Thesis for the Degree of Doctor of Philosophy

Portfolio Optimization and Statistics in Stochastic Volatility Markets

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Large financial portfolios often contain hundreds of stocks. The aim of this thesis is to find explicit optimal trading strategies that can be applied to portfolios of that size for different n-stock extensions of the model by Barndorff-Nielsen and Shephard [3]. A main ambition is that the number of parameters in our models do not grow too fast as the number of stocks n grows. This is necessary to obtain stable parameter estimates when we fit the models to data, and n is relatively large. Stability over the parameter estimates is needed to obtain accurate estimates of the optimal strategies. Statistical methods for fitting the models to data are also given.

The thesis consists of three papers. Paper I presents an n-stock extension to the model in [3] where the dependence between different stocks lies strictly in the volatility. The model is primarily intended for stocks that are dependent, but not too dependent, such as stocks from different branches of industry. We develop optimal portfolio theory for the model, and indicate how to do the statistical analysis. In Paper II we extend the model in Paper I further, to model stronger dependence. This is done by assuming that the diffusion components of the stocks contain one Brownian motion that is unique for each stock, and a few Brownian motions that all stocks share. We then develop portfolio optimization theory for this extended model. Paper III presents statistical methods to estimate the model in [3] from data. The model in Paper II is also considered. It is shown that we can divide the centered returns by a constant times the daily number of trades to get normalized returns that are *i.i.d.* and $N(0,1)$. It is a key feature of the Barndorff-Nielsen and Shephard model that the centered returns divided by the volatility are also *i.i.d.* and $N(0,1)$. This suggests that we identify the daily number of trades with the volatility, and model the number of trades within the framework of Barndorff-Nielsen and Shephard. Our approach is easier to implement than the quadratic variation method, requires much less data, and gives stable parameter estimates. A statistical analysis is done which shows that the model fits the data well.

Key words: Stochastic control, portfolio optimization, verification theorem, Feynman-Kac formula, stochastic volatility, non-Gaussian Ornstein-Uhlenbeck process, estimation, number of trades

This thesis consists of the following papers:

Paper I: News-generated dependence and optimal portfolios for n stocks in a market of Barndorff-Nielsen and Shephard type, to appear in *Mathematical Finance.*

Paper II: Portfolio optimization and a factor model in a stochastic volatility market, submitted.

Paper III: The estimation of a stochastic volatility model based on the number of trades, submitted.

Contents

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1 Introduction

Investing in the stock market can be a pain free way to get rich fast. All you have to do is to buy the right stock at the right time for a lot of money that you don't necessarily have to own. However, no one knows what "the right stock" or "the right time" is, except in retrospective. Fortunately, the humble investor can find other, more feasible, goals than "to get rich fast." For example, a trader can try to maximize her expected *utility* from investing. The concept of utility is an attempt to capture the risk aversion of a trader: The more money a trader has, the less interested she will be in an extra $100SEK$.

The idea of portfolio optimization is natural. A trader has a certain amount of money and wants to invest it in a way that maximizes her expected utility. In other words, she wants to do what she feels is best for her on average. In fact, there is nothing about this optimality condition that is specific to finance.

The optimal allocation of capital to different assets is a fundamental problem in finance. The first contribution to the area was by Markowitz [19]. He suggested that an investor should consider not only the expected rate of return of the stocks, but also the amount of fluctuation, or *volatility,* of the stock prices. This lead to optimal portfolios that diversified the capital between different assets, instead of investing all the money in the stock with the highest mean rate of return. Later, Merton solved related problems in continuous time in [20] and [21]. Merton assumed that stocks behave as multi-variate geometric Brownian motions. This implies that the volatilities are constant. The geometric Brownian motion is the classic stock price model in stochastic finance.

It is a well-known empirical fact that many characteristics of stock price data are not captured by the geometric Brownian motion, and many alternatives have been proposed. A successful approach that captures several key features of financial data was presented by Barndorff-Nielsen and Shephard in [3]. They suggested a stochastic volatility model based on linear combinations of Ornstein-Uhlenbeck processes with dynamics

$$dy = -\lambda y\left(t\right)dt + dz\left(t\right),$$

where z is a subordinator and $\lambda > 0$. A subordinator is a Lévy process with increasing paths. This framework allows us to model several of the observed features in financial time series, such as semi-heavy tails, volatility clustering, and skewness. Further, it is analytically tractable, see for example [2], [4], [7], [22], and [24]. We consider some n-stock extensions of this model.

Large financial portfolios often contain hundreds of stocks. The aim of this thesis is to find explicit optimal trading strategies that can be applied to portfolios of that size for different n-stock extensions of the model in [3]. A primary objective is that the number of parameters in our models do not grow too fast as the number of stocks n grows. This is necessary to obtain stable parameter estimates when we fit the models to data, and n is relatively large. Stability over the parameter estimates is needed to obtain accurate estimates of the optimal strategies. We also give statistical methods for fitting the models to data.

Paper I presents an n-stock extension to the Barndorff-Nielsen and Shephard model where the dependence between different stocks lies in that they partly share the Ornstein-Uhlenbeck processes of the volatility. The model is mainly intended for stocks that are dependent, but not too dependent, such as stocks that are not in the same branch of industry. We develop portfolio optimization portfolio theory, and indicate how to do the statistical analysis for the model. In Paper II we extend the model in Paper I further, so that it can model stronger dependence between different stocks. This is done by introducing a factor structure in the diffusion components. The idea of a factor structure is that the diffusion components of the stocks contain one Brownian motion that is unique for each stock, and a few Brownian motions that all stocks share. We then develop optimal portfolio theory for this extended model. Paper III presents statistical methods to estimate the model in [3] from data. We also consider the model from Paper II. It is shown that we can divide the centered returns by a constant times the daily number of trades to get normalized returns that are *i.i.d.* and $N(0,1)$. It is an important theoretical feature of the stochastic volatility framework of Barndorff-Nielsen and Shephard that the centered returns divided by the volatility are also *i.i.d.* and $N(0,1)$. This suggests that we identify the daily number of trades with the volatility, and model the number of trades within the framework of Barndorff-Nielsen and Shephard. Our approach gives more stable parameter estimates than if we analyzed only the marginal distribution of the returns directly with the standard maximum likelihood approach. Further, it is easier to implement than the quadratic variation method, and requires much less data. A statistical analysis is done which shows that the model fits the data well.

In Section 2 of this summary we recapitulate some results from classical continuous time portfolio optimization, and the ideas from stochastic control used to derive them. Section 3 discusses "stylized" facts of stock price data. Further, we indicate why the classical models lack all these characteristics. Section 4 introduces the stochastic volatility model of [3]. Finally, in Section 5 we present the three papers that constitute this thesis.

2 Portfolio optimization

The first papers on continuous time portfolio optimization are due to Merton ([20] and [21]). We present in this section a version of Merton's problem in its classical setting.

Merton modelled the stock prices as multi-variate geometric Brownian motions, which for two stocks S_1, S_2 takes the form

$$\begin{aligned} S_1(t) &= S_1(0)\exp\left((\mu_1 - \tfrac{1}{2}\sigma_{11}^2 - \tfrac{1}{2}\sigma_{12}^2)t + \sigma_{11}W_1(t) + \sigma_{12}W_2(t)\right), \qquad (2.1)\\ S_2(t) &= S_2(0)\exp\left((\mu_2 - \tfrac{1}{2}\sigma_{21}^2 - \tfrac{1}{2}\sigma_{22}^2)t + \sigma_{21}W_1(t) + \sigma_{22}W_2(t)\right). \end{aligned}$$

Here μ_i, $i = 1, 2$, are constants, W_i, $i = 1, 2$, are independent Brownian motions, and σ is a *volatility matrix*. The matrix σ gives the dependence between the two stocks.

In portfolio optimization one has to choose a *value function* to optimize. One of the most widely used optimal value functions is

$$V(t,w) = \sup_{\pi} \mathbb{E}\left[U\left(W^{\pi}(T)\right)|t,\ W(t) = w\right],$$

where T is a future date in time, $W(T)$ is our wealth at time T, $U(\cdot)$ is our *utility function, and* π is a *trading strategy.* The utility function is a measure of how much we want to risk to obtain more wealth. It is typically assumed to be concave and increasing. The concavity means that the more money an investor gets, the less interested she will be in obtaining a little more. The condition that the utility function should be increasing implies that the investor always prefers more to less. Merton suggested the utility function

$$U(w) = \frac{1}{\gamma} w^{\gamma},$$

for $0 < \gamma < 1$. The trading strategies π are recipes for how we are going to allocate our wealth between different assets. This formulation of the portfolio optimization problem means that we seek the trading strategies such that we obtain the maximum expected utility from wealth on a future day T.

We outline now the stochastic control approach to finding this optimal value function V. First, one assumes that

$$\sup_{\pi}\left\{\lim_{t\downarrow 0}\frac{\mathbb{E}\left[V\left(t,W(t)\right)\right]-V(0,w)}{t}\right\} = 0.$$

This "derivative" serves as a necessary condition for optimality. It can be evaluated using Itô's formula which gives an equation called the Hamilton-Jacobi-Bellman (HJB) equation. So far, we have only found an equation whose solution we *guess* is the optimal value function. The next step is to prove a *verification theorem.* This theorem says that a solution to the HJB-equation is in fact equal to the optimal value function. Hence, we have verified that our guess was correct. The last and final step is then to actually find the solution to the HJB equation. This is typically quite hard, since the HJB-equation is nonlinear. However, it can be done in the setting of this section, and the optimal trading strategies turn out to be

$$\pi = \left(\sigma\sigma'\right)^{-1}\left(\mu - r\underline{1}\right)\frac{1}{1-\gamma},$$

where $\underline{1}$ is a vector of ones.

Portfolio optimization with more general stock price models than the geometric Brownian motion has been treated in a number of recent articles. In [7], a one-stock portfolio problem in the model in [3] is solved. In the papers [13], [14], [23], and [26], the stochastic volatility depends on a Brownian motion which is correlated to the diffusion process of the risky asset. The paper [9] model the volatility as a continuous-time Markov chain with finite state-space, which is independent of the rest of the model. In [5], [6], and [12], different portfolio problems are treated when the stocks are driven by general Lévy processes, and [10] look at portfolio optimization in a market with Markov-modulated drift process. Further, [15] derive explicit solutions for log-optimal portfolios in complete markets in terms of the semimartingale characteristics of the price process, and [18] show that there exists a unique solution to the optimal investment problem for any arbitrage-free model if and only if the utility function has asymptotic elasticity strictly less than one.

3 "Stylized" features of stock returns

The standard approach to analyze financial data is to look at the *increments of the returns process* $R(t) := \log(S(t)/S(0))$ for the stock S. We assume that we are observing *returns* $R(\Delta), R(2\Delta) - R(\Delta), ..., R(k\Delta) - R((k-1)\Delta)$, where Δ is one day, and $k+1$ is the number of consecutive trading days in our period of observation. It is widely agreed that the returns of financial data have, among other things, the following characteristic features:

- The returns are not normally distributed. Instead, they are peaked around zero, skew, and have heavier tails than the normal distribution.
- The volatility of the returns changes stochastically over time, and appears to be clustered. That is, there seems to be a random succession of periods with high return variance and periods with low return variance.
- The autocorrelation function for absolute returns is clearly positive even for long lags.

We now give a brief indication that these empirical facts hold. The empirical density function of the returns in Figure 3.1 seems consistent with the first listed feature. It shows a clear non-normality, and appears to be both peaked around zero, skew, and more heavy-tailed than the normal distribution. In Figure 3.2, the volatility of the returns is evidently not constant. The most obvious example of a volatility cluster is the latter part of 2002. Hence these data give no reason to doubt the second "stylized" fact. Further, the autocorrelation function of the absolute returns in Figure 3.3 is positive, and so appears compatible with the last condition.

The stock price model in Equation (2.1), which is used in the classical portfolio optimization problem above, does not capture any of the features listed above: The returns in this model are *i.i.d.* and normally distributed, and the volatility is constant. A common approach to improve the geometric Brownian motion as a stock price model is to assume that the volatility is stochastic.

4 Stochastic volatility models in finance

There has been published some different models that include stochastic volatility in stock price dynamics, see for example [3], [11], [16], and [17]. This thesis

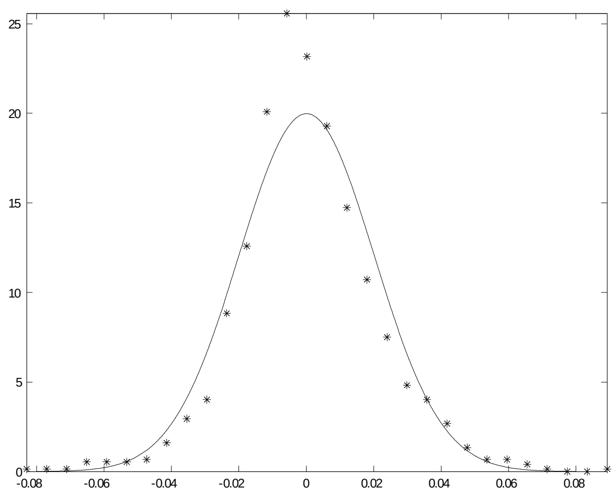

Figure 3.1: Stars indicate the empirical density function for daily returns for Volvo B during 1999-08-16 to 2004-08-16. The solid line is the estimated normal density function to the same data set.

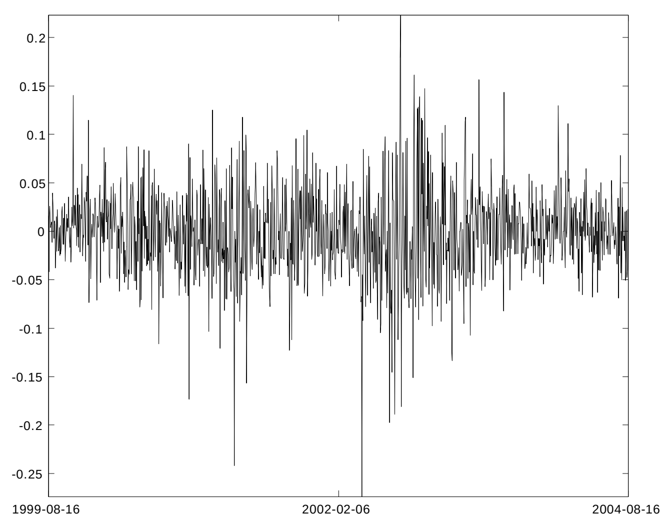

Figure 3.2: Returns for Ericsson from 1999-08-16 to 2004-08-16.

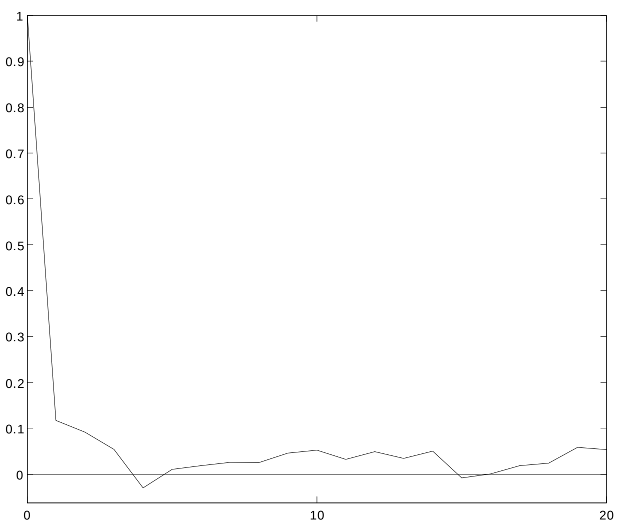

Figure 3.3: Empirical autocorrelation function for absolute returns for SKF from 1999-08-16 to 2004-08-16.

builds upon extensions of the model in [3]. In this model the volatility $\sigma^2(\cdot)$ of a stock is defined as a linear combination of non-Gaussian Ornstein-Uhlenbeck processes of the form

$$Y_j(t) = y_j e^{-\lambda_j t} + \int_0^t e^{-\lambda_j(t-u)} dZ_j(\lambda_j u), \quad t \geq 0. \tag{4.1}$$

Here $y_j := Y_j(0)$, and y_j has the stationary marginal distribution of the process and is independent of $Z_j(t) - Z_j(0)$, $t \geq 0$. The process Z_j is a subordinator, that is, a Lévy process with positive increments. The stock price process S then takes the form

$$S(t) = S(0) \exp\left(\int_0^t \left(\mu + \left(\beta - \tfrac{1}{2}\right)\sigma^2(u)\right) du + \int_0^t \sigma(u)\, dW(u) \right),$$

for some Brownian motion W. It can be shown that the returns in this model can get marginal distributions from the Generalized Hyperbolic (GH) distribution. The GH family is quite general, and includes many distributions that have been used to model financial return data, for example the normal inverse Gaussian (NIG) distribution, see [1], [3], and [25]. The use of subordinators allows for sudden increases in the volatility $\sigma^2(\cdot)$, which can be interpreted as the release of unexpected information. Further, since the Ornstein-Uhlenbeck processes Y_j decrease exponentially, the effect of large jumps in the volatility $\sigma^2(\cdot)$ "lingers." This models volatility clustering. In essence, the model captures all "stylized" facts of financial data listed above. A further advantage of the Barndorff-Nielsen and Shephard model is that it is analytically tractable; Option pricing is treated in [22], portfolio optimization for one stock and a bond in [7], and inference techniques are developed, for example, in [4] and [24].

5 Summary of papers

Most published papers on portfolio optimization with more general stock price models than the geometric Brownian motion consider only the case of one stock and a bond. However, large financial portfolios often contain hundreds of stocks. We want to develop explicit optimal trading strategies that can be applied to portfolios of this size for different n-stock extensions of the model in [3]. This requires careful modeling of the stock price and volatility dynamics. It is necessary to have many more observations than parameters to obtain stable parameter estimates. Therefore, we can not use the standard approach: An explicit stochastic volatility matrix, and n Brownian motions in the diffusion components of all n stocks. The reason is that the number of parameters in such a model would grow very fast as the number of stocks grows. We want to capture the essence of the dependence between different stocks, but still be able to estimate the model accurately from data.

5.1 Paper I

In this paper we consider Merton's portfolio optimization problem in a Barndorff-Nielsen and Shephard market. An investor is allowed to trade in n stocks and a risk-free bond, and wants to maximize her expected utility from wealth at the terminal date T. The case with only one stock was solved in [7]. The dependence between stocks is assumed to be that they partly share the Ornstein-Uhlenbeck processes of the volatility. We refer to these as *news processes*. This gives the interpretation that dependence between stocks lies solely in their reactions to the same *news*. The model is primarily intended for assets which are dependent, but not too dependent, such as stocks from different branches of industry. We show that this dependence generates covariance between the returns of different stocks, and give statistical methods for both the fitting and verification of the model to data. The model retains all the features of the univariate model in [3].

The stochastic optimization problem is solved via dynamic programming and the associated HJB integro-differential equation. By use of a verification theorem, we identify the optimal expected utility from terminal wealth as the solution of a second-order integro-differential equation. The investor is allowed to have restrictions on the fractions of wealth held in each stock, but also borrowing and short-selling constraints on the entire portfolio. For power utility, we then compute the solution to this equation via a Feynman-Kac representation, and obtain explicit optimal allocation strategies. A main advantage with the model is that the optimal strategies are functions of only $2n$ model parameters and the volatility of each stock. This is a desirable feature which allows us to obtain good estimates of the optimal strategies even when n is large. All results are derived under exponential integrability assumptions on the Lévy measures of the subordinators.

5.2 Paper II

The model in Paper I has a weak point: To obtain strong correlations between the returns of different stocks, the marginal distributions have to be very skew. This might not fit data. In the first part of Paper II, we try to deal with this weakness.

We introduce in Paper II a more general n-stock extension of the model in Paper I. It is a primary focus that the number of parameters does not grow too fast as the number of stocks grows. This is necessary to obtain accurate parameter estimates when we fit the model to data, and n is relatively large. Accurate parameter estimates is needed to obtain good estimates of the optimal strategies. Therefore, we do not use the standard approach with n Brownian motions in the diffusion components of all n stocks. Instead, we define the stochastic volatility matrix implicitly by a *factor structure*. The idea of a factor structure is that the diffusion components of the stocks contain one Brownian motion that is unique for each stock, and a few Brownian motions that all stocks share. The latter are called *factors*. Hence, the dependence between stocks lies both in the stochastic volatility, and in the Brownian motions. A factor model has fewer parameters than a standard model. The reason is that the number of factors can be chosen a lot smaller than the number of stocks. We show that this model can obtain strong correlations between the returns of the stocks without affecting their marginal distributions.

In the second part we consider an investor who wants to maximize her utility from terminal wealth by investing in n stocks and a bond. We allow for the investor to have restrictions on the fractions of wealth held in each stock, as well as borrowing and short-selling restrictions on the entire portfolio. The stochastic optimization problem is solved via dynamic programming and the associated HJB integro-differential equation. We use a verification theorem to identify the optimal expected utility from terminal wealth as the solution of a second-order integro-differential equation. We then compute the solution to this equation via a Feynman-Kac representation for power utility, and obtain explicit optimal allocation strategies. All results are derived under exponential integrability assumptions on the Lévy measures of the subordinators.

5.3 Paper III

A drawback with the Barndorff-Nielsen and Shephard model has been the difficulty to estimate the parameters of the model from data. Perhaps the most intuitive approach to do this is to analyze the quadratic variation of the stock price process, see [4]. This makes it in theory possible to recover the volatility process from observed stock prices. However, in reality the model does not hold on the microscale, and even if is only regarded as an approximation this approach still requires very much data. In addition, it is hard to implement in a statistically sound way due to peculiarities in intraday data. For example, the stock market is closed at night, and there is more intense trading on certain hours of the day. None of these features are present in the mathematical model.

In this paper we develop statistical methods for estimating the models in [3] and Paper II from data. The models are discretized under the assumption that the Wiener integrals in the Barndorff-Nielsen and Shephard model

$$\int_{t-\Delta}^{t} \sigma(s)\, dB(s) \approx \sigma(t)\, \varepsilon,$$

for $\varepsilon \in N(0,1)$. In addition, we impose some restrictions on the volatility in order to be able to estimate the model from data. We argue that it is inappropriate to estimate the GH-distribution directly from financial return data. The reason is that the GH-distribution is "almost" overparameterized. To overcome this problem, we verify that we can divide the centered returns by a constant times the number of trades in a trading day to get a sample that is *i.i.d.* and $N(0,1)$. It is an important feature of the stochastic volatility framework in [3] that the centered returns divided by the volatility are also *i.i.d.* and $N(0,1)$. This implies that we identify the daily number of trades with the volatility, and model the number of trades within the model in [3]. Our approach gives more stable parameter estimates than if we analyzed only the marginal distribution of the returns directly with the standard maximum likelihood method. Further, it is easier to implement than the quadratic variation method, and requires much less data. It gives also an economical interpretation of the discretely observed linear combination of non-Gaussian Ornstein-Uhlenbeck processes that define the stochastic volatility. In addition, our approach implies that we can view the continuous time volatility as the *intensity* with which trades arrive. A statistical analysis is performed on data from the OMX Stockholmsbörsen. The results indicate a good model fit.

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News-generated dependence and optimal portfolios for n stocks in a market of Barndorff-Nielsen and Shephard type

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Abstract

We consider Merton's portfolio optimization problem in a Black and Scholes market with non-Gaussian stochastic volatility of Ornstein-Uhlenbeck type. The investor can trade in n stocks and a risk-free bond. We assume that the dependence between stocks lies in that they partly share the Ornstein-Uhlenbeck processes of the volatility. We refer to these as *news processes*, and interpret this as that dependence between stocks lies solely in their reactions to the same *news*. The model is primarily intended for assets which are dependent, but not too dependent, such as stocks from different branches of industry. We show that this dependence generates covariance, and give statistical methods for both the fitting and verification of the model to data. Using dynamic programming, we derive and verify explicit trading strategies and Feynman-Kac representations of the value function for power utility. A primary advantage with the model is that the optimal strategies are functions of only $2n$ model parameters and the volatility of each stock. This allows us to obtain accurate estimates of the optimal strategies even when n is large.

1 Introduction

A classical problem in mathematical finance is the question of how to optimally allocate capital between different assets. In a Black and Scholes market with constant coefficients, this was solved by Merton in [16] and [17]. Recently, [6] solved a similar problem for one stock and a bond in the more general market model of [3]. In [3], Barndorff-Nielsen and Shephard propose modeling the volatility in asset price dynamics as a weighted sum of non-Gaussian Ornstein-Uhlenbeck (OU) processes of the form

$$dy\,(t) = -\lambda y\,(t)\,dt + dz\,(t)\,,$$

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where z is a subordinator and $\lambda > 0$. This framework is a powerful modeling tool that allows us to capture several of the observed features in financial time series, such as semi-heavy tails, volatility clustering, and skewness. We extend the model by introducing a new dependence structure, in which the dependence between assets lies in that they share some of the OU processes of the volatility. We will refer to the OU processes as *news processes*, which implies the interpretation that the dependence between financial assets is reactions to the same *news*. We show that this dependence generates covariance, and give statistical methods for both the fitting and verification of the model to data. The model is primarily intended for assets which are not too dependent, such as stocks from different branches of industry.

