4: *The horizontal line is he median of the estimated fatigue limit distribution. The other two lines are the estimated curved S-N curve for probabilities 0.1 and 0.5. The censored specimens are denoted as 'o' and the uncensored one as '+'.*

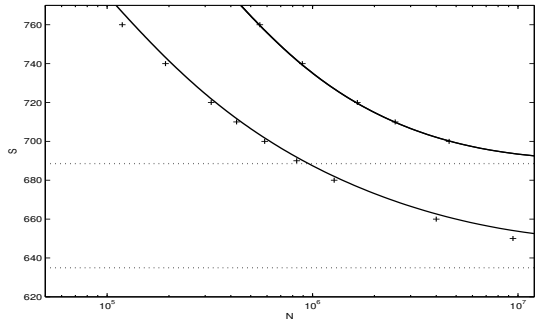

Figure 5: *The predicted life is '+', the solid line is the estimated curved S-N curve, which is the same as in Figure 4. The dotted line is the 50 % and 10 % quantile for the fatigue limit distribution using* $\bar{L}_{p_s}$.

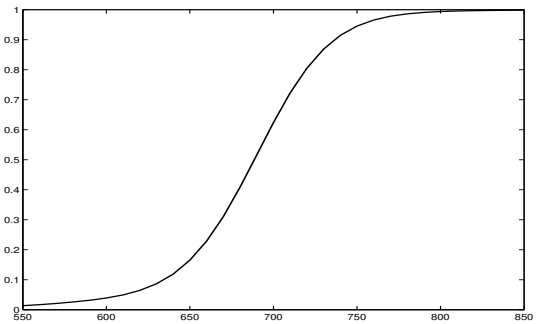

Figure 6: *The integral of* $\bar{L}_{p_s}\Big(s|\mathbf{y}\Big)$ *which is used to estimate the quantile of the fatigue limit distribution.*

8 Simulations

To see how good the predictions are, a large number of staircase tests were simulated. The procedure which was used to simulate the staircase tests was as follows:
1. Choose a stress start level s_1, step length s_{step} and a stress value n_{ro} where the test is interrupted.
2. Choose S_∞ randomly from a normal distribution with the expectation μ and the variance σ^2.
3. If $s_i < S_\infty$, then the specimen is censored and the logarithm of the lives length is put to y_{ro}. Otherwise choose the logarithm of the lives length, y_i, from a normal distribution with the expectation $\alpha + \beta x_i$ and the variance τ^2.
4. If $y_i > y_{ro}$, then set $y_i = y_{ro}$.
5. Let the next s_{i+1} be equal to $s_i + s_{step}$ if the y_i is censored, that is equal to y_{ro}. Otherwise let the next s_{i+1} be equal to $s_i - s_{step}$.
6. Restart the procedure from step 2.

Using the method above, 350 tests with 30 specimens in each test were simulated as the procedure described above. The parameters were chosen so that the simulated staircase test would correspond to the staircase test in the example in Section 6. The parameter values were chosen as s_1=540

MPa, s_{step}=20 MPa, n_{ro}=2·10^6, μ = 584, σ = 33, $\nu(x)$ =43-4.8·x and τ=0.8.

For each of these tests the predicted life is calculated by using the profile predictive likelihood and is compared to a life chosen randomly from the distribution $G(y;x,\theta)$ with the density $g(y;x,\theta)$ given in (4). The parameter in θ is the same as in the simulated stair case test. The number of times the predicted life is longer than the life chosen randomly from $G(y;x,\theta)$ is calculated and this number is denoted k. If both the predicted life and the life from $G(y;x,\theta)$ is $2 \cdot 10^6$, it is not possible to say which is longest. Let the number of times that both lives are $2 \cdot 10^6$ be b. For the case when $b > 0$, the upper limit for k is $k + b$ and the lower limit is k.

The results from the simulations are presented in Table 2 and show that the method of profile predicted likelihood seems to be useful. If $q = 0.1$ and s=550 it follows from the simulations that q is between 0.09 and 0.37. This is due to that b is large, the number of times both lives are $2 \cdot 10^6$.