In this extended model we consider an investor who wants to maximize her utility from terminal wealth by investing in n stocks and a bond. This problem is an n-stock extension of [6]. We allow for the investor to have restrictions on the fractions of wealth held in each stock, as well as borrowing and short-selling restrictions on the entire portfolio. For simplicity of notation, we have formulated and solved the problem for two stocks and a bond. However, the general case is completely analogous. The stochastic optimization problem is solved via dynamic programming and the associated Hamilton-Jakobi-Bellman (HJB) integro-differential equation. By use of a verification theorem, we identify the optimal expected utility from terminal wealth as the solution of a second-order integro-differential equation. For power utility, we then compute the solution to this equation via a Feynman-Kac representation, and obtain explicit optimal allocation strategies. These strategies are functions of only $2n$ model parameters and the volatility of each stock. This is a desirable feature which allows us to do portfolio optimization with a large number of stocks. All results are derived under exponential integrability assumptions on the Lévy measures of the subordinators.

Recently, portfolio optimization under stochastic volatility has been treated in a number of articles. In [11], [13], and [23], the stochastic volatility depends on a stochastic factor that is correlated to the diffusion process of the risky asset. The paper [8] models the stochastic factor as a continuous-time Markov chain with finite state-space. This process is assumed to be independent of the diffusion process. Both [8] and [23] use an approach to solve their portfolio optimization problems that is similar to ours. The paper [19] uses partial observation to solve a portfolio problem with a stochastic volatility process driven by a Brownian motion correlated to the dynamics of the risky asset. Going beyond the classical geometric Brownian motion, [4], [5], and [10] treat different portfolio problems when the risky assets are driven by Lévy processes, and [9] look at portfolio optimization in a market with unobservable Markov-modulated drift process. Further, [14] derive explicit solutions for log-optimal portfolios in terms of the semimartingale characteristics of the price process. For an introduction to the market model of Barndorff-Nielsen and Shephard we refer to [2] and [3]. For option pricing in this context, see [18].

This paper has six sections. In Section 2 we give a rigorous formulation of the market and the portfolio optimization problem. We also discuss the market model and the implications of the dependence structure. In Section 3 we derive some useful results on the stochastic volatility model, and on moments of the wealth process. We prove our verification theorem in Section 4, and use it in Section 5 to verify the solution we have obtained. Section 6 states our results, without proofs, in the general setting.

2 The optimization problem

In this section we define, and discuss, the market model. We also set up our optimization problem.

2.1 The market model

For $0 \leq t \leq T < \infty$, we assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the usual conditions. Introduce m independent subordinators Z_j, and denote their Lévy measures by $l_j(dz)$, $j = 1, ..., m$. Remember that a subordinator is defined to be a Lévy process taking values in $[0, \infty)$, which implies that its sample paths are increasing. The Lévy measure l of a subordinator satisfies the condition

$$\int_{0+}^{\infty} \min(1, z) l(dz) < \infty.$$

We assume that we use the cádlág version of Z_j. Let B_i, $i = 1, 2$, be two Wiener processes independent of all the subordinators. We now introduce our stochastic volatility model. It is an extension of the model proposed by Barndorff-Nielsen and Shephard in [3] to the case of two stocks, under a special dependence structure. To begin with, our model is identical to theirs. We will discuss the differences as they occur.

The next extension of the model, to n stocks, is only a matter of notation. Denote by Y_j, $j = 1, ..., m$, the OU stochastic processes whose dynamics are governed by

$$dY_j(t) = -\lambda_j Y_j(t) dt + dZ_j(\lambda_j t), \tag{2.1}$$

where the *rate of decay* is denoted by $\lambda_j > 0$. The unusual timing of Z_j is chosen so that the marginal distribution of Y_j will be unchanged regardless of the value of λ_j. To make the OU processes and the Wiener processes simultaneously adapted, we use the filtration

$$\{\sigma(B_1(t), B_2(t), Z_1(\lambda_1 t), ..., Z_m(\lambda_m t))\}_{0 \leq t \leq T}.$$

From now on we view the processes Y_j, $j = 1, ..., m$ in our model as *news processes* associated to certain events, and the jump times of Z_j, $j = 1, ..., m$ as *news* or *the release of information* on the market. The stationary process Y_j can be represented as

$$Y_j(s) = \int_{-\infty}^{t} \exp(u) dZ_j(\lambda_j s + u), \quad s \geq t,$$

but can also be written as

$$Y_j\left(s\right) = y_j e^{-\lambda_j(s-t)} + \int_t^s e^{-\lambda_j(s-u)} dZ_j(\lambda_j u), \quad s \geq t, \tag{2.2}$$

where $y_j := Y_j\left(t\right)$, and y_j has the stationary marginal distribution of the process and is independent of $Z_j\left(s\right) - Z_j\left(t\right), \ s \geq t$. In particular, if $y_j = Y_j\left(t\right) \geq 0$, then $Y_j\left(s\right) \geq 0$, since Z_j is non-decreasing. We set $Z_j\left(0\right) = 0$, $j = 1, ..., m$, and set $y := \left(y_1, ..., y_m\right)$. We assume the usual risk-free bond dynamics

$$dR\left(t\right) = rR\left(t\right) dt,$$

with interest rate $r > 0$. Define the two stocks S_1, S_2 to have the dynamics

$$dS_i\left(t\right) = \left(\mu_i + \beta_i \sigma_i\left(t\right)\right) S_i\left(t\right) dt + \sqrt{\sigma_i(t)} S_i\left(t\right) dB_i\left(t\right). \tag{2.3}$$

Here μ_i are the *constant mean rates of return*, and β_i are *skewness* parameters. We will call $\mu_i + \beta_i \sigma_i\left(t\right)$ the *mean rate of return* for stock i at time t. For notational simplicity in our portfolio problem we denote the volatility processes by σ_i instead of the more customary σ_i^2. We define σ_i as

$$\sigma_i\left(s\right) := \sigma_i^{t,y}\left(s\right) := \textstyle\sum_{j=1}^m \omega_{i,j} Y_j\left(s\right), \quad s \in \left[t, T\right], \tag{2.4}$$

where $\omega_{i,j} \geq 0$ are weights summing to one for each i. The notation $\sigma_i^{t,y}$ denotes conditioning on $Y\left(t\right)$. Our model is here not the same as just two separate models of Barndorff-Nielsen and Shephard type. The difference is that the volatility processes depend on the same news processes. These volatility dynamics gives us the stock price processes

$$S_i\left(s\right) = S_i\left(t\right) \exp\left(\int_t^s \left(\mu_i + \left(\beta_i - \tfrac{1}{2}\right) \sigma_i\left(u\right)\right) du + \int_t^s \sqrt{\sigma_i\left(u\right)} dB_i\left(u\right)\right). \tag{2.5}$$

This stock price model does not have statistically independent increments and it is non-stationary. It also allows for the increments of the *returns* $R_i\left(t\right) := \log\left(S_i\left(t\right)/S_i\left(0\right)\right), \ i = 1, 2$, to have semi-heavy tails as well as both volatility clustering and skewness. The increments of the returns R_i are stationary since

$$R_i\left(s\right) - R_i\left(t\right) = \log\left(\frac{S_i\left(s\right)}{S_i\left(0\right)}\right) - \log\left(\frac{S_i\left(t\right)}{S_i\left(0\right)}\right) = \log\left(\frac{S_i\left(s\right)}{S_i\left(t\right)}\right) =^{\mathcal{L}} R_i\left(s - t\right),$$

where " $=^{\mathcal{L}}$ " denotes equality in law.

2.2 Discussion of the market model

This section aims to show that the dependence structure proposed in Section 2.1 is not only simple from a statistical point of view, but also has very appealing economical interpretations.

The paper [3] suggests a model with n stocks with dynamics

$$dS\left(t\right) = \left\{\mu + \beta\Sigma\left(t\right)\right\} S\left(t\right) dt + \Sigma\left(t\right)^{\frac{1}{2}} S\left(t\right) dB(t),$$

where Σ is a time-varying stochastic volatility matrix, μ and β are vectors, and B is a vector of independent Wiener processes. This model includes ours as a special case with Σ being a diagonal matrix. However, in the classical Black and Scholes market, dependence is modelled by covariance. In the case of two stocks this means that for $s \geq t$,

$$S_1(s) = S_1(t) \exp\left(\left(\mu_1 - \tfrac{1}{2}\sigma_{11} - \tfrac{1}{2}\sigma_{12}\right)(s-t) + \sqrt{\sigma_{11}} B_1(s) + \sqrt{\sigma_{12}} B_2(s)\right),$$

and

$$S_2(s) = S_2(t) \exp\left(\left(\mu_2 - \tfrac{1}{2}\sigma_{21} - \tfrac{1}{2}\sigma_{22}\right)(s-t) + \sqrt{\sigma_{21}} B_1(s) + \sqrt{\sigma_{22}} B_2(s)\right),$$

for a volatility matrix σ, and $B_1(t) = B_2(t) = 0$.

In our model, stock prices develop independently beside from reacting to the same news. The model is mainly intended for assets that are dependent, but not too dependent. For example, stocks from different branches of industry. From an economic viewpoint, one can expect the model parameters to be more stable than in the classical Black and Scholes market. For example, we do not require stability over expected rate of return. Instead we ask that every time the market is "nervous" to a certain degree, i.e. for every specific value of the volatility σ_i, the mean rate of return $\mu_i + \beta_i \sigma_i$ will be the same. We can interpret this as that we only need stability in how the market reacts to news. Note that we do not make a distinction between good and bad news.

As we will see, for the purpose of portfolio optimization we do not need to know the weights $\omega_{i,j}$. More importantly, the model generates a non-diagonal covariance matrix for the increments of the returns over the same time period, which is the most frequently used measure of dependence in finance. Since the returns have stationary increments, it is sufficient to show this result for R_i, $i = 1, 2$. Note that we have

$$\begin{aligned}
&Cov\left(R_1(s) - R_1(t), R_2(u) - R_2(v)\right) \\
&= Cov\left(R_1(s), R_2(u)\right) - Cov\left(R_1(s), R_2(v)\right) \\
&\quad - Cov\left(R_1(t), R_2(u)\right) + Cov\left(R_1(t), R_2(v)\right),
\end{aligned}$$

for $s, t, u, v \in [0, T]$. As will be shown below, for $s, t \in [0, T]$, we have that

$$\begin{aligned}
&Cov\left(R_1(s), R_2(t)\right) \\
&= \left(\beta_1 - \tfrac{1}{2}\right)\left(\beta_2 - \tfrac{1}{2}\right) \sum_{j=1}^{m} \omega_{1,j}\omega_{2,j} Var\left(Y_j(0)\right) \\
&\quad \times \frac{e^{-\lambda_j s} + e^{-\lambda_j t} - e^{-\lambda_j |s-t|} - 1 + 2\lambda_j \min(s,t)}{\lambda_j^2},
\end{aligned} \tag{2.6}$$

which for $s = t$ simplifies to

$$\begin{aligned}
&Cov\left(R_1(t), R_2(t)\right) \\
&= 2\left(\beta_1 - \tfrac{1}{2}\right)\left(\beta_2 - \tfrac{1}{2}\right) \sum_{j=1}^{m} \omega_{1,j}\omega_{2,j} Var\left(Y_j(0)\right) \frac{e^{-\lambda_j t} - 1 + \lambda_j t}{\lambda_j^2}.
\end{aligned} \tag{2.7}$$

This result says that the model generates a covariance matrix between returns, but we do not immediately know which correlations that can be obtained. It turns out that we can get correlations $Corr\left(R_1(t), R_2(t)\right)$ in the entire interval $(-1, 1)$.

To derive Equation (2.6), by definition of σ_i we have that

$$\begin{aligned}
&\mathbb{E}\left[R_1(s) R_2(t)\right] \\
&= \mathbb{E}\Bigg[\left(\int_0^s \mu_1 + \left(\beta_1 - \tfrac{1}{2}\right)\sigma_1(u)\,du + \int_0^s \sqrt{\sigma_1(u)}dB_1(u)\right) \\
&\qquad \times \left(\int_0^t \mu_2 + \left(\beta_2 - \tfrac{1}{2}\right)\sigma_2(u)\,du + \int_0^t \sqrt{\sigma_2(u)}dB_2(u)\right)\Bigg] \\
&= \mu_1\mu_2 st + \mu_1 s\left(\beta_2 - \tfrac{1}{2}\right)\sum_{j=1}^m \omega_{2,j}\mathbb{E}\left[\int_0^t Y_j(u)\,du\right] \\
&\quad + \mu_2 t\left(\beta_1 - \tfrac{1}{2}\right)\sum_{j=1}^m \omega_{1,j}\mathbb{E}\left[\int_0^s Y_j(u)\,du\right] \\
&\quad + \left(\beta_1 - \tfrac{1}{2}\right)\left(\beta_2 - \tfrac{1}{2}\right)\sum_{i,j=1}^m \omega_{1,i}\omega_{2,j}\mathbb{E}\left[\int_0^s Y_i(u)\,du\int_0^t Y_j(u)\,du\right].
\end{aligned}$$

Similarly,

$$\mathbb{E}\left[R_1(t)\right] = \mu_1 t + \left(\beta_1 - \tfrac{1}{2}\right)\sum_{j=1}^m \omega_{1,j}\mathbb{E}\left[\int_0^t Y_j(u)\,du\right].$$

This gives that

$$\begin{aligned}
&Cov\left(R_1(s), R_2(t)\right) \\
&= \left(\beta_1 - \tfrac{1}{2}\right)\left(\beta_2 - \tfrac{1}{2}\right)\sum_{j=1}^m \omega_{1,j}\omega_{2,j} Cov\left(\int_0^s Y_j(u)\,du, \int_0^t Y_j(u)\,du\right).
\end{aligned}$$

By stationarity, we have that $\mathbb{E}\left[Y_j(t)\right] = \mu_{Y_j}$, for some constant $\mu_{Y_j} > 0$, for all $t \in \mathbb{R}$. If we assume that $u \leq v$, the independence of the increments of Y_j gives that

$$\begin{aligned}
&Cov\left(Y_j(u), Y_j(v)\right) \\
&= \mathbb{E}\left[\left(Y_j(u) - \mu_{Y_j}\right)\left(Y_j(v) - \mu_{Y_j}\right)\right] \\
&= \mathbb{E}\left[e^{-\lambda_j(v-u)}Y_j(u)^2 + Y_j(u)\int_u^v e^{-\lambda_j(v-s)}dZ(\lambda_j s)\right] - \mu_{Y_j}^2 \\
&= e^{-\lambda_j(v-u)}\mathbb{E}\left[Y_j(0)^2\right] - e^{-\lambda_j(v-u)}\mu_{Y_j}^2 \\
&= e^{-\lambda_j(v-u)} Var\left(Y_j(0)\right).
\end{aligned}$$

The same calculations for $v \leq u$ shows that

$$Cov\left(Y_j(u), Y_j(v)\right) = e^{-\lambda_j|v-u|} Var\left(Y_j(0)\right),$$

and we get

$$\begin{aligned}
&Cov\left(\int_0^s Y_j(u)\,du, \int_0^t Y_j(u)\,du\right) \qquad (2.8)\\
&= \int_0^s \int_0^t Cov\left(Y_j(u), Y_j(v)\right) du dv\\
&= Var\left(Y_j(0)\right) \frac{e^{-\lambda_j s} + e^{-\lambda_j t} - e^{-\lambda_j |s-t|} - 1 + 2\lambda_j \min(s,t)}{\lambda_j^2}.
\end{aligned}$$

By Itô's isometry (see [24]) we get, similarly as above,

$$Var\left(R_i(t)\right) = \sum_{j=1}^{m} \left(2\left(\beta_i - \tfrac{1}{2}\right)^2 \omega_{i,j}^2 Var\left(Y_j(0)\right) \frac{e^{-\lambda_j t} - 1 + \lambda_j t}{\lambda_j^2} + \omega_{i,j}\mu_{Y_j} t\right),$$

for $i = 1, 2$. This gives

$$\begin{aligned}
&Corr\left(R_1(s), R_2(t)\right)\\
&= \frac{1}{2} \frac{\left(\beta_1 - \frac{1}{2}\right)\left(\beta_2 - \frac{1}{2}\right)}{\left|\beta_1 - \frac{1}{2}\right|\left|\beta_2 - \frac{1}{2}\right|} \sum_{j=1}^{m} \omega_{1,j}\omega_{2,j} Var\left(Y_j(0)\right)\\
&\quad \times \frac{e^{-\lambda_j s} + e^{-\lambda_j t} - e^{-\lambda_j |s-t|} - 1 + 2\lambda_j \min(s,t)}{\lambda_j^2}\\
&\quad \times \frac{1}{\sqrt{\sum_{j=1}^{m}\left(\omega_{1,j}^2 Var\left(Y_j(0)\right) \frac{e^{-\lambda_j s} - 1 + \lambda_j s}{\lambda_j^2} + \frac{\omega_{1,j}\mu_{Y_j} s}{2\left(\beta_1 - \frac{1}{2}\right)^2}\right)}}\\
&\quad \times \frac{1}{\sqrt{\sum_{j=1}^{m}\left(\omega_{2,j}^2 Var\left(Y_j(0)\right) \frac{e^{-\lambda_j t} - 1 + \lambda_j t}{\lambda_j^2} + \frac{\omega_{2,j}\mu_{Y_j} t}{2\left(\beta_2 - \frac{1}{2}\right)^2}\right)}},
\end{aligned}$$

and, for $s = t$,

$$\begin{aligned}
&Corr\left(R_1(t), R_2(t)\right)\\
&= \frac{\left(\beta_1 - \frac{1}{2}\right)\left(\beta_2 - \frac{1}{2}\right)}{\left|\beta_1 - \frac{1}{2}\right|\left|\beta_2 - \frac{1}{2}\right|}\\
&\quad \times \sum_{j=1}^{m} \omega_{1,j}\omega_{2,j} Var\left(Y_j(0)\right) \frac{e^{-\lambda_j t} - 1 + \lambda_j t}{\lambda_j^2}\\
&\quad \times \frac{1}{\sqrt{\sum_{j=1}^{m}\left(\omega_{1,j}^2 Var\left(Y_j(0)\right) \frac{e^{-\lambda_j t} - 1 + \lambda_j t}{\lambda_j^2} + \frac{\omega_{1,j}\mu_{Y_j} t}{2\left(\beta_1 - \frac{1}{2}\right)^2}\right)}}
\end{aligned}$$

$$\times \frac{1}{\sqrt{\sum_{j=1}^{m} \left(\omega_{2,j}^2 Var\left(Y_j\left(0\right)\right) \frac{e^{-\lambda_j t} - 1 + \lambda_j t}{\lambda_j^2} + \frac{\omega_{2,j} \mu_{Y_j} t}{2\left(\beta_2 - \frac{1}{2}\right)^2} \right)}}.$$

There is always a trade-off between accuracy and applicability when designing models. An obvious advantage of our model is that we do not have to estimate a stochastic volatility matrix, and hence we need less data to obtain good estimates of the model parameters. A drawback is that, to obtain high correlations, we need the model to be very skew. This might not fit observed data. Another drawback is that we do not distinguish between good and bad news. An alternative stock price model would be

$$dS_i\left(t\right) = \left(\mu_i + \beta_i^1 \sigma_i^1\left(t\right) - \beta_i^2 \sigma_i^2\left(t\right)\right) S_i\left(t\right) dt + \sqrt{\sigma_i(t)} S_i\left(t\right) dB_i\left(t\right),$$

where $\beta_i^1, \beta_i^2 > 0$, and σ_i^1, σ_i^2 are linear combinations of the news processes such that $\sigma_i^1 + \sigma_i^2 = \sigma_i$. We have chosen to not use this model as it would be hard to estimate from data. For example, the marginal distributions of the returns will no longer fit in the framework of Barndorff-Nielsen and Shephard. We are also required to obtain estimates of the "positive" respectively "negative" volatilities in the statistical estimation of the model.

2.3 Statistical methodology

In this section we describe a methodology for fitting the model to return data. We will do this for a Normalized Inverse Gaussian distribution (NIG), which has been shown to fit financial data well, see e.g. [1], [3], and [21]. Our choice plays no formal role in the analysis. We assume that we are observing $R_i\left(\Delta\right), R_i\left(2\Delta\right) - R_i\left(\Delta\right), ..., R_i\left(k\Delta\right) - R_i\left(\left(k-1\right)\Delta\right)$, where Δ is one day, and $k+1$ is the number of consecutive trading days in our period of observation.

The NIG-distribution has parameters $\alpha = \sqrt{\beta^2 + \gamma^2}$, β, μ, and δ. Its density function is

$$\begin{aligned} & f_{NIG}\left(x;\alpha,\beta,\mu,\delta\right) \\ &= \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} - \beta\mu\right) q\left(\frac{x-\mu}{\delta}\right)^{-1} K_1\left(\delta\alpha q\left(\frac{x-\mu}{\delta}\right)\right) e^{\beta x}, \end{aligned}$$

where $q\left(x\right) = \sqrt{1+x^2}$ and K_1 denotes the modified Bessel function of the third kind with index 1. The domain of the parameters is $\mu \in \mathbb{R}$, $\delta, \gamma > 0$, and $0 \leq |\beta| \leq \alpha$.

A standard result is that if we take σ to have an Inverse Gaussian distribution (IG), and draw a $N\left(0,1\right)$-distributed random variable ε, then $x = \mu + \beta\sigma + \sqrt{\sigma}\varepsilon$ will be NIG-distributed. The IG-distribution has density function

$$f_{IG}\left(x;\delta,\gamma\right) = \tfrac{\delta}{\sqrt{2\pi}} \exp\left(\delta\gamma\right) x^{-\frac{3}{2}} \exp\left(-\tfrac{1}{2}\left(\delta^2 x^{-1} + \gamma^2 x\right)\right), \quad x > 0,$$

where δ and γ are the same as in the NIG-distribution. The existence and integrability of Lévy measures l_j such that the volatility processes σ_i will have

IG-distributed marginals is not obvious. See [2] and [22] for this theory. The Lévy density l of the subordinator Z of an IG-distributed news process Y is

$$l(x) = (2\pi)^{-\frac{1}{2}} \frac{\delta}{2} \left(x^{-1} + \gamma^2\right) x^{-\frac{1}{2}} e^{-\frac{\gamma^2 x}{2}},$$

where (δ, γ) are the parameters of the IG-distribution, see [3].

The method described in [3], which we further extend, uses that the marginal distributions of the volatility processes σ_i are invariant to the rates of decay λ_j. These parameters λ_j are then used to fit the autocorrelation function of the σ_i, $\rho_{\sigma_i}(h) = Cov(\sigma_i(h), \sigma_i(0)) / Var(\sigma_i(0))$, $h \in \mathbb{R}$, to log-return data.

For simplicity of exposition we will assume that we only need one λ to correctly model the autocorrelation function of both stocks. However, for reasons to be explained later, we will assume that $m = 3$, and that all $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. For our model calculations show that, for general m,

$$\rho_{\sigma_i}(h) = \omega_{i,1} \exp(-\lambda_1 |h|) + \ldots + \omega_{i,m} \exp(-\lambda_m |h|),$$

where the $\omega_{i,j} \geq 0$, are the weights from the volatility processes that sum to one. Observe that since we have assumed the rates of decay λ_j to be equal, we immediately get that $\rho_{\sigma_i}(h) = \exp(-\lambda |h|)$. We proved this simpler result in Subsection 2.2. The proof of the general case is analogous.

We assume that we have fitted NIG-distributions to the empirical marginal distributions of two stocks, and that we have found a λ such that our model has the right autocorrelation function. This can be done by empirically calculating the autocorrelation functions $\rho_{\sigma_i}(h)$ for different values of h, and then find a λ so that the theoretical and empirical autocorrelation functions match. We denote the IG-parameters of the volatility processes σ_i by (δ_i, γ_i), $i = 1, 2$. By Equation (2.7) we can now fit the covariance of the model to the empirical covariance from the return data. This can be done by letting the two stocks "share" the news process Y_3, and each have one of the news processes Y_i, $i = 1, 2$, "of their own." In general, this is done for each rate of decay. We formulate this mathematically as

$$\begin{aligned} \sigma_1 &= \omega_{1,1} Y_1 + \omega_{1,3} Y_3 \sim IG(\delta_1, \gamma_1) \\ \sigma_2 &= \omega_{2,1} Y_2 + \omega_{2,3} Y_3 \sim IG(\delta_2, \gamma_2). \end{aligned}$$

We now state two properties of IG-distributed random variables that we will need below. For $X \sim IG(\delta_X, \gamma_X)$, we have that

$$aX \sim IG\left(a^{\frac{1}{2}} \delta_X, a^{-\frac{1}{2}} \gamma_X\right).$$

Furthermore, if $Y \sim IG(\delta_Y, \gamma_Y)$ and is independent of X and we assume that $\gamma_X = \gamma_Y =: \gamma$, we have that $X + Y \sim IG(\delta_X + \delta_Y, \gamma)$. Because of this formula we can let

$$\begin{aligned} \omega_{1,1} Y_1 &\sim IG(\delta_{1,1}, \gamma_1) \\ \omega_{1,3} Y_3 &\sim IG(\delta_{1,3}, \gamma_1), \end{aligned}$$

where

$$\delta_{1,1}+\delta_{1,3}=\delta_1, \tag{2.9}$$

and

$$\omega_{2,1}Y_2 \sim IG(\delta_{2,1},\gamma_2)$$
$$\omega_{2,3}Y_3 \sim IG(\delta_{2,3},\gamma_2),$$

where

$$\delta_{2,1}+\delta_{2,3}=\delta_2, \tag{2.10}$$

We see, by the scaling property of the IG-distribution, that the two expressions for the distribution of Y_3,

$$Y_3 \sim IG\left(\omega_{1,3}^{-\frac{1}{2}}\delta_{1,3},\omega_{1,3}^{\frac{1}{2}}\gamma_1\right), \tag{2.11}$$

and

$$Y_3 \sim IG\left(\omega_{2,3}^{-\frac{1}{2}}\delta_{2,3},\omega_{2,3}^{\frac{1}{2}}\gamma_2\right), \tag{2.12}$$

must be identical. With the aid of Equation (2.8), in which we use that the variance $Var(Y(0))$ of a stationary inverse Gaussian process Y is δ/γ^3, we see that Equation (2.7) becomes

$$2\left(\beta_1-\tfrac{1}{2}\right)\left(\beta_2-\tfrac{1}{2}\right)\frac{\omega_{1,3}}{\omega_{2,3}}\frac{\delta_{2,3}}{\gamma_2^3}\frac{e^{-\lambda_j\Delta}-1+\lambda_j\Delta}{\lambda_j^2}=C \tag{2.13}$$

where C is the covariance that we want the returns to have. It is now straightforward to check that for reasonably small C there are non-unique choices of $\omega_{i,j}$ such that we can obtain both the right autocorrelation function of σ_i and a specific covariance for the returns. The autocorrelation function parameter λ is already correct by assumption, and we constructed the news processes Y_j so that their marginal distribution would not depend on it. Hence we only have to take care of the covariance of the returns R_i. We do this by using Equations (2.9),..., (2.13). Note that there is nothing crucial in our choice of covariance as measure of dependence, nor does it matter how many different rates of decay we use.