Table 2: Results from 350 simulated tests.

q	s	k	$k+b$	k/n	$(k+b)/n$
0.5	600	167	168	0.48	0.48
0.1	600	38	38	0.11	0.11
0.1	550	31	123	0.09	0.35

In Figure 7- 9 the results from the first 100 simulations are shown. The '+' is the predicted life using profile predicted likelihood and 'o' is life simulated from $G(y;x,\theta)$ where $x = ln(s)$. In Figure 7 the stress level is 600 and q is 0.5, so half of the '+' should be above 'o' and half below. In this Figure, 49 % are above. In Figure 8 the stress level is 600 and $q = 0.10$ 10% of the '+' are above the 'o' as it should be. In figure 9 is then the stress level is 550 and $q = 0.10$. In this figure there are great many '+' and 'o' which is $2 \cdot 10^6$ at the same time. Therefore it is slightly problematic to calculate the number of '+' above the 'o'.

In Figure 9 we also see that there is a large spread of the predicted life. This depends on the large uncertainty for low stress values and quantiles mentioned above.

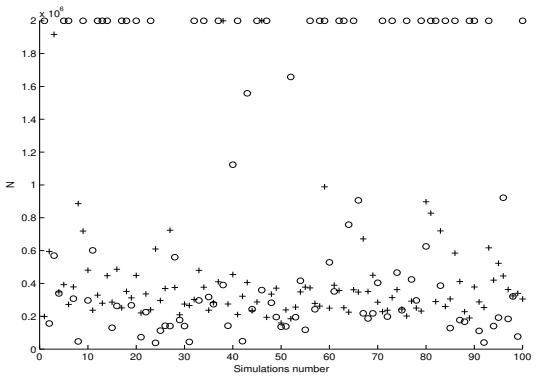

Figure 7: *Result of 100 simulations, 'o' simulated life, '+' predicted life, s=600 and* $q = 0.50$.

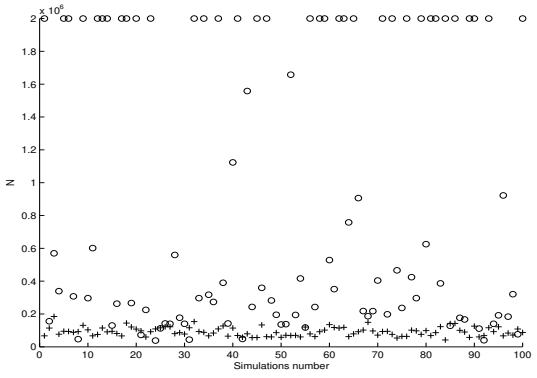

Figure 8: *Result of 100 simulations, 'o' simulated life, '+' predicted life, s=600 and* $q = 0.10$.

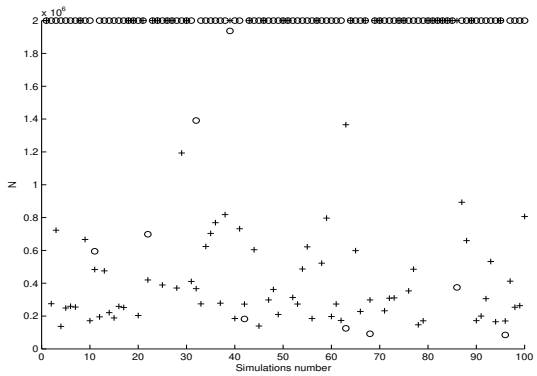

Figure 9: *Result of 100 simulations, 'o' simulated life, '+' predicted life, s=550 and* $q = 0.10$.

9 Conclusion

From S-N data the fatigue limit distribution and the finite life distribution are estimated. These two distributions are combined to estimate the whole curved S-N curve at the same time.

The life is predicted in two different ways. Either it is predicted directly from the probability of not surviving where the uncertainties in the estimated parameters are not taken into account, or it is predicted when the uncertainty is taken into account by using the method of predictive profile likelihood. There is quite a large uncertainty when the curved S-N curve is estimated from a staircase test, which one must be aware of when predicting the life.

The quantile of the fatigue limit distribution is also predicted using predictive profile likelihood, s_p. The predictive quantile for the fatigue limit distribution is the same as when the life is predicted to be n_{ro} given that the stress level is s_p, if the run-out level is chosen large enough. Although there are very few simulations they indicate that the described method for estimating curved S-N curves can be used.

10 Acknowledgments

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