We now give a simple approach to determine how well our model captures the true covariance. We begin by fitting a marginal distribution to return data, thereby obtaining the parameters μ_i and β_i, $i=1,2$. Since we have that the return processes R_i, $i=1,2$, are semimartingales, their quadratic variations, denoted by $[\cdot]$, are $\int_t^s \sigma_i(u)\,du$, $s\geq t$. That is, for a sequence of random partitions tending to the identity, we have

$$[\log(S_i/S_i(t))](s)=\int_t^s \sigma_i(u)\,du,$$

where convergence is uniformly on compacts in probability. This is a standard result in stochastic calculus. For each trading day we now empirically calculate the integrated volatility, that is, we calculate the quadratic variation of the observed returns over a trading day and, by the formula above, use that as a constant approximation of the volatility during that day. If we do this for a number of trading days, we get approximations of the volatility processes σ_i for that period of time. Using the fitted parameters μ_i, β_i and generated $N(0,1)$-distributed variables in Equation (2.5), we can now simulate "alternative" returns. We then calculate the covariance-matrix of both the return data set and the simulated alternative returns and compare them statistically.

2.4 The control problem

A main purpose of this paper is to find trading strategies that optimizes the trader's expected utility from wealth in a deterministic future point in time. The utility is measured by a *utility function* U chosen by the trader. This utility function U is a measure of the trader's aversion towards risk, in that it concretizes how much the trader is willing to risk to obtain a certain level of wealth. Our approach to finding these trading strategies, and the value function V, is dynamic programming and stochastic control. We will make use of many of the results in [6], since most of their ideas are applicable in our setting. However, we need to adapt their results to our case.

In this section we set up the control problem under the stock price dynamics of Equation (2.3). Recall that σ_1 and σ_2, are weighted sums of the news processes, see Equation (2.4). We begin by defining a *value function* V as the maximum amount of expected utility that we can obtain from a trading strategy, given a certain amount of capital. We then set up the associated Hamilton-Jakobi-Bellman equation of the value function V. This equation is a central part of our problem, as it is, in a sense, an optimality condition. Most of the later sections will be devoted to finding and verifying solutions to it.

Denote by $\pi_i(t)$ the fraction of wealth invested in stock i at time t, and set $\pi = (\pi_1, \pi_2)$. The fraction of wealth held in the risk-free asset is $(1 - \pi_1 - \pi_2)$. We allow no short-selling of stocks or bond, which implies the conditions $\pi_i \in [0,1]$, $i = 1, 2$, and $\pi_1 + \pi_2 \leq 1$, a.s., for all $t \leq s \leq T$. However, these restrictions are partly for mathematical convenience. We could equally well have chosen constants $a_i, b_i, c, d \in \mathbb{R}$, $a_i < b_i$, $c < d$, such that the constraints would have taken the form $\pi_i \in [a_i, b_i]$, $i = 1, 2$, and $c \leq \pi_1 + \pi_2 \leq d$, a.s., for all $t \leq s \leq T$. The analysis is analogous in this case, but more notationally complex. This general setting allows us to consider, for example, law enforced restrictions on the fraction of wealth held in a specific stock, as well as short-selling and borrowing of capital. We state the main results in this setting in Section 6.

The wealth process W is defined as

$$W(s) = \frac{\pi_1(s) W(s)}{S_1(s)} S_1(s) + \frac{\pi_2(s) W(s)}{S_2(s)} S_2(s) + \frac{(1 - \pi_1(s) - \pi_2(s)) W(s)}{R(s)} R(s),$$

where $\pi_i(s) W(s) / S_i(s)$ is the number of shares of stock i which is held at time s. We also assume that the portfolio is self-financing in the sense that no capital is entered or withdrawn. This can be formulated mathematically as

$$W(s) = W(t) + \sum_{i=1}^{2} \int_t^s \frac{\pi_i(u) W(u)}{S_i(u)} dS_i(u) + \int_t^s \frac{(1 - \pi_1(u) - \pi_2(u)) W(u)}{R(u)} dR(u),$$

for all $s \in [t, T]$. See [15] for a motivating discussion. The self-financing condition gives the wealth dynamics for $t \leq s \leq T$ as

$$\begin{aligned} dW(s) = {} & W(s)\pi_1(s)(\mu_1 + \beta_1\sigma_1(s) - r)\,ds \\ & + W(s)\pi_2(s)(\mu_2 + \beta_2\sigma_2(s) - r)\,ds + rW(s)ds \\ & + \pi_1(s)\sqrt{\sigma_1(s)}W(s)dB_1(s) + \pi_2(s)\sqrt{\sigma_2(s)}W(s)dB_2(s), \end{aligned} \tag{2.14}$$

with initial wealth $W(t) = w$.

The following definition of the set of admissible controls now seems natural.

Definition 2.1 *The set $\mathcal{A}_t$ of admissible controls is given by $\mathcal{A}_t := \{\pi = (\pi_1, \pi_2) : \pi_i$ is progressively measurable, $\pi_i(s) \in [0, 1]$, $i = 1, 2$, and $\pi_1 + \pi_2 \leq 1$ a.s. for all $t \leq s \leq T$, and a unique solution W^π of Equation (2.14) exists$\}$.*

An investment strategy $\pi = \{\pi(s) : t \leq s \leq T\}$ is said to be *admissible* if $\pi \in \mathcal{A}_t$. Later we will need some exponential integrability conditions on the Lévy measures. We therefore assume that the following holds:

Condition 2.1 *For constants $c_j > 0$ to be specified below,*

$$\int_{0+}^{\infty} (e^{c_j z} - 1)\, l_j(dz) < \infty, \; j = 1, ..., m.$$

Recall that the Lévy density l of the subordinator Z of an IG-distributed news process Y is

$$l(x) = (2\pi)^{-\frac{1}{2}} \frac{\delta}{2} \left(x^{-1} + \gamma^2\right) x^{-\frac{1}{2}} e^{-\frac{\gamma^2 x}{2}},$$

where (δ, γ) are the parameters of the IG-distribution. Hence Condition 2.1 is satisfied for $c_j < \gamma^2/2$.

We know from the theory of subordinators that we have

$$\mathbb{E}\left[e^{aZ_j(\lambda_j t)}\right] = \exp\left(\lambda_j \int_{0+}^{\infty} (e^{az} - 1)\, l_j(dz) t\right) \tag{2.15}$$

as long as $a \leq c_j$ with c_j from Condition 2.1 holds.

Denote $(0, \infty)$ by $\mathbb{R}_+$ and $[0, \infty)$ by $\mathbb{R}_{0+}$, and assume that $y = (y_1, ..., y_m) \in \mathbb{R}_{0+}^m$. Define the domain D by

$$D := \{(w, y) \in \mathbb{R}_+ \times \mathbb{R}_{0+}^m\}.$$

We will seek to maximize the *value function*

$$J(t, w, y; \pi) = \mathbb{E}^{t,w,y}\left[U\left(W^\pi(T)\right)\right],$$

where the notation $\mathbb{E}^{t,w,y}$ means expectation conditioned by $W(t) = w$, and $Y_j(t) = y_j$, $j = 1, ..., m$. The function U is the investor's utility function. It is assumed to be concave, non-decreasing, bounded from below, and of sublinear growth in the sense that there exists positive constants k and $\gamma \in (0,1)$ so that $U(w) \leq k(1 + w^\gamma)$ for all $w \geq 0$. Hence our stochastic control problem is to determine the *optimal value function*

$$V(t,w,y) = \sup_{\pi \in \mathcal{A}_t} J(t,w,y;\pi), \ (t,w,y) \in [0,T] \times \bar{D}, \tag{2.16}$$

and an investment strategy $\pi^* \in \mathcal{A}_t$, the optimal investment strategy, such that

$$V(t,w,y) = J(t,w,y;\pi^*).$$

The HJB equation associated to our stochastic control problem is

$$\begin{aligned} 0 = v_t + &\max_{\substack{\pi_i \in [0,1], i=1,2, \\ \pi_1+\pi_2 \leq 1}} \left\{ \left(\pi_1 \left(\mu_1 + \beta_1 \sigma_1 - r\right) + \pi_2 \left(\mu_2 + \beta_2 \sigma_2 - r\right)\right) w v_w \right. \\ &\left. + \frac{1}{2} \left(\pi_1^2 \sigma_1 + \pi_2^2 \sigma_2\right) w^2 v_{ww} \right\} + r w v_w - \sum_{j=1}^m \lambda_j y_j v_{y_j} \\ &+ \sum_{j=1}^m \lambda_j \int_0^\infty \left(v\left(t,w,y+z \cdot e_j\right) - v\left(t,w,y\right)\right) l_j(dz), \end{aligned} \tag{2.17}$$

for $(t,w,y) \in [0,T) \times D$. We observe that we have the terminal condition

$$V(T,w,y) = U(w), \quad \text{for all } (w,y) \in \bar{D}, \tag{2.18}$$

and the boundary condition

$$V(t,0,y) = U(0), \quad \text{for all } (t,y) \in [0,T] \times \mathbb{R}^m_{0+}. \tag{2.19}$$

3 Preliminary estimates

This section aims at relating the existence of exponential moments of Y to exponential integrability conditions on the Lévy measures, as well as developing moment estimates for the wealth process and showing that the value function is well-defined. The proof of Lemma 3.1 can also be found in [6].

Lemma 3.1 *Assume Condition 2.1 holds with $c_j = \xi_j/\lambda_j$ for $\xi_j > 0$. Then*

$$\mathbb{E}\left[\exp(\xi_j \int_t^s Y_j(u)du)\right] \leq \exp\left(\frac{\xi_j}{\lambda_j} y_j + \lambda_j \int_{0+}^\infty \left\{\exp\left(\frac{\xi_j z}{\lambda_j}\right) - 1\right\} l_j(dz)(s-t)\right)$$

Proof. We get from the dynamics (2.1) of Y_j that

$$\begin{aligned} \lambda_j \int_t^s Y_j(u)du &= y_j + Z_j(\lambda_j s) - Z_j(\lambda_j t) - Y_j(s) \\ &\leq y_j + Z_j(\lambda_j s) - Z_j(\lambda_j t) \\ &=^{\mathcal{L}} y_j + Z_j(\lambda_j (s-t)), \end{aligned}$$

since $Y_j(s) \geq 0$ when $y_j = Y_j(t) \geq 0$, and " $=^{\mathcal{L}}$ " denotes equality in law. Recall that we have defined $Z_j(0) = 0$. The result follows from Equation (2.15). ■

We know that $U(w) \geq U(0)$ since U is non-decreasing. This gives that $\mathbb{E}[U(W^\pi(T))] \geq U(0)$, for $\pi \in \mathcal{A}_t$, which implies that $V(t,w,y) \geq U(0)$. The sublinear growth condition of U gives that

$$V(t,w,y) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E}[U(W^\pi(T))] \leq k\left(1 + \sup_{\pi \in \mathcal{A}_t} \mathbb{E}\left[W^\pi(T)^\theta\right]\right).$$

This means that we obtain an upper bound to the optimal value function if we have control of the wealth process.

Lemma 3.2 *Assume Condition 2.1 holds with*

$$c_j = \tfrac{2\theta(|\beta_1|+\theta)\omega_{1,j}+2\theta(|\beta_2|+\theta)\omega_{2,j}}{\lambda_j}, \quad j = 1,...,m,$$

for some $\theta > 0$. *Then*

$$\sup_{\pi \in \mathcal{A}_t} \mathbb{E}^{t,w,y}\left[(W^\pi(s))^\theta\right] \leq w^\theta \exp\left(2\theta \sum_{j=1}^m \frac{(|\beta_1|+\theta)\,\omega_{1,j} + (|\beta_2|+\theta)\,\omega_{2,j}}{\lambda_j} y_j + C(\theta)(s-t)\right),$$

where

$$C(\theta) = \theta\left(|\mu_1 - r| + |\mu_2 - r| + r\right) + \frac{1}{2}\sum_{j=1}^m \lambda_j \int_{0+}^\infty \left(\exp\left(2\theta \frac{(|\beta_1|+\theta)\,\omega_{1,j} + (|\beta_2|+\theta)\,\omega_{2,j}}{\lambda_j} z\right) - 1\right) l_j(dz).$$

Proof. The proof is analogous to [6, Lemma 3.3]. Hence, we only sketch the details. We have by Equation (2.14) and Itô's formula that

$$W^\pi(s) = w\exp\left(\int_t^s \alpha\left(u, \sigma_1(u), \sigma_2(u)\right) du + \int_t^s \pi_1(u)\sqrt{\sigma_1(u)}dB_1(u) + \int_t^s \pi_2(u)\sqrt{\sigma_2(u)}dB_2(u)\right),$$

where

$$\alpha(u,\sigma_1,\sigma_2) = \pi_1(u)(\mu_1 + \beta_1\sigma_1 - r) + \pi_2(u)(\mu_2 + \beta_2\sigma_2 - r) + r - \frac{1}{2}(\pi_1(u))^2\sigma_1 - \frac{1}{2}(\pi_2(u))^2\sigma_2.$$

Define

$$X(s) = \exp\left(\int_t^s 2\theta\pi_1(u)\sqrt{\sigma_1(u)}dB_1(u) + \int_t^s 2\theta\pi_2(u)\sqrt{\sigma_2(u)}dB_2(u) - \frac{1}{2}\int_t^s (2\theta)^2(\pi_1(u))^2\sigma_1(u)\,du - \frac{1}{2}\int_t^s (2\theta)^2(\pi_2(u))^2\sigma_2(u)\,du\right).$$

We can prove that X is a martingale. This can be used together with Hölder's inequality to get the result. ■

From now on we assume that Condition 2.1 holds with

$$c_j = \frac{2\theta(|\beta_1|+\theta)\omega_{1,j}+2\theta(|\beta_2|+\theta)\omega_{2,j}}{\lambda_j}, \quad j = 1, ..., m.$$

This ensures that the value function is well-defined.

4 A verification theorem

We state and prove the following verification theorem for our stochastic control problem.

Theorem 4.1 *Assume that*

$$v(t,w,y) \in C^{1,2,1}([0,T) \times (0,\infty) \times [0,\infty)^m) \cap C([0,T] \times \bar{D})$$

is a solution of the HJB equation (2.17) with terminal condition (2.18) and boundary condition (2.19). For $j = 1, ..., m$, *assume*

$$\sup_{\pi\in\mathcal{A}_t} \int_0^T \int_{0+}^{\infty} \mathbb{E}\left[\left|v\left(s, W^{\pi}\left(s\right), Y\left(s-\right)+z\cdot e_j\right) - v\left(s, W^{\pi}\left(s\right), Y\left(s-\right)\right)\right|\right] l_j(dz)ds < \infty,$$

and

$$\sup_{\pi\in\mathcal{A}_t} \int_0^T \mathbb{E}\left[\left(\pi_i(s)\right)^2 \sigma_i(s) \left(W^{\pi}(s)\right)^2 \left(v_w\left(s, W^{\pi}\left(s\right), Y\left(s\right)\right)\right)^2\right] ds < \infty,$$

$i = 1, 2$. *Then*

$$v(t,w,y) \geq V(t,w,y), \quad \text{for all } (t,w,y) \in [0,T] \times \bar{D}.$$

If, in addition, there exist measurable functions $\pi_i^*(t,w,y) \in [0,1]$, $i = 1, 2$, *being the maximizers for the max-operator in Equation (2.17), then* $\pi^* = (\pi_1^*, \pi_2^*)$ *defines an optimal investment strategy in feedback form if Equation (2.14) admits a unique solution* W^{π^*} *and*

$$V(t,w,y) = v(t,w,y) = \mathbb{E}^{t,w,y}\left[U\left(W^{\pi^*}(T)\right)\right], \text{ for all } (t,w,y) \in [0,T] \times \bar{D}.$$

The notation $C^{1,2,1}([0,T) \times (0,\infty) \times [0,\infty)^m)$ *means twice continuously differentiable in* w *on* $(0,\infty)$ *and once continuously differentiable in* t, y *on* $[0,T) \times [0,\infty)^m$ *with continuous extensions of the derivatives to* $t = 0$ *and* $y_j = 0$, $j = 1, ..., m$.

Proof. The proof is similar to Theorem 4.1 in [6]. Therefore we omit some details. Let $(t,w,y) \in [0,T) \times D$ and $\pi \in \mathcal{A}_t$, and introduce the operator

$$\mathcal{M}^{\pi}v := \left(\pi_1\left(\mu_1 + \beta_1\sigma_1 - r\right) + \pi_2\left(\mu_2 + \beta_2\sigma_2 - r\right)\right) wv_w$$

$$+\frac{1}{2}\left(\pi_1^2\sigma_1+\pi_2^2\sigma_2\right)w^2v_{ww}+rwv_w-\sum_{j=1}^{m}\lambda_j y_j v_{y_j}$$
$$+\sum_{j=1}^{m}\lambda_j\int_{0+}^{\infty}\left(v(t,w,y+z\cdot e_j)-v(t,w,y)\right)l_j\left(dz\right).$$

Itô's formula gives that

$$\begin{aligned}
&\mathbb{E}\left[v(s,W^{\pi}(s),Y(s))\right]\\
&=v(t,w,y)+\mathbb{E}\left[\int_t^s\left(v_t+\mathcal{L}^{\pi}v\right)(u,W^{\pi}(u),Y(u))\,du\right]\\
&\le v(t,w,y)+\mathbb{E}\left[\int_t^s\left(v_t+\max_{\substack{\pi_i\in[0,1],i=1,2,\\ \pi_1+\pi_2\le 1}}\mathcal{L}^{\pi}v\right)(u,W^{\pi}(u),Y(u))\,du\right]\\
&=v(t,w,y),
\end{aligned}$$

We get now that

$$v(t,w,y)\ge\mathbb{E}\left[U\left(W^{\pi}(T)\right)\right],$$

for all $\pi\in\mathcal{A}_t$, by putting $s=T$ and invoking the terminal condition for v. The first conclusion in the theorem now follows by observing that the result holds for $t=T$ and $w=0$.

We prove the second part by verifying that π^* is an admissible control. Since π^* is a maximizer,

$$\max_{\substack{\pi_i\in[0,1],i=1,2,\\ \pi_1+\pi_2\le 1}}\mathcal{L}^{\pi}v=\mathcal{L}^{\pi^*}v,$$

which for $s=T$ gives that

$$v\left(t,w,y\right)=\mathbb{E}\left[U\left(W^{\pi^*}\left(T\right)\right)\right]\le V\left(t,w,y\right).$$

This proves the theorem. ■

5 Explicit solution

In this section we construct and verify an explicit solution to the control problem (2.16), as well as an explicit optimal control π^*, when the utility function is of the form

$$U(w)=\gamma^{-1}w^{\gamma},\quad\gamma\in(0,1).$$

5.1 Reduction of the HJB equation

In this subsection we reduce the HJB equation (2.17) to a first-order integro-differential equation by making a conjecture that the value function v has a certain form.

We conjecture that the value function has the form

$$v(t,w,y) = \gamma^{-1} w^{\gamma} h(t,y), \quad (t,w,y) \in [0,T] \times \bar{D},$$

for some function $h(t,y)$. We define the function $\Pi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ as

$$\Pi(\sigma_1,\sigma_2) = \max_{\substack{\pi_i \in [0,1], i=1,2 \\ \pi_1+\pi_2 \leq 1}} \{\pi_1 (\mu_1 + \beta_1\sigma_1 - r) + \pi_2 (\mu_2 + \beta_2\sigma_2 - r) \\ - \tfrac{1}{2} \left(\pi_1^2 \sigma_1 + \pi_2^2 \sigma_2\right)(1-\gamma)\} + r. \tag{5.1}$$

If we insert the conjectured value function into the HJB equation (2.17) we get a first-order integro-differential equation for h as

$$0 = h_t(t,y) + \gamma \Pi(\sigma_1,\sigma_2) h(t,y) - \sum_{j=1}^{m} \lambda_j y_j h_{y_j}(t,y) \\ + \sum_{j=1}^{m} \lambda_j \int_{0+}^{\infty} \left(h(t, y + z\cdot e_j) - h(t,y)\right) l_j(dz), \tag{5.2}$$

where $(t,y) \in [0,T) \times [0,\infty)^m$. The terminal condition becomes

$$h(T,y) = 1, \quad \forall y \in [0,\infty)^m,$$

since $v(T,w,y) = U(w) = \gamma^{-1} w^{\gamma}$.

For our purposes, we will need Π to be continuously differentiable. This follows from Danskin's theorem, see for example [7, Theorem 4.13 and Remark 4.14]. Calculations give that our candidates for optimal fractions of wealth are

$$\pi_i^*(\sigma_i) = \frac{1}{1-\gamma}\left(\frac{\mu_i - r}{\sigma_i} + \beta_i\right), \tag{5.3}$$

whenever $\bar{\pi}_i \in (0,1)$ and $\bar{\pi}_1 + \bar{\pi}_2 < 1$, and

$$\pi_i^* = 0, \tag{5.4}$$

when $\bar{\pi}_i \leq 0$. When $\bar{\pi}_1 + \bar{\pi}_2 \geq 1$, the optimal fractions of wealth are

$$\pi_1^*(\sigma_1,\sigma_2) = \frac{1}{(1-\gamma)}\left(\frac{(\mu_1 + \beta_1\sigma_1 - r) - (\mu_2 + \beta_2\sigma_2 - r)}{(\sigma_1+\sigma_2)}\right) + \frac{\sigma_2}{(\sigma_1+\sigma_2)}, \tag{5.5}$$

and

$$\pi_2^* = 1 - \pi_1^*. \tag{5.6}$$

Note that this strategy only depends on the parameters μ_i, β_i, and the volatility for each stock.

Remark 5.1 *Note that we can find a constant $\alpha > 0$ such that*

$$|\Pi(\sigma_1,\sigma_2)| \leq \alpha + |\beta_1|\sigma_1 + |\beta_2|\sigma_2.$$

5.2 Verification of explicit solution

In this subsection we define a Feynman-Kac formula that we verify as a classical solution to the related forward problem of Equation (5.2). We indicate then how we can show that our conjectured solution v coincides with the optimal value function V.

Define the function $h(t,y)$ by

$$h(t,y) = \mathbb{E}^{t,y}\left[\exp\left(\int_t^T \gamma\Pi(\sigma_1(s), \sigma_2(s))\, ds\right)\right], \quad (t,y) \in [0,T] \times \mathbb{R}^m_{0+}.$$

We prefer to re-write the function h on a form that is simpler for us to handle. By the stationarity of Y, we have that

$$\begin{aligned} h(t,y) &= \mathbb{E}^{t,y}\left[\exp\left(\int_t^T \gamma\Pi(\sigma_1(s), \sigma_2(s))\, ds\right)\right] \qquad (5.7) \\ &= \mathbb{E}^{0,y}\left[\exp\left(\int_0^{T-t} \gamma\Pi(\sigma_1(s), \sigma_2(s))\, ds\right)\right], \end{aligned}$$

for $(t,y) \in [0,T] \times \mathbb{R}^m_{0+}$.We define now

$$g(t,y) := h(T-t,y) = \mathbb{E}^y\left[\exp\left(\int_0^t \gamma\Pi(\sigma_1(s), \sigma_2(s))\, ds\right)\right].$$

Note that $g(0,y) = 1$. The only difference between the two functions is the direction of the time variable t. We show now that g is well-defined under an exponential growth hypothesis in σ_1 and σ_2.

Lemma 5.1 *Assume Condition 2.1 holds with* $c_j = \frac{\gamma}{\lambda_j}(|\beta_1|\,\omega_{1,j} + |\beta_2|\,\omega_{2,j})$ *for* $j = 1, ..., m$. *Then*

$$g(t,y) \leq \exp\left(kt + \gamma\sum_{j=1}^m \frac{(|\beta_1|\,\omega_{1,j} + |\beta_2|\,\omega_{2,j})}{\lambda_j} y_j\right),$$

for some positive constant k.

Proof. From Remark 5.1 we know that

$$|\Pi(\sigma_1, \sigma_2)| \leq \alpha + |\beta_1|\,\sigma_1 + |\beta_2|\,\sigma_2$$

for some constant $\alpha > 0$. Therefore,

$$\begin{aligned} g(t,y) &= \mathbb{E}^y\left[\exp\left(\int_0^t \gamma\Pi(\sigma_1(s), \sigma_2(s))\, ds\right)\right] \\ &\leq \mathbb{E}^y\left[\exp\left(\int_0^t \gamma\alpha + \gamma|\beta_1|\,\sigma_1(s) + \gamma|\beta_2|\,\sigma_2(s)\, ds\right)\right] \\ &\leq e^{\gamma\alpha t}\mathbb{E}^y\left[\prod_{j=1}^m e^{\gamma(|\beta_1|\omega_{1,j} + |\beta_2|\omega_{2,j})\int_0^t Y_j^{y_j}(s)ds}\right]. \end{aligned}$$

By independence of the Y_j, $j=1,...,m$, we get the result by Lemma 3.1. ■

We will need that g is continuously differentiable in y for h to satisfy Equation (5.2).

Lemma 5.2 *Assume Condition 2.1 holds with* $c_j = \frac{\gamma}{\lambda_j}\left(|\beta_1|\,\omega_{1,j} + |\beta_2|\,\omega_{2,j}\right)$, $j=1,...,m$. *Then* $g \in C^{0,1}\left(([0,T]) \times \mathbb{R}^m_{0+}\right)$; *that is,* $g(\cdot, y)$ *is continuous for all* $y \in \mathbb{R}^m_{0+}$ *and* $g(t,\cdot)$ *is once continuously differentiable for all* $t \in [0,T]$.

Proof. The proof is analogous to Lemma 5.3 in [6]. ■

To prove that g solves the suitably modified Equation (5.2), we need the following result concerning the expectation of the jumps of g. In our view, the proof of Lemma 5.4 in [6] is incorrect. We therefore give a different proof.

Lemma 5.3 *Assume Condition 2.1 holds with* $c_j = \frac{\gamma}{\lambda_j}\left(|\beta_1|\,\omega_{1,j} + |\beta_2|\,\omega_{2,j}\right) + \frac{\gamma(1-\gamma)}{2}\left(\omega_{1,j} + \omega_{2,j}\right)$ *for* $j=1,...,m$. *Then*

$$\sum_{j=1}^{m} \mathbb{E}\left[\int_0^T \int_{0+}^{\infty} \left|g\left(t, Y(u) + z \cdot e_j\right) - g\left(t, Y(u)\right)\right| l_j(dz)\, du\right] < \infty.$$

Proof. Since $\Pi'_{y_j} \le c_j \lambda_j / \gamma$, we have that

$$\begin{aligned}
&\left|g\left(t, y + z \cdot e_j\right) - g\left(t, y\right)\right| \\
&\le \left|\mathbb{E}^y\left[\exp\left(\int_0^t \gamma \Pi\left(\sigma_1^{y+z\cdot e_j}(s), \sigma_2^{y+z\cdot e_j}(s)\right) ds\right)\right.\right. \\
&\qquad \left.\left. - \exp\left(\int_0^t \gamma \Pi\left(\sigma_1^{y}(s), \sigma_2^{y}(s)\right) ds\right)\right]\right| \\
&\le \mathbb{E}^y\left[\exp\left(\int_0^t \gamma \Pi\left(\sigma_1^{y}(s), \sigma_2^{y}(s)\right) + c_j \lambda_j z e^{-\lambda_j s} ds\right)\right. \\
&\qquad \left. - \exp\left(\int_0^t \gamma \Pi\left(\sigma_1^{y}(s), \sigma_2^{y}(s)\right) ds\right)\right] \\
&\le \exp\left(k_1 t + \gamma \sum_{j=1}^{m} \frac{\left(|\beta_1|\,\omega_{1,j} + |\beta_2|\,\omega_{2,j}\right)}{\lambda_j} y_j\right)\left(\exp\left(c_j z\right) - 1\right),
\end{aligned}$$

for $k_1 > 0$ by Lemma 5.1. Since

$$Y_j^{y_j}(s) \le y_j + Z_j\left(\lambda_j s\right),$$

we have that

$$\mathbb{E}\left[\int_0^T \int_{0+}^{\infty} \left|g\left(t, Y(s) + z \cdot e_j\right) - g\left(t, Y(s)\right)\right| l_j(dz)\, ds\right]$$

$$\leq \int_0^T e^{k_1 t + k_2 \sum_{j=1}^m y_j} \mathbb{E}\left[\exp\left(\gamma \sum_{j=1}^m \frac{(|\beta_1|\,\omega_{1,j} + |\beta_2|\,\omega_{2,j})\, Z_j(\lambda_j s)}{\lambda_j}\right)\right] ds$$
$$\times \int_{0+}^{\infty} (\exp(c_j z) - 1)\, l_j(dz)$$
$$< \infty,$$

by the assumptions, for $k_2 > 0$. ■

We give now a proposition that shows that $g(t,y)$ is a classical solution to the related forward problem of Equation (5.2). Its proof is very much the same as in [6, Proposition 5.5], and we omit it.

Proposition 5.1 *Assume there exists $\varepsilon > 0$ such that Condition 2.1 is satisfied with $c_j = \frac{2\gamma}{\lambda_j}(|\beta_1|\,\omega_{1,j} + |\beta_2|\,\omega_{2,j}) + \frac{\gamma(1-\gamma)}{2}(\omega_{1,j} + \omega_{2,j}) + \varepsilon$ for $j = 1, ..., m$, and some $\varepsilon > 0$. Then $g(t,\cdot)$ belongs to the domain of the infinitesimal generator of Y and*

$$0 = g_t(t,y) - \gamma \Pi(\sigma_1,\sigma_2)\, g(t,y) + \sum_{j=1}^m \lambda_j y_j g_{y_j}(t,y) \tag{5.8}$$
$$- \sum_{j=1}^m \lambda_j \int_{0+}^{\infty} (g(t, y + z\cdot e_j) - g(t,y))\, l_j(dz)$$

for $(t,y) \in (0,T] \times [0,\infty)^m$. Moreover, g_t is continuous, so that

$$g \in C^{1,1}((0,T] \times [0,\infty)^m).$$

From our conjecture of the form of the value function we have now our explicit solution candidate, namely

$$v(t,w,y) = \gamma^{-1} w^{\gamma} h(t,y). \tag{5.9}$$

The candidate for the optimal feedback control π^* is given by Equations (5.3) to (5.6).

Assume now that

$$c_j = \frac{8\gamma}{\lambda_j}((|\beta_1| + 4\gamma)\,\omega_{1,j} + (|\beta_2| + 4\gamma)\,\omega_{2,j}) + \frac{\gamma(1-\gamma)}{2}(\omega_{1,j} + \omega_{2,j}),$$

for $j = 1, ..., m$. We note that this implies that the optimal value function V is well-defined, and we can easily proceed as in [6] to check that all the assumptions in Theorem 4.1 are satisfied. Hence we have proved that our conjectured solution coincides with the optimal value function, which is what we wanted to show.

6 Generalizations

In this section we state, without proofs, the most important results for the case of n stocks,

$$\pi_i(s) \in [a_i, b_i], \quad i = 1, ..., n,$$

and

$$c \leq \sum_{i=1}^{n} \pi_i \leq d.$$

It can be seen that the additional difficulty in this setting is merely notational.

The HJB equation associated to this stochastic control problem is

$$0 = v_t + \max_{\substack{\pi_i \in [a_i, b_i], i=1,...,n, \\ c \leq \sum_{i=1}^{n} \pi_i \leq d}} \left\{ w v_w \sum_{i=1}^{n} \pi_i \left(\mu_i + \beta_i \sigma_i - r \right) + \frac{1}{2} w^2 v_{ww} \sum_{i=1}^{n} \pi_i^2 \sigma_i \right\}$$

$$+ r w v_w - \sum_{j=1}^{m} \lambda_j y_j v_{y_j} + \sum_{j=1}^{m} \lambda_j \int_0^{\infty} \left(v\left(t, w, y + z \cdot e_j\right) - v\left(t, w, y\right) \right) l_j(dz),$$

for $(t, w, y) \in [0, T) \times D$. We still have the terminal condition

$$V(T, w, y) = U(w), \quad \text{for all } (w, y) \in \bar{D},$$

and the boundary condition

$$V(t, 0, y) = U(0), \quad \text{for all } (t, y) \in [0, T] \times \mathbb{R}_+^m.$$

The solution to this equation can be shown to be

$$v\left(t, w, y\right) = \gamma^{-1} w^{\gamma} h\left(t, y\right) = \gamma^{-1} w^{\gamma} \mathbb{E}^y \left[e^{\int_t^T \gamma \Pi\left(\sigma_1^y(s), ..., \sigma_n^y(s)\right) ds} \right],$$

where Π is defined as

$$\Pi\left(\sigma_1, ..., \sigma_n\right) \tag{6.1}$$

$$= \max_{\substack{\pi_i \in [a_i, b_i], i=1,...,n, \\ c \leq \sum_{i=1}^{n} \pi_i \leq d}} \left\{ \sum_{i=1}^{n} \pi_i \left(\mu_i + \beta_i \sigma_i - r \right) - \frac{1-\gamma}{2} \sum_{i=1}^{n} \pi_i^2 \sigma_i \right\} + r.$$

The optimal fractions of wealth are given by the parameters $\pi^* = (\pi_1^*, ..., \pi_n^*)$ that obtain $\Pi\left(\sigma_1, ..., \sigma_n\right)$ in Equation (6.1).

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Portfolio optimization and a factor model in a stochastic volatility market

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Abstract

The aim of this paper is to find explicit optimal portfolio strategies for a n-stock stochastic volatility model. We introduce an extension of the stochastic volatility model proposed in [2]. It is a modification of [18], and characterizes the dependence by the use a factor structure. The idea of a factor structure is that the diffusion components of the stocks contain one Brownian motion that is unique for each stock, and a few Brownian motions that all stocks share. Hence, the dependence between stocks lies both in the stochastic volatility, and in the Brownian motions in the diffusion components. The model in the present paper can obtain strong correlations between the returns for different stocks without affecting their marginal distributions. This was not possible in [18]. Further, the number of model parameters does not grow too fast as the number of stocks n grows. This allows us to obtain stable parameter estimates for relatively large n. We use dynamic programming to solve Merton's optimization problem for power utility, with utility drawn from terminal wealth. Explicit optimal portfolios for n stocks are obtained, which is of large practical importance. A method to fit this model to data is given in the companion paper [19].

1 Introduction

We consider a version of the problem of optimal allocation of capital between different assets. This was solved by Merton in [20] and [21] for a Black and Scholes market with constant coefficients. Recently, [5] solved a similar problem for one stock and a bond in the more general market model of Barndorff-Nielsen and Shephard [2]. This model assumes that the volatility in asset price dynamics be modelled as a weighted sum of non-Gaussian Ornstein-Uhlenbeck (OU)

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processes of the form

$$dy(t) = -\lambda y(t)\, dt + dz(t),$$

where z is a subordinator and $\lambda > 0$. This idea allows us to capture several of the observed features in financial time series, such as semi-heavy tails, volatility clustering, and skewness. A multi-stock extension of [5] was considered by [18]. In that paper, the dependence between stocks is that they share some of OU processes of the volatility. This is given the interpretation that the stocks react to the same *news.* The model was primarily intended for stocks that are dependent, but not too dependent, such as stocks from different branches of industry. It retains all the features of the univariate model of [2]. Further advantages are that it requires little data and gives explicit optimal portfolio strategies. The disadvantage of the model used in [18] is that to obtain strong correlations between the returns of different stocks, the marginal distributions have to be very skew. This might not fit data. In the present paper, we deal with this disadvantage. It is a primary objective that the number of parameters in our model do not grow too fast as the number of stocks n grows. In other words, the parameter estimates must be stable for relatively large n. This feature is necessary since we want to be able to apply the optimal strategies to portfolios that contain a considerable number of stocks. Therefore, we do not use the standard approach: An explicit stochastic volatility matrix, and n Brownian motions in the diffusion components of all n stocks. Instead, we define the stochastic volatility matrix implicitly by a *factor structure.* The idea of a factor structure is that the diffusion components of the stocks contain one Brownian motion that is unique for each stock, and a few Brownian motions that all stocks share. The latter are called *factors.* This means that the dependence between stocks lies both in the stochastic volatility, and in the Brownian motions. A factor model has fewer parameters than a standard model. The reason is that the number of factors can be chosen a lot smaller than the number of stocks. We show that this model can obtain strong correlations between the returns of the stocks without affecting their marginal distributions.

The object of this paper is to find *explicit optimal allocation strategies* for the factor model described above. We consider an investor who wants to maximize her utility from terminal wealth by investing in n stocks and a bond. This problem is an extension of [18] and [5]. We allow for the investor to have restrictions on the fractions of wealth held in each stock, as well as borrowing and short-selling restrictions on the entire portfolio. The stochastic optimization problem is solved via dynamic programming and the associated Hamilton-Jakobi-Bellman (HJB) integro-differential equation. We use a verification theorem to identify the optimal expected utility from terminal wealth as the solution of a second-order integro-differential equation. We then compute the solution to this equation via a Feynman-Kac representation for power utility. Thus explicit optimal allocation strategies are obtained, which from an applied perspective is a key feature of the solution. All results are derived under exponential integrability assumptions on the Lévy measures of the subordinators.

Portfolio optimization with stochastic volatility has been treated in a number of articles. In [11], [13], and [27], the stochastic volatility depends on a stochastic factor, correlated to the diffusion process of the risky asset. The paper [7] models the stochastic factor as a continuous-time Markov chain with finite state-space that is assumed to be independent of the diffusion process. The approach to solve the portfolio optimization problems in [7] and [27] is related to the approach in this paper. In [24], partial observation is used to solve a portfolio problem with a stochastic volatility process driven by a Brownian motion correlated to the dynamics of the risky asset. The papers [3], [4], and [9] treat different portfolio problems when the risky assets are driven by Lévy processes, and [8] look at portfolio optimization in a market with unobservable Markov-modulated drift process. Further, [14] derive explicit solutions for log-optimal portfolios in complete markets in terms of the semimartingale characteristics of the price process, and [17] show that there exists a unique solution to the optimal investment problem for any arbitrage-free model if and only if the utility function has asymptotic elasticity strictly less than one. We refer to [1] and [2] for an introduction to the market model of Barndorff-Nielsen and Shephard. See [23] for option pricing in this context.

This paper is divided into six sections. In Section 2 we give a rigorous formulation of the market model. We discuss the dependence structure of the market, but also alternative models. In Section 3 we set up the control problem. Section 4 shows that our problem is well defined. We prove our verification theorem in Section 5, and use it in Section 6 to verify the solution we have obtained.

2 The model

2.1 Model definitions

For $0 \le t \le T < \infty$, we assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$ satisfying the usual conditions. We take m independent subordinators Z_j, and denote their Lévy measures by $l_j(dz), j = 1, ..., m$. We recall that a subordinator is defined to be a Lévy process taking values in $[0, \infty)$. This implies that its sample paths are increasing. The Lévy measure l of a subordinator satisfies the condition

$$\int_{0+}^{\infty} \min(1, z) l(dz) < \infty.$$

We assume that we use the cádlág version of Z_j. We introduce now a n stock extension of the model proposed by Barndorff-Nielsen and Shephard in [2]. Our model is a generalization of that in [18].

Consider $n+q$ independent Brownian motions B_i. Denote by Y_j, $j = 1, ..., m$, the OU stochastic processes whose dynamics are governed by

$$dY_j(t) = -\lambda_j Y_j(t)dt + dZ_j(\lambda_j t), \tag{2.1}$$

where $\lambda_j > 0$ denotes the *rate of decay*. The unusual timing of Z_j is chosen so that the marginal distribution of Y_j will be unchanged regardless of the value of λ_j. We use the filtration

$$\{\mathcal{F}_t\}_{0\leq t\leq T} := \{\sigma\left(B_1\left(t\right), ..., B_{n+q}\left(t\right), Z_1\left(\lambda_1 t\right), ..., Z_m\left(\lambda_m t\right)\right)\}_{0\leq t\leq T},$$

to make the OU processes and the Wiener processes simultaneously adapted.

We view the processes $Y_j,\ j = 1, ..., m$ in our model as *news processes* associated to certain events, and the jump times of $Z_j,\ j = 1, ..., m$ as *news* or *the release of information* on the market. The stationary process Y_j can be represented as

$$Y_j\left(s\right) = y_j e^{-\lambda_j(s-t)} + \textstyle\int_t^s e^{-\lambda_j(s-u)} dZ_j(\lambda_j u), \quad s \geq t, \tag{2.2}$$

where $y_j := Y_j\left(t\right)$, and y_j has the stationary marginal distribution of the process and is independent of $Z_j\left(s\right) - Z_j\left(t\right),\ s \geq t$. Note in particular that if $y_j \geq 0$, then $Y_j\left(s\right) > 0\ \forall s \in \left[t, T\right]$, since Z_j is non-decreasing. We set $Z_j\left(0\right) = 0,\ j = 1, ..., m$, and set $y := \left(y_1, ..., y_m\right)$. We define σ_i^2 as

$$\sigma_i\left(t\right)^2 := \sigma_i^{t,y}\left(s\right)^2 := \textstyle\sum_{j=1}^m \omega_{i,j} Y_j\left(s\right), \quad s \in \left[t, T\right], \tag{2.3}$$

where $\omega_{i,j} \geq 0$ are weights summing to one for each i. The notation $\sigma_i^{t,y}\left(\cdot\right)^2$ denotes conditioning on $Y\left(t\right)$. Further, we define

$$\sigma_{i,k}\left(s\right)^2 := \sigma_{i,k}^{t,y}\left(s\right)^2 := \textstyle\sum_{j=1}^m \rho_{i,j,k}\omega_{i,j} Y_j\left(s\right), \quad s \in \left[t, T\right],$$

where $\rho_{i,j,k} \in [0, 1]$ are weights chosen so that

$$\sigma_i\left(s\right)^2 = \textstyle\sum_{k=0}^q \sigma_{i,k}\left(s\right)^2, \quad \forall s \in \left[t, T\right].$$

We will see now that the $\rho_{i,j,k}$ give the volatilities for each Brownian motion. Define the stocks $S_i,\ i = 1, ..., n$, to have the dynamics

$$dS_i\left(t\right) = S_i\left(t\right)\left(\left(\mu_i + \beta_i \sigma_i\left(t\right)^2\right) dt + \sigma_{i,0}\left(t\right) dB_i\left(t\right) + \sum_{k=1}^q \sigma_{i,k}(t) dB_{n+k}\left(t\right)\right).$$

Here μ_i are the *constant mean rates of return*, and β_i are *skewness* parameters. The Brownian motions $B_{n+k},\ k = 1, ..., q$, are referred to as the *factors,* and we will call $\mu_i + \beta_i \sigma_i\left(t\right)^2$ the *mean rate of return* for stock i at time t. Note that the choice of volatility process σ_i^2 preserves the features of the univariate model in [2]. These stock price dynamics gives us the stock price processes

$$S_i\left(s\right) = S_i\left(t\right) \exp\left(\int_t^s \left(\mu_i + \left(\beta_i - \tfrac{1}{2}\right)\sigma_i\left(u\right)^2\right) du + \int_t^s \sigma_{i,0}\left(u\right) dB_i\left(u\right) + \sum_{k=1}^q \int_t^s \sigma_{i,k}(u) dB_{n+k}\left(u\right)\right). \tag{2.4}$$

This stock price model does not have statistically independent increments. It allows for the increments of the *returns* $R_i(t) := \log(S_i(t)/S_i(0))$, $i = 1, ..., n$, to have semi-heavy tails as well as both volatility clustering and skewness. The increments of the returns R_i are stationary since

$$R_i(s) - R_i(t) = \log\left(\frac{S_i(s)}{S_i(0)}\right) - \log\left(\frac{S_i(t)}{S_i(0)}\right) = \log\left(\frac{S_i(s)}{S_i(t)}\right) =^{\mathcal{L}} R_i(s-t),$$

where " $=^{\mathcal{L}}$ " denotes equality in law.

We assume the usual risk-free bond dynamics

$$dR(t) = rR(t)\,dt,$$

with interest rate $r > 0$.

The idea of this model is to model the dependence between stocks in two ways. First of all the stocks share the news processes Y_j, $j = 1, ..., m$. This implies that the volatilities of different stocks will be similar. Second, we characterize the dependence further by letting stocks depend on common factors. We will show below that this allows us to obtain high correlations between the returns for different stocks without affecting their marginal distributions. Our n-stock extension preserves the qualities of the univariate model. In addition, the number of factors can be chosen to be a lot less than the number of stocks. This means that much less data is required to estimate the model compared to if an explicit volatility matrix would have been used. This is an important feature, since financial data can typically not be assumed to be stationary for long periods of time. The concept of characterizing dependence by factors is not new. For example, it is used in the Factor-ARCH model (see [10]). It is also indicated in [2].

2.2 The dependence

In this section we describe briefly how to estimate the one-stock model from data. We then calculate explicit formulas for the covariances and correlations for the increments of the returns between different stocks.

We assume that we are observing returns $R_i(\Delta), R_i(2\Delta) - R_i(\Delta), ..., R_i(k\Delta) - R_i((k-1)\Delta)$, for stock $i = 1, ..., n$, where Δ e.g. is one day, and $k+1$ is the number of trading days in our period of observation. We recall the standard result that if we take σ^2 to have a Generalized Inverse Gaussian distribution (GIG), and draw an independent $N(0,1)$-distributed random variable ε, then $x = \mu + \beta\sigma^2 + \sigma\varepsilon$ will have a Generalized Hyperbolic distribution (GH). This class is quite flexible and contains many of the most frequently used marginal distributions in finance. We see by this result that if we choose GIG-marginals of our continuous time volatility processes σ_i^2, we will obtain approximately GH-marginals of the increments of the returns R_i, $i = 1, ..., n$. The existence and integrability of Lévy measures l_j such that the volatility processes σ_i^2 will have GIG-distributed marginals is not obvious. See [1] and [26, Section 17] for this theory. We estimate the rates of decay λ_j by using the autocorrelation function of the volatility processes σ_i^2. The autocorrelation ρ is defined by

$$\rho_\sigma(h) = \frac{Cov\left(\sigma(h)^2, \sigma(0)^2\right)}{Var\left(\sigma(0)^2\right)}, \quad h \in \mathbb{R}.$$

Straightforward calculations show that

$$\rho_{\sigma_i}(h) = \omega_{i,1}\exp\left(-\lambda_1 |h|\right) + ... + \omega_{i,m}\exp\left(-\lambda_m |h|\right),$$

where the $\omega_{i,j} \geq 0$, are the weights from the volatility processes. The weights sum to one for each $i = 1, ..., n$.

Our model generates a non-diagonal covariance matrix for the increments of the returns over the same time period, which is the most frequently used measure of dependence in finance. It is an important feature of the model in the present paper that we can estimate the covariances of the increments of the returns between different stocks from data without affecting the marginal distributions. This was not possible for strong correlations in the model in [18]. It is sufficient to show this result for the returns R_i, $i = 1, 2$, since the returns have stationary increments. First note that

$$\begin{aligned} &Cov\left(R_1(s) - R_1(t), R_2(u) - R_2(v)\right) \\ &= Cov\left(R_1(s), R_2(u)\right) - Cov\left(R_1(s), R_2(v)\right) \\ &\quad - Cov\left(R_1(t), R_2(u)\right) + Cov\left(R_1(t), R_2(v)\right), \end{aligned}$$

for $s, t, u, v \in [0, T]$.

We now calculate the correlation between the returns R_i, $i = 1, 2$. It turns out that

$$\begin{aligned} &Corr\left(R_1(s), R_2(t)\right) \\ &= \Bigg(\left(\beta_1 - \tfrac{1}{2}\right)\left(\beta_2 - \tfrac{1}{2}\right)\sum_{j=1}^{m}\omega_{1,j}\omega_{2,j}Var\left(Y_j(0)\right) \\ &\quad \times \frac{e^{-\lambda_j s} + e^{-\lambda_j t} - e^{-\lambda_j |s-t|} - 1 + 2\lambda_j \min(s,t)}{\lambda_j^2} \\ &\quad + \sum_{k=1}^{q}\mathbb{E}\left[\int_0^{\min(s,t)}\sigma_{1,k}(u)\sigma_{2,k}(u)du\right]\Bigg) \\ &\quad \times \frac{1}{\sqrt{\sum_{j=1}^m \left(2\left(\beta_1 - \frac{1}{2}\right)^2 \omega_{1,j}^2 Var\left(Y_j(0)\right)\frac{e^{-\lambda_j s} - 1 + \lambda_j s}{\lambda_j^2} + \omega_{1,j}\mu_{Y_j}s\right)}} \\ &\quad \times \frac{1}{\sqrt{\sum_{j=1}^m \left(2\left(\beta_2 - \frac{1}{2}\right)^2 \omega_{2,j}^2 Var\left(Y_j(0)\right)\frac{e^{-\lambda_j t} - 1 + \lambda_j t}{\lambda_j^2} + \omega_{2,j}\mu_{Y_j}t\right)}}. \end{aligned}$$

We have by definition of σ_i^2 and Itô's isometry (see [28])

$$\begin{aligned}
&\mathbb{E}\left[R_1(s)\,R_2(t)\right]\\
&=\mathbb{E}\Bigg[\Bigg(\int_0^s\left(\mu_1+\left(\beta_1-\tfrac{1}{2}\right)\sigma_1(u)^2\right)du+\int_0^s\sigma_{1,0}(u)\,dB_1(u)\\
&\qquad+\sum_{k=1}^q\int_0^s\sigma_{1,k}(u)dB_{n+k}(u)\Bigg)\\
&\qquad\times\Bigg(\int_0^t\left(\mu_2+\left(\beta_2-\tfrac{1}{2}\right)\sigma_2(u)^2\right)du+\int_0^t\sigma_{2,0}(u)\,dB_2(u)\\
&\qquad+\sum_{k=1}^q\int_0^t\sigma_{2,k}(u)dB_{n+k}(u)\Bigg)\Bigg]\\
&=\mu_1\mu_2st+\mu_1s\left(\beta_2-\tfrac{1}{2}\right)\sum_{j=1}^m\omega_{2,j}\mathbb{E}\left[\int_0^tY_j(u)\,du\right]\\
&\qquad+\mu_2t\left(\beta_1-\tfrac{1}{2}\right)\sum_{j=1}^m\omega_{1,j}\mathbb{E}\left[\int_0^sY_j(u)\,du\right]\\
&\qquad+\left(\beta_1-\tfrac{1}{2}\right)\left(\beta_2-\tfrac{1}{2}\right)\sum_{i,j=1}^m\omega_{1,i}\omega_{2,j}\mathbb{E}\left[\int_0^sY_i(u)\,du\int_0^tY_j(u)\,du\right]\\
&\qquad+\sum_{k=1}^q\mathbb{E}\left[\int_0^{\min(s,t)}\sigma_{1,k}(u)\sigma_{2,k}(u)du\right].
\end{aligned}$$

Similarly,

$$\mathbb{E}\left[R_1(t)\right]=\mu_1t+\left(\beta_1-\tfrac{1}{2}\right)\sum_{j=1}^m\omega_{1,j}\mathbb{E}\left[\int_0^tY_j(u)\,du\right].$$

This gives that

$$\begin{aligned}
&Cov\left(R_1(s),R_2(t)\right)\\
&=\left(\beta_1-\tfrac{1}{2}\right)\left(\beta_2-\tfrac{1}{2}\right)\sum_{j=1}^m\omega_{1,j}\omega_{2,j}Cov\left(\int_0^sY_j(u)\,du,\int_0^tY_j(u)\,du\right)\\
&\qquad+\sum_{k=1}^q\mathbb{E}\left[\int_0^{\min(s,t)}\sigma_{1,k}(u)\sigma_{2,k}(u)du\right].
\end{aligned}$$

By stationarity, we have that $\mathbb{E}\left[Y_j(t)\right]=\mu_{Y_j}$, for some constant $\mu_{Y_j}>0$, for all $t\in\mathbb{R}$. If we assume that $u\leq v$, the independence of the increments of Y_j gives that

$$
\begin{aligned}
&Cov\left(Y_j\left(u\right), Y_j\left(v\right)\right) \\
&= \mathbb{E}\left[\left(Y_j\left(u\right) - \mu_{Y_j}\right)\left(Y_j\left(v\right) - \mu_{Y_j}\right)\right] \\
&= \mathbb{E}\left[e^{-\lambda_j(v-u)}Y_j\left(u\right)^2 + Y_j\left(u\right)\int_u^v e^{-\lambda_j(v-s)}dZ\left(\lambda_j s\right)\right] - \mu_{Y_j}^2 \\
&= e^{-\lambda_j(v-u)}\mathbb{E}\left[Y_j\left(0\right)^2\right] - e^{-\lambda_j(v-u)}\mu_{Y_j}^2 \\
&= e^{-\lambda_j(v-u)}Var\left(Y_j\left(0\right)\right).
\end{aligned}
$$

The same calculations for $v \leq u$ shows that

$$
Cov\left(Y_j\left(u\right), Y_j\left(v\right)\right) = e^{-\lambda_j|v-u|}Var\left(Y_j\left(0\right)\right),
$$

and we get

$$
\begin{aligned}
&Cov\left(\int_0^s Y_j\left(u\right)du, \int_0^t Y_j\left(u\right)du\right) \\
&= \int_0^s \int_0^t Cov\left(Y_j\left(u\right), Y_j\left(v\right)\right)dudv \\
&= Var\left(Y_j\left(0\right)\right)\frac{e^{-\lambda_j s} + e^{-\lambda_j t} - e^{-\lambda_j|s-t|} - 1 + 2\lambda_j \min\left(s,t\right)}{\lambda_j^2}.
\end{aligned}
\tag{2.5}
$$

Finally, we get by Itô's isometry (see [28]), similar to above, that

$$
Var\left(R_i\left(t\right)\right) = \sum_{j=1}^m \left(2\left(\beta_i - \tfrac{1}{2}\right)^2 \omega_{i,j}^2 Var\left(Y_j\left(0\right)\right)\frac{e^{-\lambda_j t} - 1 + \lambda_j t}{\lambda_j^2} + \omega_{i,j}\mu_{Y_j}t\right),
$$

for $i = 1, 2$.

2.3 Alternative n-stock extensions

We introduced in [18] the notion of *news-generated dependence*. We meant by this that the dependence between stocks lies in that they react to the same news. In other words, the stocks share news processes. A natural extension of this model is to distinguish between *good* and *bad* news. This can be done, for example, by considering the stock dynamics

$$
dS_i\left(t\right) = \left(\mu_i + \beta_i\sigma_i\left(t\right)^2\right)S_i\left(t\right)dt + \sigma_i(t)S_i\left(t\right)dB_i\left(t\right) + S_i\left(t\right)\sum_{j=1}^m \frac{\eta_{i,j}}{Y_j\left(t\right)}dZ_j\left(\lambda_j t\right),
$$

or

$$
dS_i\left(t\right) = \left(\mu_i + \sum_{j=1}^m \eta_{i,j}Y_j\left(t\right)\right)S_i\left(t\right)dt + \sigma_i(t)S_i\left(t\right)dB_i\left(t\right),
$$

for constants $\eta_{i,j} \in \mathbb{R}$, $j = 1, ..., m$, $i = 1, ..., n$. It is the signs of $\eta_{i,j}$ that decides whether a news process Y_j is associated with good or bad news. The first model is a modification of a model proposed in [2]. The difference is mainly that we have normalized the jumps in the stocks price process so that they are always less than the stock price itself. We do this to avoid that the stock prices become negative as a result of large negative jumps. The second model is a generalization of the model in [18], which is the special case where $\eta_{i,j} = \beta_i \omega_{i,j}$. The optimal portfolio problem for this model can be solved using analogous techniques as in [18]. These two extensions both seem reasonable from a modelling perspective, but we have chosen to not pursue them any further. The reason is that we suspect that they will be hard to estimate from data. For example, the returns of these models can not in general be written (approximately) on the form $x = \mu + \beta\sigma^2 + \sigma\varepsilon$, where μ and β are constants, ε is a $N(0,1)$-distributed random variable, and σ^2 has the marginal distribution of the volatility process of the stock price. This means that we can not obtain marginals from the generalized hyperbolic distribution (for example the NIG-distribution) just by choosing generalized inverse Gaussian marginals of the volatility processes. Further, the first model has the disadvantage that is gives discontinuous stock prices, unlike the model we have chosen to work with.

3 The control problem

We want to solve the problem of how a trader is supposed to invest in a stock market to optimize her expected utility from wealth in a deterministic future point in time. We measure utility by a *utility function* U. This function is chosen by the trader, and measures the trader's aversion towards risk in that it concretizes how much the trader is willing to risk to obtain a certain level of wealth. We use dynamic programming and stochastic control to find the maximum expected utility from terminal wealth, and the trading strategies to obtain it.

In this section we set up the control problem for the market model of 2.1. Recall that σ_i^2 are weighted sums of the news processes, see Equation (2.3). We define a *optimal value function* V as the maximum amount of expected utility that we can obtain from a trading strategy, given a certain amount of capital. We then set up the associated Hamilton-Jakobi-Bellman equation for the optimal value function V.

Denote by $\pi_i(t)$ the fraction of wealth invested in stock i at time t, and set $\pi = (\pi_1, ..., \pi_n)$. The fraction of wealth held in the risk-free asset is $(1 - \pi_1 - ... - \pi_n)$. We choose constants $a_i, b_i, c, d \in \mathbb{R}$, $a_i < b_i$, $c < d$, and let the constraints take the form $\pi_i \in [a_i, b_i]$, $i = 1, ..., n$, and $c \leq \pi_1 + ... + \pi_n \leq d$, a.s., for all $t \leq s \leq T$. This means that we can consider, for example, law enforced restrictions on the fraction of wealth held in a specific stock, as well as short-selling and borrowing of capital.

We define the wealth process W as

$$W(s) = \frac{\pi_1(s) W(s)}{S_1(s)} S_1(s) + ... + \frac{\pi_n(s) W(s)}{S_n(s)} S_n(s) + \frac{(1 - \pi_1(s) - ... - \pi_n(s)) W(s)}{R(s)} R(s),$$

where $\pi_i(s) W(s) / S_i(s)$ is the number of shares of stock i which is held at time s. We assume also that the portfolio is self-financing in the sense that no capital is entered or withdrawn. This can be formulated mathematically as

$$W(s) = W(t) + \sum_{i=1}^{n} \int_t^s \frac{\pi_i(u) W(u)}{S_i(u)} dS_i(u) + \int_t^s \frac{(1 - \pi_1(u) - \pi_2(u)) W(u)}{R(u)} dR(u),$$

for all $s \in [t, T]$. See [16] for a motivating discussion. This condition gives the wealth dynamics for the model from Subsection 2.1 for $t \leq s \leq T$ as

$$\begin{aligned} dW(s) = {} & W(s) \sum_{i=1}^{n} \pi_i(s) \left(\mu_i + \beta_i \sigma_i(s)^2 - r\right) ds \\ & + rW(s)\, ds + W(s) \sum_{i=1}^{n} \pi_i(s) \sigma_{i,0}(s)\, dB_i(s) \\ & + W(s) \sum_{i=1}^{n} \sum_{k=1}^{q} \pi_i(s) \sigma_{i,k}(s)\, dB_{n+k}(s), \end{aligned} \tag{3.1}$$

with initial wealth $W(t)$.

We now define our set of admissible controls.

Definition 3.1 *The set $\mathcal{A}_t$ of admissible controls is given by $\mathcal{A}_t := \{\pi = (\pi_1, ..., \pi_n) : \pi_i$ are adapted to $\{\mathcal{F}_s\}_{t \leq s \leq T}$, $\pi_i(s) \in [a_i, b_i]$, $i = 1, ..., n$, and $c \leq \pi_1(s) + ... + \pi_n(s) \leq d$, a.s.$\forall t \leq s \leq T$, and a unique solution $W^\pi \geq 0$ of Equation (3.1) exists$\}$.*

An investment strategy $\pi = \{\pi(s) : t \leq s \leq T\}$ is said to be *admissible* if $\pi \in \mathcal{A}_t$. The conditions in the definition are natural: We impose trading regulations both on each stock and on the entire portfolio. Further, we want our wealth to be positive, and we should always be able to tell how rich we are.

We will need later some exponential integrability conditions on the Lévy measures. We therefore assume that the following holds:

Condition 3.1 *For a constant $c_j > 0$ to be specified below,*

$$\int_{0+}^{\infty} \left(e^{c_j z} - 1\right) l_j(dz) < \infty, \ j = 1, ..., m.$$

We know from the theory of subordinators that we have

$$\mathbb{E}\left[e^{aZ_j(\lambda_j t)}\right] = \exp\left(\lambda_j t \int_{0+}^{\infty} \left(e^{az} - 1\right) l_j(dz)\right) \tag{3.2}$$

as long as $a \leq c_j$ with c_j from Condition 3.1 holds.

Denote $(0, \infty)$ by $\mathbb{R}_+$ and $[0, \infty)$ by $\mathbb{R}_{0+}$, and assume that $y = (y_1, ..., y_m) \in \mathbb{R}_+^m$. We will seek to maximize the *value function*

$$J(t, w, y; \pi) = \mathbb{E}^{t,w,y}\left[U\left(W^{\pi}\left(T\right)\right)\right],$$

where the notation $\mathbb{E}^{t,w,y}$ means expectation conditioned by $W(t) = w$, and $Y_j(t) = y_j$, $j = 1, ..., m$. The function U is the investor's utility function. It is assumed to be concave, non-decreasing, bounded from below, and of sublinear growth in the sense that there exists positive constants k and $\theta \in (0, 1)$ so that $U(w) \leq k(1 + w^{\theta})$ for all $w \geq 0$. Hence our stochastic control problem is to determine the *optimal value function*

$$V(t, w, y) = \sup_{\pi \in \mathcal{A}_t} J(t, w, y; \pi), \ (t, w, y) \in [0, T] \times \mathbb{R}_+^{m+1}, \tag{3.3}$$

and an investment strategy $\pi^* \in \mathcal{A}_t$, the optimal investment strategy, such that

$$V(t, w, y) = J(t, w, y; \pi^*).$$

The HJB equation associated to our stochastic control problem is

$$\begin{aligned} 0 = v_t + \max_{\substack{\pi_i \in [a_i, b_i], i=1,...,n, \\ c \leq \pi_1 + ... + \pi_n \leq d}} & \left\{ \sum_{i=1}^{n} \left[\pi_i \left(\mu_i + \beta_i \sigma_i^2 - r \right) w v_w + \frac{1}{2} \pi_i^2 \sigma_{i,0}^2 w^2 v_{ww} \right] \right. \\ & \left. + \frac{1}{2} \sum_{1 \leq h,i \leq n} \sum_{1 \leq k \leq q} \pi_h \pi_i \sigma_{h,k} \sigma_{i,k} w^2 v_{ww} \right\} + r w v_w - \sum_{j=1}^{m} \lambda_j y_j v_{y_j} \\ & + \sum_{j=1}^{m} \lambda_j \int_0^{\infty} \left(v\left(t, w, y + z \cdot e_j\right) - v\left(t, w, y\right) \right) l_j(dz), \end{aligned} \tag{3.4}$$

for $(t, w, y) \in [0, T) \times \mathbb{R}_+^{m+1}$. We observe that we have the terminal condition

$$V\left(T, w, y\right) = U\left(w\right), \quad \forall \left(w, y\right) \in \mathbb{R}_+^{m+1}, \tag{3.5}$$

and the boundary condition

$$V\left(t, 0, y\right) = U\left(0\right), \quad \forall \left(t, y\right) \in [0, T] \times \mathbb{R}_+^m. \tag{3.6}$$

We recall the motivation to this equation for the convenience of the reader.

Assume that the Dynamic Programming Principle holds. That is, if

$$V\left(t,w,y\right)\in C^{1,2,1}\left(\left[0,T\right]\times\mathbb{R}_{+}^{m+1}\right),$$

then for any stopping time $\tau\leq T$ $a.s.$ and $t\leq T$,

$$V\left(t,w,y\right)=\sup_{\pi\in\mathcal{A}_t}\mathbb{E}^{t,w,y}\left[V\left(W^{\pi}\left(\tau\right),Y\left(\tau\right),\tau\right)\right].$$

The notation $C^{1,2,1}\left(\left[0,T\right]\times\mathbb{R}_{+}^{m+1}\right)$ means twice continuously differentiable in w on $(0,\infty)$ and once continuously differentiable in (t,y) on $[0,T]\times\mathbb{R}_{+}^{m}$ with continuous extensions of the derivatives to $t=0$, $t=T$, $w=0$, and $y_j=0$, $j=1,...,m$. We see that if we choose stopping times τ so that $\tau\downarrow t$, then

$$\sup_{\pi\in\mathcal{A}_t}\left\{\lim_{\tau\downarrow t}\frac{\mathbb{E}^{t,w,y}\left[V\left(\tau,W\left(\tau\right),Y\left(\tau\right)\right)\right]-V\left(t,w,y\right)}{\tau-t}\right\}=0, \tag{3.7}$$

where we assume that we may change the order of the supremum operator and the limit operator. If we evaluate Equation (3.7), we get the HJB equation (3.4).

4 Well-definedness of the optimal value function

In this section we show that the optimal value function is well-defined. We start by stating a lemma that we will need both in this section and in some of the subsequent sections. It is due to [5].

Lemma 4.1 *Assume Condition 3.1 holds with $c_j=\xi_j/\lambda_j$ for $\xi_j>0$. Then*

$$\mathbb{E}^{t,y}\left[\exp(\xi_j\int_t^s Y_j(u)du)\right]\leq\exp\left(\frac{\xi_j}{\lambda_j}y_j+\lambda_j\int_{0+}^{\infty}\left\{\exp\left(\frac{\xi_j z}{\lambda_j}\right)-1\right\}l_j(dz)(s-t)\right)$$

We want to show that the optimal value function $V\left(t,w,y\right)$ is well-defined. If we use the sublinear growth condition on the utility function U, we see that

$$V\left(t,w,y\right)=\sup_{\pi\in\mathcal{A}_t}\mathbb{E}^{t,w,y}\left[U\left(W^{\pi}(T)\right)\right]\leq k\left(1+\sup_{\pi\in\mathcal{A}_t}\mathbb{E}^{t,w,y}\left[\left(W^{\pi}\left(T\right)\right)^{\theta}\right]\right).$$

This observation points out a direction for us to take in showing this property: We want to obtain an upper bound for $\sup_{\pi\in\mathcal{A}_t}\mathbb{E}^{t,w,y}\left[\left(W^{\pi}\left(s\right)\right)^{\theta}\right]$.

Lemma 4.2 *Assume Condition 3.1 holds with*

$$c_j=\hat{\pi}^2\textstyle\sum_{i=1}^{n}2\theta\left(|\beta_i|+\theta\right)\frac{\omega_{i,j}}{\lambda_j},\quad j=1,...,m,$$

where $\theta>0$. Then

$$\sup_{\pi\in\mathcal{A}_t}\mathbb{E}^{t,w,y}\left[\left(W^{\pi}\left(s\right)\right)^{\theta}\right]\leq w^{\theta}\exp\left(\theta\hat{\pi}\left(\sum_{i=1}^{n}|\mu_i-r|+r\right)(s-t)\right)$$

$$\times \exp\left(\frac{1}{2}\hat{\pi}^2 \sum_{j=1}^{m}\left(c_j y_j + \lambda_j \int_{0+}^{\infty} \left\{\exp\left(c_j z\right) - 1\right\} l_j(dz) \left(s-t\right)\right)\right).$$

Proof. Itô's formula and Equation (3.1) gives that

$$W^{\pi}(s) = w \exp\left(\int_t^s \alpha\left(u, Y\left(u\right)\right) du + \sum_{i=1}^{n} \int_t^s \pi_i\left(u\right) \sigma_{i,0}\left(u\right) dB_i\left(u\right) + \sum_{i=1}^{n}\sum_{k=1}^{q} \int_t^s \pi_i\left(u\right) \sigma_{i,k}\left(u\right) dB_{n+k}\left(u\right)\right),$$

where

$$\alpha(u, Y\left(u\right)) = \sum_{i=1}^{n}\left[\pi_i\left(u\right)\left(\mu_i + \beta_i \sigma_i\left(u\right)^2 - r\right) - \frac{1}{2}\left(\pi_i\left(u\right)\right)^2 \sigma_i^2\right] + r.$$

We now have that

$$\begin{aligned}
&\mathbb{E}^{t,w,y}\left[\left(W^{\pi}\left(s\right)\right)^{\theta}\right] \\
&= w^{\theta}\mathbb{E}^{t,w,y}\left[\exp\left(\theta \int_t^s \alpha\left(u, Y\left(u\right)\right) du + \theta \sum_{i=1}^{n} \int_t^s \pi_i\left(u\right) \sigma_{i,0}\left(u\right) dB_i\left(u\right)\right.\right. \\
&\qquad \left.\left. + \theta \sum_{i=1}^{n}\sum_{k=1}^{q} \int_t^s \pi_i\left(u\right) \sigma_{i,k}\left(u\right) dB_{n+k}\left(u\right)\right)\right].
\end{aligned}$$

We want to be able to conclude that the Wiener integrals are parts of a martingale, so we can control their expectations. Therefore, we define

$$\begin{aligned}
X(s) := \exp&\left(\sum_{i=1}^{n} \int_t^s 2\theta \pi_i\left(u\right) \sigma_{i,0}\left(u\right) dB_i\left(u\right)\right. \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{q} \int_t^s 2\theta \pi_i\left(u\right) \sigma_{i,k}\left(u\right) dB_{n+k}\left(u\right) \\
&\left. - \frac{1}{2} \sum_{i=1}^{n} \int_t^s \left(2\theta\right)^2 \left(\pi_i\left(u\right)\right)^2 \sigma_i\left(u\right)^2 du\right).
\end{aligned}$$

Due to the exponential integrability conditions on Y_j from Lemma 4.1 we have that

$$\mathbb{E}\left[\exp\left(\sum_{i=1}^{n} \int_0^T \sigma_i\left(t\right)^2 dt\right)\right] < \infty.$$

Hence the Wiener integrals are all well-defined continuous martingales. Then $X(s)$ is a martingale by Novikov's condition (see [25, p. 140]), so we have that

$$\mathbb{E}^{t,w,y}\left[X(s)\right] = 1.$$

The definition of X gives that

$$\begin{aligned}
&\mathbb{E}^{t,w,y}\left[\left(W^{\pi}\left(s\right)\right)^{\theta}\right] \\
&= w^{\theta}\mathbb{E}^{t,w,y}\left[\exp\left(\theta\int_t^s \alpha\left(u,Y\left(u\right)\right)du + \sum_{i=1}^{n}\int_t^s \theta^2\left(\pi_i\left(u\right)\right)^2\sigma_i\left(u\right)^2 du\right)X\left(s\right)^{\frac{1}{2}}\right].
\end{aligned}$$

Further, we have by Hölder's inequality and the fact that X is a martingale that

$$\begin{aligned}
&\mathbb{E}^{t,w,y}\left[\left(W^{\pi}\left(s\right)\right)^{\theta}\right] \\
&\leq w^{\theta}\mathbb{E}^{t,w,y}\left[\exp\left(2\theta\int_t^s \alpha\left(u,Y\left(u\right)\right)du + \sum_{i=1}^{n}\int_t^s 2\theta^2\left(\pi_i\left(u\right)\right)^2\sigma_i\left(u\right)du\right)\right]^{\frac{1}{2}} \\
&\qquad * \mathbb{E}^{t,w,y}\left[X\left(s\right)\right]^{\frac{1}{2}} \\
&= w^{\theta}\mathbb{E}^{t,w,y}\left[\exp\left(2\theta\int_t^s \alpha\left(u,Y\left(u\right)\right)du + \sum_{i=1}^{n}\int_t^s 2\theta^2\left(\pi_i\left(u\right)\right)^2\sigma_i\left(u\right)du\right)\right]^{\frac{1}{2}}.
\end{aligned}$$

But since $\pi_i \leq \max\left(1, |a_1|, ..., |a_n|, |b_1|, ..., |b_n|\right) =: \hat{\pi}$,

$$\begin{aligned}
&\mathbb{E}^{t,w,y}\left[\exp\left(2\theta\int_t^s \alpha\left(u,Y\left(u\right)\right)du + \sum_{i=1}^{n}\int_t^s 2\theta^2\left(\pi_i\left(u\right)\right)^2\sigma_i\left(u\right)^2 du\right)\right]^{\frac{1}{2}} \\
&\leq \exp\left(\theta\hat{\pi}\left(\sum_{i=1}^{n}|\mu_i - r| + r\right)(s-t)\right) \\
&\qquad \times \mathbb{E}^{t,w,y}\left[\exp\left(\hat{\pi}^2\sum_{i=1}^{n}\sum_{j=1}^{m}\int_t^s 2\theta\left(|\beta_i| + \theta\right)\omega_{i,j}Y_j\left(u\right)du\right)\right]^{\frac{1}{2}}
\end{aligned}$$

We can now apply Lemma 4.1 m times with $\xi_j = \hat{\pi}^2\sum_{i=1}^{n}2\theta\left(|\beta_i| + \theta\right)\omega_{i,j}$. ■

We have now proved that both the optimal value function of our control problem and the wealth process are well-defined, since

$$\begin{aligned}
V\left(t,w,y\right) \leq k\Bigg(1 + w^{\theta}\exp\left(\theta\hat{\pi}\left(\sum_{i=1}^{n}|\mu_i - r| + r\right)(s-t)\right) \\
\times \exp\left(\frac{1}{2}\hat{\pi}^2\sum_{j=1}^{m}\left(c_j y_j + \lambda_j\int_{0+}^{\infty}\left\{\exp\left(c_j z\right) - 1\right\}l_j(dz)\left(s-t\right)\right)\right)\Bigg).
\end{aligned}$$

Note that we also have that $U(0) \leq V(t,w,y)$, since U is non-decreasing.

From now on we assume that Condition 3.1 holds with

$$c_j = \sum_{i=1}^{n} 2\theta\left(|\beta_i| + \theta\right) \frac{\omega_{i,j}}{\lambda_j}, \quad j = 1, ..., m.$$

This ensures that the value function is well-defined.

5 A verification theorem

We prove in this section a verification theorem for our control problem. This theorem says essentially that if we can find a solution to our HJB equation, then that solution is the optimal value function.

Theorem 5.1 (Verification Theorem) *Assume that*

$$v(t,w,y) \in C^{1,2,1}\left([0,T] \times \mathbb{R}_+^{m+1}\right)$$

is a solution of the HJB equation (3.4) with terminal condition (3.5) and boundary condition (3.6). Assume that

$$\sup_{\pi \in \mathcal{A}_0} \int_0^T \int_{0+}^{\infty} \mathbb{E}\left[\left|v\left(t, W^{\pi}(t), Y(t-) + z \cdot e_j\right) - v\left(t, W^{\pi}(t), Y(t-)\right)\right|\right] l_j(dz)\, dt < \infty,$$

and

$$\sup_{\pi \in \mathcal{A}_0} \int_0^T \mathbb{E}\left[\left(\pi_i(t)\, \sigma_i(t)\, W^{\pi}(t)\, v_w\left(t, W^{\pi}(t), Y(t)\right)\right)^2\right] dt < \infty,$$

for $i = 1, ..., n$. Then

$$v(t,w,y) \geq V(t,w,y), \quad \forall (t,w,y) \in [0,T] \times \mathbb{R}_+^{m+1}.$$

Further, if there exist measurable functions $\pi_i^*(t,w,y) \in [a_i, b_i]$, $i = 1, ..., n$, $c \leq \pi_1^*(t,w,y) + ... + \pi_n^*(t,w,y) \leq d$, a.s., being the maximizers for the max-operator in Equation (3.4), and Equation (3.1) admits a unique solution $W^{\pi^*} \geq 0$, then π^* defines an optimal investment strategy in feedback form and

$$V(t,w,y) = v(t,w,y) = \mathbb{E}^{t,w,y}\left[U\left(W^{\pi^*}(T)\right)\right], \quad \forall (t,w,y) \in [0,T] \times \mathbb{R}_+^{m+1}.$$

Proof. Let $(t,w,y) \in [0,T) \times \mathbb{R}_+^{m+1}$ and $\pi \in \mathcal{A}_t$, and introduce the operator

$$\mathcal{M}^{\pi} v := \sum_{i=1}^{n} \left[\pi_i\left(\mu_i + \beta_i \sigma_i^2 - r\right) w v_w + \frac{1}{2}\pi_i^2 \sigma_{i,0}^2 w^2 v_{ww}\right] + \frac{1}{2} \sum_{1 \leq h,i \leq n} \sum_{1 \leq k \leq q} \pi_h \pi_i \sigma_{h,k} \sigma_{i,k} w^2 v_{ww} + r w v_w - \sum_{j=1}^{m} \lambda_j y_j v_{y_j}.$$

Itô's formula gives that

$$\begin{aligned}
& v\left(s, W^{\pi}(s), Y(s)\right) \\
& = v\left(t, w, y\right) + \int_{t+}^{s} \left\{ v_t\left(u, W^{\pi}(u), Y(u-)\right) + \mathcal{M}^{\pi} v\left(u, W^{\pi}(u), Y(u-)\right) \right\} du \\
& \quad + \sum_{i=1}^{n} \int_{t+}^{s} W^{\pi}(u)\, \pi_i(u)\, \sigma_{i,0}(u-)\, v_w\left(u, W^{\pi}(u), Y(u-)\right) dB_i(u) \\
& \quad + \sum_{i=1}^{n} \sum_{k=1}^{q} \int_{t+}^{s} W^{\pi}(u)\, \pi_i(u)\, \sigma_{i,k}(u-)\, v_w\left(u, W^{\pi}(u), Y(u-)\right) dB_{n+k}(u) \\
& \quad + \sum_{j=1}^{m} \int_{t+}^{s} \int_{0+}^{\infty} \left[v\left(u, W^{\pi}(u), Y(u-) + z \cdot e_j\right) \right. \\
& \quad \left. - v\left(u, W^{\pi}(u), Y(u-)\right) \right] N_j\left(\lambda_j du, dz\right),
\end{aligned}$$

where the N_j are the Poisson random measure in the Lévy-Khintchine representation of Z_j, $j = 1, ..., m$. We have used that $[Y_j, Y_j]^c = 0$, $j = 1, ..., m$, by Theorem 26 in [25], where $[\cdot,\cdot]^c$ denotes the continuous part of the quadratic covariation. Further, the Kunita-Watanabe inequality (see [25, p. 69]) tells us that $d[X, Y_j]^c$ is *a.e.*(path by path) absolutely continuous with respect to $d[Y_j, Y_j]^c$, $j = 1, ..., m$, for a semimartingale X. We know from the assumptions that the Itô integrals are martingales and that the integrals with respect to N_j have finite expectation. Hence we have that

$$\begin{aligned}
& \mathbb{E}^{t,w,y} \left[\int_{t+}^{s} \int_{0+}^{\infty} v\left(u, W^{\pi}(u), Y(u-) + z \cdot e_j\right) - v\left(u, W^{\pi}(u), Y(u-)\right) N_j\left(\lambda_j du, dz\right) \right] \\
& = \lambda_j \int_{t+}^{s} \int_{0+}^{\infty} \mathbb{E}^{t,w,y} \left[v\left(u, W^{\pi}(u), Y(u-) + z \cdot e_j\right) - v\left(u, W^{\pi}(u), Y(u-)\right) \right] l_j(dz)\, du,
\end{aligned}$$

for $j = 1, ..., m$, where we have used the Fubini-Tonelli theorem, the cádlág property of Y, and the fact that, for Borel sets Λ, $N_j(t, \Lambda) - t l_j(\Lambda)$ is a martingale. This gives us that

$$\begin{aligned}
& \mathbb{E}^{t,w,y} \left[v\left(s, W^{\pi}(s), Y(s)\right) \right] \\
& = v\left(t, w, y\right) + \mathbb{E}^{t,w,y} \left[\int_{t+}^{s} \left(v_t + \mathcal{L}^{\pi} v \right)\left(u, W^{\pi}(u), Y(u-)\right) du \right] \\
& \leq v\left(t, w, y\right) + \mathbb{E}^{t,w,y} \left[\int_{t+}^{s} \left(v_t + \max_{\pi \in \mathcal{A}_t} \mathcal{L}^{\pi} v \right)\left(u, W^{\pi}(u), Y(u-)\right) du \right] \\
& = v\left(t, w, y\right)
\end{aligned}$$

where

$$\mathcal{L}^{\pi} v := \mathcal{M}^{\pi} v + \sum_{j=1}^{m} \lambda_j \int_{0+}^{\infty} \left(v\left(t, w, y + z \cdot e_j\right) - v\left(t, w, y\right) \right) l_j(dz).$$

If we choose $s = T$ and use the terminal condition for v, we get that

$$v(t,w,y) \geq \mathbb{E}^{t,w,y}\left[U\left(W^{\pi}(T)\right)\right],$$

for all $\pi \in \mathcal{A}_t$. The first conclusion in the theorem now follows by observing that the result holds for $t = T$ and $w = 0$.

We have that $\pi_i^*(s, W(s), Y(s))$ are $\mathcal{F}_s$-measurable for $t \leq s \leq T$, since $\pi_i^*(t,w,y)$ is assumed to be a measurable function, $i = 1,...,n$. This, together with the assumptions gives that $\pi^*(s, W(s), Y(s))$ is an admissible control. Further,

$$\max_{\pi \in \mathcal{A}_t} \mathcal{L}^{\pi} v = \mathcal{L}^{\pi^*} v,$$

since π^* is a maximizer. The calculations in the first part of the theorem hold with equality by letting $\pi = \pi^*$, and we get that

$$v(t,w,y) = \mathbb{E}^{t,w,y}\left[U\left(W^{\pi^*}(T)\right)\right] \leq V(t,w,y).$$

This means that

$$v(t,w,y) = V(t,w,y) = \mathbb{E}\left[U\left(W^{\pi^*}(T)\right)\right],$$

for $(t,w,y) \in [0,T] \times \mathbb{R}_{0+}^{m+1}$, since the equality holds for $t = T$ and $w = 0$ by the terminal and boundary conditions (3.5) and (3.6). ■

In the next two sections we verify two Feynman-Kac formulas as solutions to our problem.

6 An explicit solution

In this section we derive a solution to our optimal control problem when

$$U(w) = \tfrac{w^{\gamma}}{\gamma} \quad \gamma \in (0,1).$$

We impose also a condition on the weights $\rho_{i,j,k}$.

Condition 6.1 *We assume that for every $h, i = 1,...,n$, and $k = 1,...,q$, if $\sigma_{h,k}\sigma_{i,k} > 0$, then*

$$\rho_{h,j,k} > 0 \Leftrightarrow \rho_{i,j,k} > 0$$

for every $j = 1,...,m$.

This condition means that each factor has some news processes Y_j associated with it, and these Y_j are part of the volatility for every stock that is affected by the factor. We obtain the solution by constructing a function, along with its associated controls, such that all the assumptions in our verification theorem are satisfied. More precisely, we need to verify that a well-defined function

$$v(t,w,y) \in C^{1,2,1}\left([0,T] \times \mathbb{R}_{+}^{m+1}\right)$$

is a solution to the HJB equation (3.4) with terminal condition (3.5) and boundary condition (3.6), that

$$\sup_{\pi\in\mathcal{A}_0}\int_0^T\int_{0+}^{\infty}\mathbb{E}\left[\left|v\left(t,W^{\pi}\left(t\right),Y\left(t-\right)+z\cdot e_j\right)-v\left(t,W^{\pi}\left(t\right),Y\left(t-\right)\right)\right|\right]l_j\left(dz\right)dt<\infty,$$

and that

$$\sup_{\pi\in\mathcal{A}_0}\int_0^T\mathbb{E}\left[\left\{\pi_i\left(t\right)\sigma_i\left(t\right)W^{\pi}\left(t\right)v_w\left(t,W^{\pi}\left(t\right),Y\left(t\right)\right)\right\}^2\right]dt<\infty,$$

for $i=1,...,n$. Our first step will be to reduce the HJB equation to a related equation that is simpler to handle.

6.1 Reduction of the HJB equation

We conjecture that the solution to the HJB equation (3.4) is of the form

$$v\left(t,w,y\right)=\tfrac{w^{\gamma}}{\gamma}h\left(t,y\right),\quad\left(t,w,y\right)\in\left[0,T\right]\times\mathbb{R}_+^{m+1}$$

where h is some function of t,y. It is obvious that v is continuous in w. If we insert the function v in the HJB equation (3.4), we get the associated equation

$$\begin{aligned}0=h_t\left(t,y\right)+\gamma\max_{\substack{\pi_i\in[a_i,b_i],i=1,\dots,n,\\ c\le\pi_1+\ldots+\pi_n\le d}}&\left\{\sum_{i=1}^{n}\left[\pi_i\left(\mu_i+\beta_i\sigma_i^2-r\right)-\frac{1}{2}\left(1-\gamma\right)\pi_i^2\sigma_{i,0}^2\right]\right.\\ &\left.-\frac{1}{2}\left(1-\gamma\right)\sum_{1\le h,i\le n}\sum_{1\le k\le q}\pi_h\pi_i\sigma_{h,k}\sigma_{i,k}+r\right\}h\left(t,y\right)-\sum_{j=1}^{m}\lambda_j y_j h_{y_j}\left(y,t\right)\\ &+\sum_{j=1}^{m}\lambda_j\int_{0+}^{\infty}\left(h\left(t,w,y+z\cdot e_j\right)-h\left(t,w,y\right)\right)l_j(dz),\end{aligned}\tag{6.1}$$

with the terminal condition

$$h\left(T,y\right)=1,\quad\forall y\in\mathbb{R}_+^m.\tag{6.2}$$

In other words, we have replaced the problem of finding a solution to the HJB equation (3.4) by the presumably simpler problem of finding a solution to Equation (6.1). Our next step is to find a well-defined function h that satisfies Equation (6.1).

6.2 A solution to the reduced HJB equation

In this subsection we define a Feynman-Kac formula and show that it is well-defined and continuously differentiable. We show also that it solves the reduced

HJB equation (6.1).

We set

$$h(t,y)=\mathbb{E}^{t,y}\left[\exp\left(\int_t^T \Pi\left(Y^y(s)\right)ds\right)\right],\quad (t,y)\in[0,T]\times\mathbb{R}_+^m, \tag{6.3}$$

as our candidate solution, where $y=Y(0)$, with vector notation. Note that this function satisfies the terminal condition (6.2) since $h(T,y)=1$. The function $\Pi:\mathbb{R}_+^m\to\mathbb{R}$ is defined as

$$\Pi(y)=\max_{\substack{\pi_i\in[a_i,b_i],i=1,\dots,n,\\ c\le\pi_1+\dots+\pi_n\le d}}\left\{\sum_{i=1}^{n}\left[\pi_i\left(\mu_i+\beta_i\sigma_i^2-r\right)-\frac{1}{2}(1-\gamma)\pi_i^2\sigma_{i,0}^2\right]\right. \tag{6.4}$$
$$\left.-\frac{1}{2}(1-\gamma)\sum_{1\le h,i\le n}\sum_{1\le k\le q}\pi_h\pi_i\sigma_{h,k}\sigma_{i,k}\right\}+r.$$

For technical reasons, we prefer to re-write the function h on a form that is simpler for us to deal with. By the stationarity of Y, we have that

$$h(t,y)=\mathbb{E}^{t,y}\left[\exp\left(\int_t^T\gamma\Pi\left(Y^y(s)\right)ds\right)\right]=\mathbb{E}^{0,y}\left[\exp\left(\int_0^{T-t}\gamma\Pi\left(Y^y(s)\right)ds\right)\right],$$

for $(t,y)\in[0,T]\times\mathbb{R}_+^m$. We define now

$$g(t,y):=h(T-t,y)=\mathbb{E}^{y}\left[\exp\left(\int_0^{t}\gamma\Pi\left(Y^y(s)\right)ds\right)\right].$$

The only difference between the two functions is the direction of the time variable t.

We will now show that g is well-defined.

Lemma 6.1 *Assume Condition 3.1 holds with* $c_j=\frac{\gamma\hat{\pi}}{\lambda_j}\sum_{i=1}^n|\beta_i|\,\omega_{i,j},\ j=1,\dots,m$. *Then*

$$g(t,y)\le\exp\left(kt+\gamma\hat{\pi}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{|\beta_i|\,\omega_{i,j}y_j}{\lambda_j}\right),$$

for $(t,y)\in[0,T]\times\mathbb{R}_+^m$, *and some constant* $k>0$.

Proof. It is straightforward to see that we can write Π as

$$\Pi(y)=\max_{\substack{\pi_i\in[a_i,b_i],i=1,\dots,n,\\ c\le\pi_1+\dots+\pi_n\le d}}\left\{\tau(y)^T\pi-\frac{1}{2}\pi^TQ(y)\pi\right\}+r \tag{6.5}$$

where Q is a positive definite matrix for every $y\in\mathbb{R}_+^m$, and $\tau_i=\left(\mu_i+\beta_i\sigma_i^2-r\right)$.

It follows by the positive definiteness of Q that we can find a constant $\alpha > 0$ such that

$$\left|\Pi\left(y\right)\right| \leq \alpha + \hat{\pi}\sum_{i=1}^{n}\left|\beta_i\right|\sigma_i^2, \tag{6.6}$$

where we recall that $\hat{\pi} = \max\left(1, \left|a_1\right|, ..., \left|a_n\right|, \left|b_1\right|, ..., \left|b_n\right|\right)$. Hence, we have

$$\begin{aligned} g\left(t,y\right) &= \mathbb{E}^y\left[\exp\left(\int_0^t \gamma\Pi\left(Y^y\left(s\right)\right)ds\right)\right] \\ &\leq \mathbb{E}^y\left[\exp\left(\gamma\int_0^t \alpha + \hat{\pi}\sum_{i=1}^{n}\left|\beta_i\right|\sigma_i\left(s\right)^2 ds\right)\right] \\ &\leq e^{\gamma\alpha t}\mathbb{E}^y\left[\prod_{j=1}^{m}\exp\left(\gamma\hat{\pi}\sum_{i=1}^{n}\left|\beta_i\right|\omega_{i,j}\int_0^t Y_j\left(s\right)ds\right)\right]. \end{aligned}$$

Further, we have by the independence of the Y_j, $j = 1, ..., m$, and by Lemma 4.1 that

$$\begin{aligned} g\left(t,y\right) \leq e^{\gamma\alpha t}\prod_{j=1}^{q}\exp\Bigg(&\frac{\gamma\hat{\pi}\sum_{i=1}^{n}\left|\beta_i\right|\omega_{i,j}}{\lambda_j}y_j \\ &+\lambda_j\int_{0+}^{\infty}\left\{\exp\left(\frac{\gamma\hat{\pi}\sum_{i=1}^{n}\left|\beta_i\right|\omega_{i,j}}{\lambda_j}z\right)-1\right\}l_j\left(dz\right)t\Bigg). \end{aligned}$$

The result follows from Condition 3.1. ■

We show now that g is continuously differentiable in y.

Lemma 6.2 *Assume Condition 3.1 holds with* $c_j = \frac{\gamma\hat{\pi}}{\lambda_j}\sum_{i=1}^{n}\left|\beta_i\right|\omega_{i,j}$, $j = 1, ..., m$. *Then*

$$g \in C^{0,1}\left(\left[0,T\right]\times\mathbb{R}_+^m\right).$$

Proof. Define the compact intervals $A_n = \left[\frac{1}{n}, n\right]$, $n = 1, 2, ...$. Let $(t, y) \in [0, T] \times A_n$ and set

$$F\left(t,y\right) = \exp\left(\int_0^t \gamma\Pi\left(Y^y\left(s\right)\right)ds\right).$$

We have then that

$$\frac{\partial F\left(t,y\right)}{\partial y_j} = \left(\frac{\partial}{\partial y_j}\int_0^t \gamma\Pi\left(Y^y\left(s\right)\right)ds\right)e^{\int_0^t \gamma\Pi\left(Y^y\left(s\right)\right)ds},$$

for each $j = 1, ..., m$. We know from Equation (6.5) that for fixed y, Π is a quadratic program with linear constraints. It is well known from optimization theory that this problem has a unique solution, see for example [22, Theorem 2.1]. We can now apply Danskin's theorem, see for example [6, Theorem 4.13, Remark 4.14], to conclude that Π is continuously differentiable. Since $y \in A_n$, we can also conclude that $\nabla_y \Pi$ is bounded on A_n. From [12, Theorem 2.27(b)], and the fact that

$$\lambda_j \int_0^t Y_j(u)du \leq y_j + Z_j(\lambda_j t)$$

we have that

$$\begin{aligned}
\left|\frac{\partial F(t,y)}{\partial y_j}\right| &= \left|\left(\int_0^t \gamma \Pi'_{y_j}\left(Y^y(s)\right)e^{-\lambda_j s}ds\right)e^{\int_0^t \gamma\Pi(Y^y(s))ds}\right| \\
&\leq \left|k_1 t e^{k_2 t}\left(\prod_{j=1}^m \exp\left(\gamma\hat{\pi}\sum_{i=1}^n |\beta_i|\,\omega_{i,j}\int_0^t Y_j(s)\,ds\right)\right)\right| \\
&\leq \left|e^{k_3 t + k_4 \sum_{j=1}^m y_j}\left(\prod_{j=1}^m \exp\left(\gamma\hat{\pi}\sum_{i=1}^n \frac{|\beta_i|\,\omega_{i,j}}{\lambda_j} Z_j(\lambda_j t)\right)\right)\right|.
\end{aligned}$$

But

$$\exp\left(\gamma\hat{\pi}\sum_{i=1}^n \frac{|\beta_i|\,\omega_{i,j}}{\lambda_j} Z_j(\lambda_j t)\right)$$

is integrable by Condition 3.1. Hence, we have that $|\partial F(t,y)/\partial y_j|$ is uniformly bounded in y on A_n, and we can apply [12, Theorem 2.27(b)] to show that $g(t,y) = \mathbb{E}[F(t,y)]$ is differentiable in y on A_n, and that

$$\tfrac{\partial g(t,y)}{\partial y_j} = \mathbb{E}\left[\tfrac{\partial F(t,y)}{\partial y_j}\right], \quad \forall y \in A_n, \quad j = 1, ..., m.$$

Note that $|\partial F(t,y)/\partial y_j|$ is continuous in t and y. We get now by using [12, Theorem 2.27(a)] that $\partial g(t,y)/\partial y_j$ is continuous in $(t,y) \in [0,T] \times A_n$. We conclude the proof by observing that differentiability and continuity are local notions, and that $\lim_{n\to\infty} A_n = \mathbb{R}_+^m$. ■

We show now that g is a classical solution to the related forward problem of Equation (6.1).

Proposition 6.1 *Assume that Condition 3.1 holds with*

$$\begin{aligned}
c_j = \varepsilon + \frac{\gamma}{\lambda_j}\Bigg(& 2\hat{\pi}\sum_{i=1}^n |\beta_i|\,\omega_{i,j} \\
&+ \frac{(1-\gamma)\hat{\pi}^2}{2}\left(\sum_{1\leq h,i\leq n}\sum_{1\leq k\leq q}\sqrt{\rho_{h,j,k}\omega_{h,j}\rho_{i,j,k}\omega_{i,j}} + \sum_{i=1}^n \rho_{i,j,0}\omega_{i,j}\right)\Bigg),
\end{aligned}$$

for $j = 1, ..., m$. *Then* $g(t,\cdot)$ *belongs to the domain of the infinitesimal generator* *of* Y *and*

$$0 = -g_t(t,y) + \gamma\Pi(y)\, g(t,y) - \sum_{j=1}^{m} \lambda_j y_j g_{y_j}(y,t) \tag{6.7}$$
$$+\sum_{j=1}^{m} \lambda_j \int_{0+}^{\infty} \left(g(t,w,y+z\cdot e_j) - g(t,w,y)\right) l_j(dz),$$

for $(t,y) \in [0,T]\times\mathbb{R}_+^m$*, with initial value*

$$g(0,y) = 1, \quad y \in \mathbb{R}_+^m. \tag{6.8}$$

We have also that g_t *is continuous, so that* $g \in C^{1,1}\left([0,T]\times\mathbb{R}_+^m\right)$.

We will need some integrability and continuity conditions on the Lévy measure integrals in order to prove this theorem.

Lemma 6.3 *Assume that Condition 3.1 holds with*

$$c_j = \frac{\gamma}{\lambda_j}\left(\hat{\pi}\sum_{i=1}^{n} |\beta_i|\,\omega_{i,j} + \frac{(1-\gamma)\,\hat{\pi}^2}{2}\left(\sum_{1\le h,i\le n}\sum_{1\le k\le q}\sqrt{\rho_{h,j,k}\omega_{h,j}\rho_{i,j,k}\omega_{i,j}} + \sum_{i=1}^{n}\rho_{i,j,0}\omega_{i,j}\right)\right),$$

for $j = 1, ..., m$*. Then*

$$\sum_{j=1}^{m}\mathbb{E}\left[\int_0^T\int_{0+}^{\infty} \left|g(t,Y(s)+z\cdot e_j) - g(t,Y(s))\right| l_j(dz)\, ds\right] < \infty,$$

for all $(t,y)\in[0,T]\times\mathbb{R}_+^m$*. In addition, the integral*

$$\int_{0+}^{\infty} \left|g(t,y+z\cdot e_j) - g(t,y)\right| l_j(dz)$$

is continuous in $(t,y)\in[0,T]\times\mathbb{R}_+^m$.

Proof. We know that $\sqrt{x}$ is strictly concave and increasing. This gives that

$$\frac{\partial}{\partial y_j}\left(\sigma_{h,k}\sigma_{i,k}\right) \le \frac{\partial}{\partial y_j}\left(\sqrt{\rho_{h,j,k}\omega_{h,j}\rho_{i,j,k}\omega_{i,j}}\, y_j\right),$$

for every h,i,j,k. We have then by inspection that $\nabla_y\Pi$ is bounded and that

$\left|\Pi'_{y_j}\right| \leq c_j\lambda_j/\gamma$, which is due to Condition 6.1. This gives that

$$\begin{aligned}
&\left|g\left(t, y+z\cdot e_j\right)-g\left(t,y\right)\right| \\
&\leq \left|\mathbb{E}^y\left[\exp\left(\int_0^t \gamma\Pi\left(Y^{y+z\cdot e_j}\left(s\right)\right)ds\right)-\exp\left(\int_0^t \gamma\Pi\left(Y^y\left(s\right)\right)ds\right)\right]\right| \\
&\leq \mathbb{E}^y\left[\exp\left(\int_0^t \gamma\Pi\left(Y^y\left(s\right)\right)+c_j\lambda_j z e^{-\lambda_j s}ds\right)-\exp\left(\int_0^t \gamma\Pi\left(Y^y\left(s\right)\right)ds\right)\right] \\
&\leq \exp\left(k_1 t+\gamma\hat{\pi}\sum_{j=1}^m\sum_{i=1}^n \frac{\left|\beta_i\right|\omega_{i,j}y_j}{\lambda_j}\right)\left(\exp\left(c_j z\right)-1\right),
\end{aligned}$$

for $k_1 > 0$ by Lemma 6.1. We have then that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\int_{0+}^\infty \left|g\left(t,Y\left(s\right)+z\cdot e_j\right)-g\left(t,Y\left(s\right)\right)\right| l_j\left(dz\right)ds\right] \\
&\leq \int_0^T e^{k_1 t+k_2\sum_{j=1}^m y_j}\mathbb{E}\left[\exp\left(\gamma\hat{\pi}\sum_{j=1}^m\sum_{i=1}^n \frac{\left|\beta_i\right|\omega_{i,j}Z_j\left(\lambda_j s\right)}{\lambda_j}\right)\right]ds \\
&\qquad \int_{0+}^\infty \left(\exp\left(c_j z\right)-1\right) l_j\left(dz\right),
\end{aligned}$$

for $k_2 > 0$, since

$$Y_j^{y_j}\left(s\right) \leq y_j + Z_j\left(\lambda_j s\right).$$

The integral with respect to the Lévy measure is finite by assumption. We recall from Equation (3.2) that

$$\mathbb{E}\left[e^{aZ_j(\lambda_j t)}\right] = \exp\left(\lambda_j t\int_{0+}^\infty \left(e^{az}-1\right) l_j\left(dz\right)\right),$$

as long as $a \leq c_j$ with c_j from Condition 3.1 holds. Hence, the expectation part is finite as well, which concludes the first part of the proof. The second part can be proved by similar techniques. ■

We are now ready to prove Proposition 6.1.

Proof Proposition 6.1. We observe that the assumptions in Lemmas 6.1, 6.2, and 6.3 are satisfied. This gives that if g solves Equation (6.7) then g_t is continuous in $(t,y) \in (0,T)\times\mathbb{R}_+^m$, with continuous extensions to $t=0$ and $t=T$.

We know that Y_j $j=1,...,m$, are adapted, cádlág, and have paths of finite variation on compacts since $Y_j\left(t\right) \leq Z_j\left(\lambda_j t\right)$. We have from [25, Theorem 26] that Y_j $j=1,...,m$, are quadratic pure jump semimartingales. Since $y \to g\left(t,y\right)$ is continuously differentiable, Itô's formula (see [25, Theorem 33]) gives that the mapping $s \to g(t, Y(s))$ is a semimartingale with dynamics

$$g(t, Y(s)) = g(t, y) + \sum_{j=1}^{m} \lambda_j \int_0^s Y_j(u)\, g_{y_j}(t, Y(u))\, du$$
$$+ \sum_{j=1}^{m} \int_0^s \int_{0+}^{\infty} \left(g(t, Y(u-) + z \cdot e_j) - g(t, Y(u-))\right) N_j(\lambda_j du, dz),$$

where N_j is the Poisson random measure in the Lévy-Khintchine representation of $Z_j,\ j = 1, ..., m$. If we take expectation on both sides and apply Fubini's theorem (see [12, Theorem 2.37]), we get

$$\frac{\mathbb{E}\left[g(t, Y(s)) - g(t, y)\right]}{s}$$
$$= -\sum_{j=1}^{m} \frac{\lambda_j}{s} \int_0^s \mathbb{E}\left[Y_j(u)\, g_{y_j}(t, Y(u))\right] du$$
$$+ \sum_{j=1}^{m} \frac{\lambda_j}{s} \int_0^s \mathbb{E}\left[\int_{0+}^{\infty} g(t, Y(u-) + z \cdot e_j) - g(t, Y(u-))\, l_j(dz)\, du\right].$$

We see from Lemma 6.3 that $\mathbb{E}\left[Y_j(u)\, g_{y_j}(t, Y(u))\right] \in L^1([0, s], Leb)$, since $\mathbb{E}\left[g(t, Y(s))\right] < \infty$ by Lemma 6.1 and

$$Y_j(t) \le y_j + Z_j(\lambda_j t), \quad \forall t \in [0, T].$$

Hence, if we note that the $Y_j\ j = 1, ..., m$, are cádlág, that $y \mapsto g(t, y)$ is continuously differentiable, and that

$$\int_{0+}^{\infty} g(t, y + z \cdot e_j) - g(t, y)\, l_j(dz)$$

is continuous, by letting $s \downarrow 0$ we get by the Fundamental Theorem of Calculus for Lebesque Integrals (see [12, Theorem 3.35]) that $g(t, \cdot)$ is in the domain of the infinitesimal generator of Y. We denote the infinitesimal generator by $\mathcal{G}$, which gives that

$$\mathcal{G}g(t, y) = -\sum_{j=1}^{m} \lambda_j y_j g_{y_j}(t, y) + \sum_{j=1}^{m} \lambda_j \int_{0+}^{\infty} \left(g(t, y + z \cdot e_j) - g(t, y)\right) l_j(dz).$$

The Markov property of Y together with the law of total expectation yields

$$\mathbb{E}\left[g(t, Y(s))\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left[e^{\int_0^t \gamma \Pi\left(Y^{Y^y(s)}(u)\right) du}\right]\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\mathbb{E}\left[e^{\int_0^t \gamma \Pi(Y^y(u+s))du} | \mathcal{F}_s\right]\right] \\
&= \mathbb{E}\left[e^{\int_s^{t+s} \gamma \Pi(Y^y(u))du}\right] \\
&= \mathbb{E}\left[e^{\int_0^{t+s} \gamma \Pi(Y^y(u))du} e^{-\int_0^s \gamma \Pi(Y^y(u))du}\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{\mathbb{E}\left[g\left(t, Y\left(s\right)\right) - g\left(t, y\right)\right]}{s} \\
&= \frac{1}{s}\mathbb{E}\left[e^{\int_0^{t+s} \gamma \Pi(Y^y(u))du} e^{-\int_0^s \gamma \Pi(Y^y(u))du} - e^{\int_0^{t} \gamma \Pi(Y^y(u))du}\right] \\
&= \frac{1}{s}\mathbb{E}\left[e^{\int_0^{t+s} \gamma \Pi(Y^y(u))du} e^{-\int_0^s \gamma \Pi(Y^y(u))du} - e^{\int_0^{t+s} \gamma \Pi(Y^y(u))du}\right] \\
&\quad + \frac{1}{s}\left\{\mathbb{E}\left[e^{\int_0^{t+s} \gamma \Pi(Y^y(u))du}\right] - \mathbb{E}\left[e^{\int_0^{t} \gamma \Pi(Y^y(u))du}\right]\right\} \\
&= \mathbb{E}\left[e^{\int_0^{t+s} \gamma \Pi(Y^y(u))du} \frac{1}{s}\left\{e^{-\int_0^s \gamma \Pi(Y^y(u))du} - 1\right\}\right] \\
&\quad + \frac{g\left(t+s, y\right) - g\left(t, y\right)}{s}.
\end{aligned}
$$

For simplicity of calculations, we assume that $t + s \in [0, T]$. We can verify that

$$
\begin{aligned}
&e^{\int_0^{t+s} \gamma \Pi(Y^y(u))du} \frac{1}{s}\left\{e^{-\int_0^s \gamma \Pi(Y^y(u))du} - 1\right\} \\
&\to -\gamma \Pi\left(y\right) e^{\int_0^{t} \gamma \Pi(Y^y(u))du},
\end{aligned}
$$

as $s \downarrow 0$. We need to show now that we can interchange limit and integration. We define the function

$$
f\left(s\right) = e^{-\int_0^s \gamma \Pi(Y^y(u))du}.
$$

From the mean value theorem and the linear growth assumption on Π we get that

$$
\begin{aligned}
&\frac{1}{s}\left|f\left(s\right) - f\left(0\right)\right| \\
&\leq \frac{1}{s}\sup_{v \in [0,s]}\left|f'_+\left(v\right)\right| s \\
&= \sup_{v \in [0,s]}\left|\gamma \Pi\left(Y^y\left(v\right)\right) e^{-\int_0^v \gamma \Pi(Y^y(u))du}\right| \\
&\leq \sup_{v \in [0,s]}\left|\left(\gamma \alpha + \gamma \hat{\pi}\sum_{i=1}^n \left|\beta_i\right| \sigma_i^y\left(v\right)^2\right)\right| e^{\gamma \int_0^T \alpha + \hat{\pi}\sum_{i=1}^n \left|\beta_i\right| \sigma_i^y(u)^2 du}.
\end{aligned}
$$

Since each Z_j is a non-decreasing process,

$$\sup_{v\in[0,T]}\left(\sum_{i=1}^{n}|\beta_i|\,\sigma_i^y(v)^2\right)$$
$$\leq\sum_{i=1}^{n}|\beta_i|\,\sigma_i^2+\sum_{i=1}^{n}\sum_{j=1}^{m}|\beta_i|\,\omega_{i,j}Z_j(\lambda_j T),$$

which implies

$$e^{\int_0^{t+s}\gamma\Pi(Y^y(u))du}\frac{1}{s}\left\{e^{-\int_0^s\gamma\Pi(Y^y(u))du}-1\right\}$$
$$\leq\left(\sum_{i=1}^{n}|\beta_i|\,\sigma_i^2+\sum_{i=1}^{n}\sum_{j=1}^{m}|\beta_i|\,\omega_{i,j}Z_j(\lambda_j T)\right)$$
$$\times e^{2\gamma\int_0^T\alpha+\hat{\pi}\sum_{i=1}^{n}|\beta_i|\sigma_i^y(u)^2du}.$$

Note that there exists a positive constant k_ε such that $z\leq k_\varepsilon e^{\varepsilon z}$, for all $z\geq 0$. We have by the independence of the Y_j, Lemma 4.1 and the Dominated Convergence Theorem that

$$\mathbb{E}\left[e^{\int_0^{t+s}\gamma\Pi(Y^y(u))du}\frac{1}{s}\left\{e^{-\int_0^s\gamma\Pi(Y^y(u))du}-1\right\}\right]$$
$$=-\gamma\Pi(y)\,g(t,y).$$

We can show analogously that g_t exists. We have then that

$$\mathcal{G}g(t,y)=-\gamma\Pi(y)\,g(t,y)+g_t(t,y),$$

which concludes the proof. ■

We have found a function

$$v(t,w,y)\in C^{1,2,1}\left([0,T]\times\mathbb{R}_+^{m+1}\right)$$

that is a solution to the HJB equation (3.4) with terminal condition (3.5) and boundary condition (3.6). All that is left for us to do is to show that the remaining conditions in the verification theorem 5.1 hold.

6.3 Verification of the explicit solution

We show in this subsection that

$$\sup_{\pi\in\mathcal{A}_0}\int_0^T\int_{0+}^{\infty}\mathbb{E}\left[\left|v\left(t,W^\pi(t),Y(t-)+z\cdot e_j\right)-v\left(t,W^\pi(t),Y(t-)\right)\right|\right]l_j(dz)\,dt<\infty, \tag{6.9}$$

and that

$$\sup_{\pi\in\mathcal{A}_0}\int_0^T \mathbb{E}\left[\left\{\pi_i\left(t\right)\sigma_i\left(t\right)W^{\pi}\left(t\right)v_w\left(t,W^{\pi}\left(t\right),Y\left(t\right)\right)\right\}^2\right]dt<\infty, \tag{6.10}$$

for $i=1,...,n$. Once this is done, all the conditions from the verification theorem 5.1 are satisfied for our candidate solution v. Hence we have solved our problem of finding the optimal value function. Note that the conditions we verify are slightly modified versions of Equations (6.9) and (6.10), since our conjectured solution v is of the form

$$v\left(t,w,y\right)=\frac{w^{\gamma}}{\gamma}h\left(t,y\right).$$

Lemma 6.4 *Assume that Condition 3.1 holds with*

$$c_j=\frac{\gamma}{\lambda_j}\left(4\hat{\pi}^2\sum_{i=1}^{n}\left(\left|\beta_i\right|+2\gamma\right)\omega_{i,j}\right.$$
$$\left.+\frac{\left(1-\gamma\right)\hat{\pi}^2}{2}\left(\sum_{1\leq h,i\leq n}\sum_{1\leq k\leq q}\sqrt{\rho_{h,j,k}\omega_{h,j}\rho_{i,j,k}\omega_{i,j}}+\sum_{i=1}^{n}\rho_{i,j,0}\omega_{i,j}\right)\right),$$

for $j=1,...,m$. Then

$$\sup_{\pi\in\mathcal{A}_0}\int_0^T\int_{0+}^{\infty}\mathbb{E}\left[W^{\pi}\left(t\right)^{\gamma}\left|h\left(t,Y\left(t-\right)+z\cdot e_j\right)-h\left(t,Y\left(t-\right)\right)\right|\right]l_j\left(dz\right)dt<\infty,$$

for all $\pi\in\mathcal{A}_0$.

Proof. Since $\nabla_y\Pi$ is bounded and $\left|\Pi'_{y_j}\right|\leq c_j\lambda_j/\gamma$, we have analogous to Lemma 6.1 that

$$\begin{aligned}&\int_0^T\int_{0+}^{\infty}\mathbb{E}\left[W^{\pi}\left(t\right)^{\gamma}\left|h\left(t,Y\left(t-\right)+z\cdot e_j\right)-h\left(t,Y\left(t-\right)\right)\right|\right]l_j\left(dz\right)dt\\ &\leq\int_{0+}^{\infty}\left(\exp\left(c_jz\right)-1\right)l_j\left(dz\right)\\ &\quad\times\int_0^T e^{k_1t+k_2\sum_{j=1}^{m}y_j}\\ &\quad\quad\times\mathbb{E}\left[W^{\pi}\left(t\right)^{\gamma}\exp\left(\gamma\hat{\pi}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{\left|\beta_i\right|\omega_{i,j}Z_j\left(\lambda_jt\right)}{\lambda_j}\right)\right]dt.\end{aligned}$$

If we apply Hölder's inequality with $p = q = 2$ to the expectation part, we get

$$\begin{aligned}&\mathbb{E}\left[W^{\pi}(t)^{\gamma}\exp\left(\gamma\hat{\pi}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{|\beta_i|\,\omega_{i,j}Z_j(\lambda_j t)}{\lambda_j}\right)\right]\\&=\mathbb{E}\left[W^{\pi}(t)^{2\gamma}\right]^{1/2}\mathbb{E}\left[\exp\left(2\gamma\hat{\pi}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{|\beta_i|\,\omega_{i,j}Z_j(\lambda_j t)}{\lambda_j}\right)\right]^{1/2}.\end{aligned}$$

We recall from Equation (3.2) and Lemma 4.2 that both expectations are finite.
■

Lemma 6.5 *Assume that Condition 3.1 holds with*

$$\begin{aligned}c_j=\frac{\gamma}{\lambda_j}&\left(8\hat{\pi}^2\sum_{i=1}^{n}(|\beta_i|+4\gamma)\,\omega_{i,j}\right.\\&\left.+\frac{(1-\gamma)\,\hat{\pi}^2}{2}\left(\sum_{1\le h,i\le n}\sum_{1\le k\le q}\sqrt{\rho_{h,j,k}\omega_{h,j}\rho_{i,j,k}\omega_{i,j}}+\sum_{i=1}^{n}\rho_{i,j,0}\omega_{i,j}\right)\right),\end{aligned}\tag{6.11}$$

for $j = 1, ..., m$. *Then*

$$\int_0^T\mathbb{E}\left[\{\pi_i(t)\,\sigma_i(t)\,W^{\pi}(t)^{\gamma}\,h(t,Y(t))\}^2\right]dt<\infty.$$

Proof. It follows from the definitions of h and g that they have the same growth. We have then by Lemma 6.1 that

$$\begin{aligned}&\int_0^T\mathbb{E}\left[\{\pi_i(t)\,\sigma_i(t)\,W^{\pi}(t)^{\gamma}\,h(t,Y(t))\}^2\right]dt\\&\le\hat{\pi}^2k_{\varepsilon}e^{kT}\int_0^T\mathbb{E}\left[W^{\pi}(t)^{2\gamma}\exp\left(\left(2\gamma\hat{\pi}+\frac{\varepsilon}{2}\right)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{|\beta_i|\,\omega_{i,j}Y_j}{\lambda_j}\right)\right]dt,\end{aligned}$$

since we can find positive constants k_ε such that $\sigma_i^2 \le k_\varepsilon \exp\left(\frac{\varepsilon}{2}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{|\beta_i|\omega_{i,j}Y_j}{\lambda_j}\right)$, for every $i = 1, ..., n$. We can now apply Hölder's inequality with $p = q = 2$. This gives that

$$\begin{aligned}&\mathbb{E}\left[W^{\pi}(t)^{2\gamma}\exp\left(\left(2\gamma\hat{\pi}+\frac{\varepsilon}{2}\right)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{|\beta_i|\,\omega_{i,j}Y_j}{\lambda_j}\right)\right]\\&\le\mathbb{E}\left[W^{\pi}(t)^{4\gamma}\right]^{1/2}\mathbb{E}\left[\exp\left((4\gamma\hat{\pi}+\varepsilon)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{|\beta_i|\,\omega_{i,j}Y_j}{\lambda_j}\right)\right]^{1/2},\end{aligned}$$

which are both finite by the assumptions, for ε sufficiently small. ■

To conclude, we assume that Condition 3.1 holds with c_j as in Equation (6.11). This ensures that all our results are valid for ε sufficiently small. Further, we note that there exist measurable functions $\pi_i^*(y)$ that are the maximizers for the max-operator in Equation (3.4). These optimal allocation strategies are the solution of the quadratic program of Equation (6.4). This gives that Equation (3.1) admits a unique positive solution W^{π^*} by [25, Ch. 2, Thm. 37]. Hence, all the assumptions in Theorem 5.1 are satisfied, and we have solved the problem.

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The estimation of a stochastic volatility model based on the number of trades

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Abstract

This paper presents statistical methods for fitting the stochastic volatility model of Barndorff-Nielsen and Shephard [5] to data. We also consider the factor model in [13], which is an n-stock extension of the model in [5]. We argue that the straightforward approach to estimating the Generalized Hyperbolic (GH) distribution from financial return data is inappropriate because the GH-distribution is "almost" overparameterized. To overcome this problem, we verify that we can divide the centered returns with a constant times the number of trades in a trading day to obtain normalized returns that are *i.i.d.* and $N(0,1)$. It is a key theoretical feature of the framework in [5] that the centered returns divided by the volatility are also *i.i.d.* and $N(0,1)$. This suggests that we identify the daily number of trades with the volatility, and model the number of trades with the model in [5]. Hence, we get an economical interpretation of the non-Gaussian Ornstein-Uhlenbeck processes that define the stochastic volatility in the model, but also stable parameter estimates. Further, our approach is easier to implement than the quadratic variation method, and requires much less data. An illustrative statistical analysis is performed on data from the OMX Stockholmsbörsen. The results indicate a good model fit.

1 Introduction

It is a well-known empirical fact that many characteristics of stock price data are not captured by the classical Black and Scholes model. Many alternatives that seek to overcome these flaws have been proposed. A common approach is to assume that the volatility is stochastic. Barndorff-Nielsen and Shephard

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[5] model the stochastic volatility in asset price dynamics as a weighted sum of non-Gaussian Ornstein-Uhlenbeck processes of the form

$$dy = -\lambda y(t)\,dt + dz(t),$$

where z is a subordinator and $\lambda > 0$. It turns out that this framework allows us to capture several of the observed features in financial time series, such as semi-heavy tails, volatility clustering, and skewness. Further, it is analytically tractable, see for example [3], [6], [12], [13], [14], and [15]. A drawback with this volatility model has been the difficulty to estimate the parameters of the model from data. Perhaps the most intuitive approach to do this is to analyze the quadratic variation of the stock price process, see [4]. This makes it in theory possible to recover the volatility process from observed stock prices. However, in reality the model does not hold on the microscale, and even if is only regarded as an approximation this approach still requires very much data. In addition, it is hard to implement in a statistically sound way due to peculiarities in intraday data. For example, the stock market is closed at night, and there is more intense trading on certain hours of the day. None of these features are present in the mathematical model. We therefore propose a different strategy that only uses daily data.

This paper builds on a model proposed in [13]. The model in [13] is a further development of the n-stock stochastic volatility model in [12], which in turn was an extension of [5]. In [12], the stocks are assumed to share some of the OU processes of the volatility. This is given the interpretation that the stocks react to the same *news.* The model is primarily intended for stocks that are dependent, but not too dependent, such as stocks from different branches of industry. This model retains the features of the univariate model of [5]. In addition, the model requires little data since no volatility matrix has to be estimated, and it gives explicit optimal portfolio strategies. The disadvantage is that to obtain strong correlations between the returns $R_i(t) = \log(S_i(t)/S_i(t-1))$ of different stocks S_i, we need the marginal distributions to be very skew. This might not fit data. The paper [13] makes an attempt to remedy this. In [13], the stochastic volatility matrix is defined implicitly by a factor structure. The idea of a factor structure is that the diffusion components of the stocks contain one Brownian motion that is unique for each stock, and a few Brownian motions that all stocks share. It is shown that this dependence generates covariance between the returns of different stocks, and that we can obtain strong correlations without affecting the marginal distributions of the returns of the stocks. We have chosen to not use a full explicit stochastic volatility matrix, with n Brownian motions in the diffusion components of all n stocks, since the statistical estimation of such models quickly becomes infeasible as the number of assets grows. To characterize dependence by factors is common in discrete time finance, see for example [7] and [10].

In this paper we develop statistical methods for estimating the models in [5] and [13] from data. The models are first discretized under the assumption that

$$\int_{t-\Delta}^{t} \sigma(s)\, dB(s) \approx \sigma(t)\, \varepsilon,$$

for $\varepsilon \in N(0,1)$. We argue that one can not estimate the Generalized Hyperbolic (GH) distribution directly from financial return data, due to that the GH-distribution is "almost" overparameterized. We are inspired by [1] to verify that we can divide the centered returns by a constant times the number of trades in a trading day to get a sample that is *i.i.d.* and $N(0,1)$. It is an important feature of the stochastic volatility framework in [5] that the centered returns divided by the volatility are also *i.i.d.* and $N(0,1)$. This suggests that we identify the daily number of trades with the volatility, and model the number of trades within the model in [5]. Our approach gives more stable parameter estimates than if we analyzed only the marginal distribution of the returns directly with the standard maximum likelihood approach. Further, it is easier to implement than the quadratic variation method, and requires much less data. It also implies an economical interpretation of the daily average stochastic volatility, and it hints that we can view the continuous time volatility as the *intensity* with which the trades arrive. A statistical analysis is performed on data from the OMX Stockholmsbörsen. The results indicate a good model fit.

In Section 2 we present the continuous time model. We then introduce the discrete time analogue in Section 3. Here, we also discuss our data set, the stationarity assumptions, and the GH-distribution. The data analysis is presented in Section 4. The section also contains a discussion.

2 The continuous time model

For $0 \leq t < \infty$, we assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{0 \leq t < \infty}$ satisfying the usual conditions. Introduce m independent subordinators Z_j. Recall that a subordinator is defined to be a Lévy process that takes values in $[0, \infty)$, which implies that its sample paths are increasing. We assume that we use the cádlág version of Z_j, and denote the Lévy measures of Z_j by $l_j(dz)$, $j = 1, ..., m$.

We present now a n-stock extension of the model proposed by Barndorff-Nielsen and Shephard in [5]. It is a generalization of that in [12]. Take $n+q$ independent Brownian motions B_i. Denote by Y_j, $j = 1, ..., m$, the OU stochastic processes whose dynamics are governed by

$$dY_j(t) = -\lambda_j Y_j(t) dt + dZ_j(\lambda_j t), \tag{2.1}$$

where $\lambda_j > 0$ denotes the *rate of decay*. The unusual timing of Z_j is chosen so that the marginal distribution of Y_j will be unchanged regardless of the value of λ_j. The filtration

$$\mathcal{F}_t = \sigma\left(B_1(t), ..., B_{n+q}(t), Z_1(\lambda_1 t), ..., Z_m(\lambda_m t)\right)$$

is used to make the OU processes and the Wiener processes simultaneously adapted.

We follow the interpretations in [12] and [13], and view the processes Y_j, $j = 1, ..., m$, as *news processes* associated to certain events, and the jump times of Z_j, $j = 1, ..., m$ as *news* or *the release of information* on the market. The stationary process Y_j can be represented as

$$Y_j(t) = \int_{-\infty}^{0} \exp(s)\, dZ_j(\lambda_j t + s), \quad t \geq 0.$$

It can also be written as

$$Y_j(t) = y_j e^{-\lambda_j t} + \int_0^t e^{-\lambda_j (t-s)} dZ_j(\lambda_j s), \quad t \geq 0, \tag{2.2}$$

where $y_j := Y_j(0)$, and y_j has the stationary marginal distribution of the process and is independent of $Z_j(t) - Z_j(0)$, $t \geq 0$. Note in particular that if $y_j \geq 0$, then $Y_j(t) > 0$ $\forall t \geq 0$, since Z_j is non-decreasing. We set $Z_j(0) = 0$, $j = 1, ..., m$, and write $y := (y_1, ..., y_m)$. The volatility processes σ_i^2 are defined as

$$\sigma_i(t)^2 := \sum_{j=1}^{m} \omega_{i,j} Y_j(t), \quad t \geq 0, \tag{2.3}$$

where $\omega_{i,j} \geq 0$ are weights summing to one for each i. Further,

$$\sigma_{i,k}(t)^2 := \sum_{j=1}^{m} \rho_{i,j,k} \omega_{i,j} Y_j(t), \quad t \geq 0,$$

where $\rho_{i,j,k} \in [0, 1]$ are chosen so that

$$\sigma_i(t)^2 = \sum_{k=0}^{q} \sigma_{i,k}(t)^2, \quad t \geq 0. \tag{2.4}$$

The processes $\sigma_{i,k}^2$ are the volatilities for factor k for each stock i. Define the stocks S_i, $i = 1, ..., n$, to have the dynamics

$$dS_i(t) = S_i(t)\left(\left(\mu_i + \beta_i \sigma_i(t)^2\right) dt + \sigma_{i,0}(t)\, dB_i(t) + \sum_{k=1}^{q} \sigma_{i,k}(t) dB_{n+k}(t)\right).$$

Here μ_i are the *constant mean rates of return*, and β_i are *skewness* parameters. The Brownian motions B_{n+k}, $k = 1, ..., q$, are referred to as the *factors,* and we will call $\mu_i + \beta_i \sigma_i(t)^2$ the *mean rate of return* for stock i at time t. Note that if we choose $n = 1$, we are back in the univariate model of [5]. The stock price dynamics gives us the stock price processes

$$\begin{aligned} S_i(t) = S_i(0) \exp\Bigg(&\int_0^t \left(\mu_i + \left(\beta_i - \tfrac{1}{2}\right) \sigma_i(s)^2\right) ds \\ &+ \int_0^t \sigma_{i,0}(s)\, dB_i(s) + \sum_{k=1}^{q} \int_0^t \sigma_{i,k}(s) dB_{n+k}(s) \Bigg). \end{aligned} \tag{2.5}$$

This stock price model allows for the increments of the *returns*

$$R_i^c(t) := \log\left(S_i(t) / S_i(0)\right), \quad i = 1, ..., n,$$

to have semi-heavy tails and for both volatility clustering and skewness. In addition, the increments of the returns R_i^c are stationary since

$$R_i^c(s) - R_i^c(t) = \log\left(\frac{S_i(s)}{S_i(0)}\right) - \log\left(\frac{S_i(t)}{S_i(0)}\right) = \log\left(\frac{S_i(s)}{S_i(t)}\right) =^{\mathcal{L}} R_i^c(s-t), \tag{2.6}$$

where " $=^{\mathcal{L}}$ " denotes equality in law.

The underlying idea of this model is to capture the dependence between stocks in two ways: First, by letting the stocks share the news processes Y_j, $j = 1, ..., m$, we allow for the volatilities of different stocks to be similar. Second, the diffusion components of the stocks contain one Brownian motion that is unique for each stock, and a few Brownian motions that all stocks share. It was shown in [13] that this allows us to obtain strong correlations between the returns for different stocks without affecting their marginal distributions. The model has also the feature of preserving the qualities of the univariate model. In addition, the number of factors can be chosen to be a lot less than the number of stocks. This makes it possible to consider more assets than if we modelled the covariance structure by an explicit volatility matrix, with n Brownian motions in the diffusion components of all n stocks. The reason is that the number of parameters to be estimated is smaller. The idea to characterize dependence by factors is of course not new, see for example [7] and [10].

3 The discrete time model and the data

In this section we introduce a discrete time version of our model, and present the data that we use in our analysis. Further, the obvious approach to the problem of estimating the model parameters is discussed. Finally we give an alternative approach to analyzing the data.

3.1 The discrete time model

Assume that we observe returns

$$R_i^c(\Delta), R_i^c(2\Delta) - R_i^c(\Delta), ..., R_i^c(d\Delta) - R_i^c((d-1)\Delta),$$

for stock $i = 1, ..., n$, with R^c defined by Equations (2.5) and (2.6). Here Δ is one time unit, and $d+1$ is the number of consecutive observations. We assume from now on that the time units are chosen so that $\Delta = 1$.

The approximation to assume that

$$\int_{t-\Delta}^{t} \sigma(s)\, dB(s) \approx \sigma(t)\,\varepsilon, \tag{3.1}$$

with $\varepsilon \in N(0,1)$ may be reasonable unless some λ_j are large so that the volatility processes will be volatile. This motivates the following approximate discrete time version of the model in Equations (2.5) and (2.6),

$$R_i(t) = \mu_i + \beta_i \sigma_i^2(t) + \sigma_{i,0}(t)\varepsilon_i(t) + \sum_{k=1}^{q} \sigma_{i,k}(t)\varepsilon_{n+k}(t), \tag{3.2}$$

where $t = 1, 2, ...,$ and $\varepsilon_i(\cdot)$ are sequences of independent $N(0,1)$ variables, $i = 1, ..., n$.

In this paper we will only work with the discrete time volatilities $\sigma_i^2(1), ..., \sigma_i^2(d)$, and not with the underlying news processes Y_j. This is a reasonable restriction, since we have little hope of estimating the news processes Y_j accurately. Therefore, we assume for simplicity that $\rho_{i,j,k}$ are such that the volatilities for each factor are fractions of the total volatility. That is, we require that for some $\varphi_{i,k} \geq 0$, $k = 0, 1, ..., q$, we have $\sigma_{i,k} = \varphi_{i,k}\sigma_i$, such that Equation (2.4) holds for $i = 1, ..., n$. This might not be totally realistic, but is necessary from an applied perspective due to our inability to estimate the Y_j. Inserting this into Equation (3.2) gives the discrete time model

$$R_i(t) = \mu_i + \beta_i \sigma_i^2(t) + \sqrt{1 - \sum_{k=1}^{q} \varphi_{i,k}^2}\,\sigma_i(t)\varepsilon_i(t) + \sum_{k=1}^{q} \varphi_{i,k}\sigma_i(t)\varepsilon_{n+k}(t), \tag{3.3}$$

where we have used that

$$\varphi_{i,0}^2 = 1 - \sum_{k=1}^{q} \varphi_{i,k}^2.$$

It is important to be able to estimate the dependence structure of the discrete time model from data. Since the volatility is stochastic we can not hope for constant correlation. However, the conditional correlation between the returns of different stocks turns out to be constant. In fact, for two stocks R_i $i = 1, 2$, we have that

$$\begin{aligned}
&Corr\left(R_1(t)\,R_2(t)\middle|\,\sigma_1(t), \sigma_2(t)\right) \\
&= \frac{Cov\left(R_1(t)\,R_2(t)\middle|\,\sigma_1(t), \sigma_2(t)\right)}{\sigma_1(t)\,\sigma_2(t)} \\
&= \mathbb{E}\left[\left(\sqrt{1 - \sum_{k=1}^{q} \varphi_{1,k}^2}\,\sigma_1(t)\varepsilon_1(t) + \sum_{k=1}^{q} \varphi_{1,k}\sigma_1(t)\varepsilon_{n+k}(t)\right)\right. \\
&\left.\left(\sqrt{1 - \sum_{k=1}^{q} \varphi_{2,k}^2}\,\sigma_2(t)\varepsilon_2(t) + \sum_{k=1}^{q} \varphi_{2,k}\sigma_2(t)\varepsilon_{n+k}(t)\right|\sigma_1(t), \sigma_2(t)\right] \\
&\times \frac{1}{\sigma_1(t)\,\sigma_2(t)} \\
&= \sum_{k=1}^{q} \varphi_{1,k}\varphi_{2,k}.
\end{aligned}$$

In general the conditional correlation matrix is given by

$$\Phi\Phi' + \Psi, \tag{3.4}$$

where $\Phi = (\varphi_{i,k})_{i,k=1}^{i=n,k=q}$, and Ψ is a diagonal matrix where the non-zero elements are $1 - \sum_{k=1}^{q} \varphi_{i,k}^2$. This result implies that if we find a good model for the volatility $\sigma^2(\cdot)$, we can apply standard factor analysis to the *i.i.d.* normalized returns $\xi_i(\cdot)$. This gives us the parameters $\varphi_{i,k}$, and hence also the implicit constant conditional correlation matrix that describes the correlation between the returns. Constant conditional correlation is a common feature in discrete time finance. A fundamental paper is [7].

Next, we discuss the data to be analyzed.

3.2 The stock market data

We consider five different stocks from the OMX Stockholmsbörsen. The stocks are Ericsson B, Volvo B, SKF B, Atlas Copco B, and AstraZeneca, which are given indices $i = 1, ..., 5$. These five are all large companies and are also among the most traded stocks at the exchange. We have chosen to analyze data from the time period August 1, 2003 to June 1, 2004. This choice was made primarily for two reasons. First the time series is long enough to give a fairly large amount of data, and still short enough to make it reasonable to assume stationarity. This second reason is also supported by economical considerations. The OMX Stockholmsbörsen as a whole decreased in value three years in a row from spring 2000 to spring 2003. During this time period the company Ericsson, which was the most influential stock on the exchange, had been close to bankruptcy. However, Ericsson survived and its stock started to increase in value, and in the spring of 2003 so did the exchange as a whole. We have chosen to start our period of observation after the summer of 2003. The reason for this is that by then the long period of decreasing stock prices was somewhat distant in time. We stop right before the summer of 2004. The summers are avoided since we suspect that they will give us problems with non-stationarity due to less activity on the exchange. This choice of time period of observation gives $d = 208$.

The data we have used is the daily closing prices for each stock, and the cumulative number of trades on each day. The time series were kindly given to us by SIX - Stockholm Information Exchange.

3.3 Limitations of the GH-distribution

In this section we discuss a natural way to estimate the model parameters, and why this approach is not successful. We treat only the univariate version of our model, that is $n = 1$. The Normalized Inverse Gaussian distribution (NIG) is used to illustrate the discussion. However, the reason that the approach fails is valid for the more general GH-distribution, too.

The NIG-distribution has been shown to fit financial return data well, see e.g. [2], [5], and [16]. The NIG-distribution has parameters $\alpha = \sqrt{\beta^2 + \gamma^2}$, β,

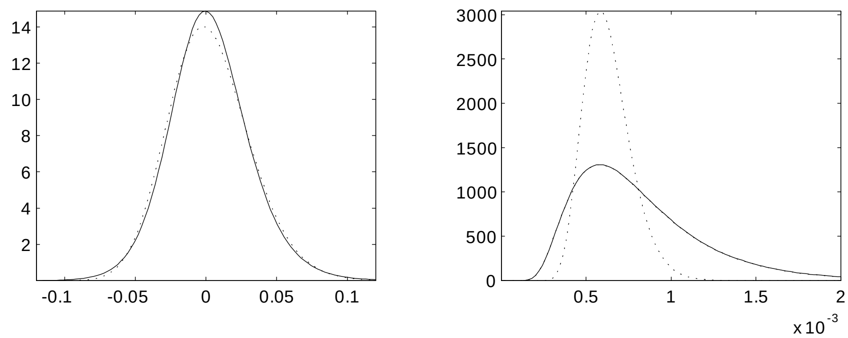

Figure 3.1: *Left:* NIG densities for Ericsson. Parameter set 1: Solid line. Parameter set 2: Dotted line. *Right:* IG densities for Ericsson. Parameter set 1: Solid line. Parameter set 2: Dotted line.

μ, and δ, and its density function is

$$f_{NIG}(x;\alpha,\beta,\mu,\delta)$$
$$= \frac{\alpha}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}-\beta\mu\right) q\left(\frac{x-\mu}{\delta}\right)^{-1} K_1\left(\delta\alpha q\left(\frac{x-\mu}{\delta}\right)\right) e^{\beta x},$$

where $q(x) = \sqrt{1+x^2}$ and K_1 denotes the modified Bessel function of the third kind with index 1. The domain of the parameters is $\mu \in \mathbb{R}$, $\gamma, \delta > 0$, and $0 \leq |\beta| \leq \alpha$. The NIG-parameters can be estimated from data by the maximum likelihood method in a straightforward way.

The Inverse Gaussian distribution (IG) is related to the NIG-distribution. The IG-distribution has density

$$f_{IG}(x;\delta,\gamma) = \frac{\delta}{\sqrt{2\pi}}\exp(\delta\gamma)\, x^{-\frac{3}{2}}\exp\left(-\tfrac{1}{2}\left(\delta^2x^{-1}+\gamma^2x\right)\right), \quad x>0,$$

where δ and γ are the same as in the NIG-distribution. It is well known that if σ^2 has an IG distribution and ξ is standard normal, then

$$r = \mu + \beta\sigma^2 + \sigma\xi \tag{3.5}$$

has a NIG distribution. Recall that the volatility σ^2 will be observable.

Suppose now that we estimate the NIG-distribution from a set of returns $r(\cdot)$ with the maximum likelihood method. This gives through Equation (3.5) a set of IG parameters. Assume further that we manage to estimate the volatility process $\sigma(\cdot)^2$ so that $\sigma(\cdot)^2$ and $r(\cdot)$ in Equation (3.5) gives normalized returns $\xi(\cdot)$ that appear much like *i.i.d.* $N(0,1)$ variables. Then we would expect to see that our estimated $\sigma(\cdot)^2$ had approximately the same IG-distribution as the NIG-distributed returns imply, since both parameter sets are estimated from the same data. It turns out that this does not hold for our data set. We illustrate this with the following example.

When we estimated the NIG parameters for the observed Ericsson returns our MATLAB routine came up with two very different maximum likelihood estimates, depending on the starting values. These were

1. $\alpha_1 = 73.8,\ \beta_1 = 14.7,\ \mu_1 = -0.0092,\ \delta_1 = 0.0591$
2. $\alpha_1 = 211,\ \beta_1 = 108,\ \mu_1 = -0.0650,\ \delta_1 = 0.114$

These parameter sets give the NIG and IG densities that are shown in Figure 3.1. We see that the NIG-distributions are virtually indistinguishable. Accordingly, the likelihood functions for the two sets were within 0.2% of each other. Still, there is a substantial difference between the IG-densities.

This example indicates that the problem with estimating the NIG-distribution from returns is not that it is hard to find good parameter estimates. Rather, there are too many of them. We also made a number of profile likelihood function plots. From those it could be seen that there were directions in the parameter space along which the likelihood function for the NIG-distribution was very flat. This made the parameter estimates very unstable. Loosely speaking, *one can obtain a good fit of the NIG-distribution to return data for many IG-distributions*. In other words, the NIG-distribution is "almost" overparameterized. We need to find a way to single out which set of NIG parameter values that are, in some sense, the correct ones.

Equation (3.5) actually holds in more general. If σ^2 has a Generalized Inverse Gaussian (GIG) distribution and ξ is standard normal, then

$$r = \mu + \beta\sigma^2 + \sigma\xi$$

has a GH-distribution. We realize from this result that the GH-distribution must have the same problem as its subset, the NIG-distribution: If we have information only about the returns of an asset, the GH-distribution is also "almost" overparameterized.

4 Analysis

In this section we outline an approach to fitting the discrete time model to data. The methods are illustrated by applying them to the data set from Subsection 3.2 and estimating the NIG and IG distributions to the returns and volatility, respectively.

4.1 Method

We propose that instead of using the returns directly one should try to estimate the model through Equation (4.1). That is, one should try to find ways to measure σ_i^2 with parameters μ_i and β_i such that the normalized returns $\xi_i(\cdot)$ are *i.i.d.* and $N(0,1)$. If we can do this, we can model the σ_i^2 within the framework in [5]. This verifies the validity of the univariate discrete time model, which allows us to understand better the structure of the process that generated the returns $R_i(\cdot)$. The effort to simultaneously fit the returns $R_i(\cdot)$ to the NIG-distribution, the volatility process $\sigma_i(\cdot)^2$ to the IG-distribution, and the normalized returns $\xi_i(\cdot)$ to the normal distribution would probably not make the fit better. Since the NIG-distribution is very flexible, we are likely to get almost as good estimates by first trying to obtain normality of $\xi_i(\cdot)$, and fit the NIG and IG distributions to the returns and the estimated volatility process later, with μ_i and β_i fixed. The understanding of the model lies first of all in getting $\xi_i(\cdot)$ and the model for $\sigma_i(\cdot)^2$ correct. To find the parameters in the distribution of $\sigma_i(\cdot)^2$ is the next priority. Equation (3.5) then gives an implied distribution of the returns $R_i(\cdot)$ that we have a good understanding of.

The analysis is done in four steps.

1. Find volatility processes σ_i^2 and parameters μ_i and β_i for each stock so that the normalized returns

$$\xi_i(\cdot) = \left(R_i(t) - \left(\mu_i + \beta_i \sigma_i(t)^2\right)\right) / \sigma_i(t) \tag{4.1}$$

become independent $N(0,1)$. Here we assume that the discrete time volatility processes σ_i^2 is a constant times the number of trades $z_i(\cdot)$ on each trading day. That is,

$$\sigma_i(\cdot)^2 = \theta_i z_i(\cdot). \tag{4.2}$$

This means that we can write the loglikelihood function L for the observations $R_i(1), ..., R_i(d)$, as

$$\begin{aligned} &L(\theta_i, \sigma_i(0)) \\ &= \log\left(\phi(\xi_i(1)) * ... * \phi(\xi_i(d))\right) \\ &= -\frac{1}{2}\sum_{t=1}^{d}\left(\frac{\left(R_i(t) - \left(\mu_i + \beta_i\sigma_i(t)^2\right)\right)^2}{\sigma_i(t)^2} + \log\left(\sigma_i(t)^2\right)\right). \end{aligned}$$

We recall that the continuous time volatility is defined as a linear combination of news processes Y_j from Equation (2.2). This implies that our discrete time volatility model in Equation (3.3) is, in a sense, an average of the continuous time volatility on that trading day. See Equation (3.1). Further, we can view the continuous time volatility as the intensity with which new trades "arrive". Note that for $X \in IG(\delta, \gamma)$, we have that $aX \in IG\left(a^{1/2}\delta, a^{-1/2}\gamma\right)$, $a > 0$. Hence our volatility model in Equation (4.2) is well-defined.

The model in Equation (4.2) is inspired by [1], which uses the cumulative number of trades as a stochastic clock. The paper [1] shows that the intraday cumulative number of trades contains enough information to allow us to obtain almost perfect predictions of the volatility in the near future. Since the concepts of stochastic time change and stochastic volatility are related, we want to use a similar idea with daily data for our discretized model. In other words, we verify that we can use a constant times the number of trades as $\sigma_i(\cdot)^2$ in Equation (4.1) to obtain $\xi_i(\cdot)$ that are *i.i.d.* and $N(0,1)$. Since Equation (4.1) holds in the model in [5], this implies that we identify the daily number of trades with the volatility, and model the number of trades within the model in [5]. If we can do this, we have asserted that our continuous time stochastic volatility model is reasonable. Further, we get an economical interpretation of the volatility. Note that it would be desirable to find a *previsible* model for $\sigma_i(\cdot)^2$ such that the normalized returns $\xi_i(\cdot)$ are *i.i.d.* and $N(0,1)$. The reason is that such a model would make it easier to apply results from, for example, portfolio optimization, option pricing theory, or risk management. However, we know that this will be hard, especially for large Δ. This is due to that a previsible model with $\sigma_i(t) \in \mathcal{F}_{t-1}$, such as the GARCH, does not take into account the information released at time t. The impact of this information could be considerable. In other words, the $\xi_i(\cdot)$ might behave like an *i.i.d.* sample, but their distribution will have thicker tails than the standard normal.

2. The next step is to find parameters α_i and δ_i so that the empirical distributions of $\sigma_i(\cdot)^2$ from Equation (4.2) fit the $IG\left(\delta_i, \sqrt{\alpha_i^2 - \beta_i^2},\right)$ distribution.

Hence, we have also specified the NIG-distribution for $R_i(\cdot)$. We could do this estimation simultaneously for IG and NIG. However, since the NIG-distribution is very special we know that even if we would get a slightly better fit this way, it would be at the cost of less understanding of the process.

3. We next use the estimates of the volatility processes σ_i^2 to estimate the rates of decay λ_j. This can be done by using the autocorrelation function of the continuous time volatility process σ_i^2. The autocorrelation ρ_{σ_i} is defined by

$$\rho_{\sigma_i}(h) = \frac{Cov\left(\sigma_i(h)^2, \sigma_i(0)^2\right)}{Var\left(\sigma_i(0)^2\right)}, \quad h \geq 0.$$

Straightforward calculations show that

$$\rho_{\sigma_i}(h) = \omega_1 \exp\left(-\lambda_1 |h|\right) + ... + \omega_m \exp\left(-\lambda_m |h|\right),$$

where the $\omega_j \geq 0$, are the weights from the volatility processes that sum to one. We estimate the rates of decay λ_j from the discrete time volatilities $\sigma_i(1), ..., \sigma_i(d)$, by minimizing the least squared distance between the theoretical and empirical autocorrelation function. We require the rates of decay λ_j to be equal for all stocks to allow for the news processes Y_j to be shared by different stocks.

4. The final step is to apply factor analysis to the normalized returns to find the correlation between the different stocks. We know from Equation (3.4) that we need the matrix $\Phi = (\varphi_{i,k})_{i,k=1}^{i=n,k=q}$ to estimate the correlation between the returns of different stocks. We recall that in factor analysis it is assumed that a vector x of observed variables with mean 0 can be written as $x = \Lambda f + e$. Here Λ is a constant $n \times q$ matrix of factor

loadings, and f and e are vectors of independent factors. It is a common assumption that the factors in f and e are $N(0,1)$ and normal with mean 0, respectively. We can see from Equation (3.3) that the normalized returns $\xi_i(\cdot)$ are on this form for each t. Hence we can apply standard factor analysis techniques to the estimated $\xi_i(\cdot)$. We choose the number of factors $q = 2$, and use MATLAB's maximum likelihood factor analysis estimation with varimax rotation to get an estimate of the factor loadings matrix $\Phi = (\varphi_{i,k})_{i,k=1}^{i=n,k=q}$, and to test the hypothesis that $q = 2$ is the correct number of factors. Varimax rotation rotates the loadings matrix with an orthogonal matrix, and attempts to make the loadings either large or small to facilitate interpretation. The factors will still be independent under this operation.

4.2 Results

We exemplify the analysis with some of the results for the AstraZeneca stock. The results were very similar for all stocks. It turns out that the simple model of Equation (4.2) seems quite sufficient: The normalized returns appear to come from an *i.i.d.* sample for all five stocks, and we obtain very good normal QQ plots, see Figure 4.2. Further, the implied NIG-distribution and the estimated IG-distribution both fit their empirical density histograms well, see Figure 4.3, and the empirical autocorrelation functions for $\xi_i(\cdot)$ and $|\xi_i(\cdot)|$ show no signs of dependence, see Figure 4.4. Further, the volatility process has the characteristic look of a OU news process, see Figure 4.1. The estimated parameter values for AstraZeneca were $\hat{\alpha}_5 = 233.0$, $\hat{\beta}_5 = 5.612$, $\hat{\mu}_5 = -5.331 * 10^{-4}$, $\hat{\delta}_5 = 0.0370$, and $\hat{\theta}_5 = 0.1962$. This completes the marginal analysis.

To fit the rates of decay λ_j, two news processes seemed to give reasonable results. The estimated parameters for AstraZeneca were $\hat{\omega}_{5,1} = 0.9224$, $\hat{\lambda}_1 = 0.9127$, $\hat{\omega}_{5,2} = 0.0776$, and $\hat{\lambda}_2 = 0.0262$, see Figure 4.5.

Factor analysis of the normalized returns yielded an estimate of the factor loadings matrix $\hat{\Phi}$ as

$$\hat{\Phi} = \begin{pmatrix} 0.3652 & 0.5940 \\ 0.4963 & 0.4533 \\ 0.7750 & 0.2779 \\ 0.8183 & 0.3745 \\ 0.1498 & 0.4614 \end{pmatrix},$$

for the orthogonal matrix

$$T = \begin{pmatrix} 0.8691 & 0.4946 \\ -0.4946 & 0.8691 \end{pmatrix}.$$

The test of the hypothesis that $q = 2$ is the correct number of factors gave the p-value $p^* = 0.5632$.

Figure 4.6 gives the factor loadings for the five stocks. Note that even though it is tempting to give the factors some interpretation, one should be cautious in doing this. The reason is that the factor loadings matrix is non-unique. Nevertheless, there appears to be one factor related to the "mechanical" part of industry, and one related to "softer" branches like medicine and telecom.

4.3 Discussion

We believe that identifying the number of trades with the discrete volatility in the model in [5] contributes to making that theory more applicable in practice. First, it gives more stable parameter estimates than if we analyzed only the marginal distribution of the returns directly with the standard maximum likelihood method. Accurate parameter estimates are important in most fields of applied risk management and mathematical finance. For example, option prices, hedging portfolios, and optimal portfolios all depend on parameters that have to be estimated from data. It also gives an economical interpretation of the news processes, which makes the understanding of the model better. Further, our approach is easier to implement than the quadratic variation method, and it requires much less data.

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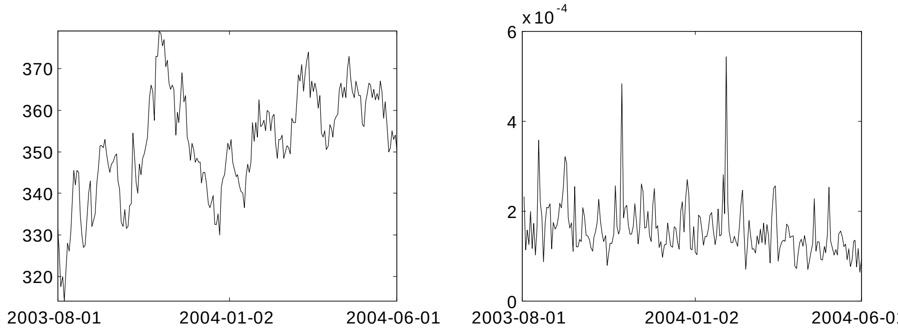

Figure 4.1: *Left:* The price process in SEK for AstraZeneca from August 4, 2003, to June 1, 2004. *Right:* The estimated volatility process $\hat{\theta} *$ (Number of trades per day) for AstraZeneca during the same time period.

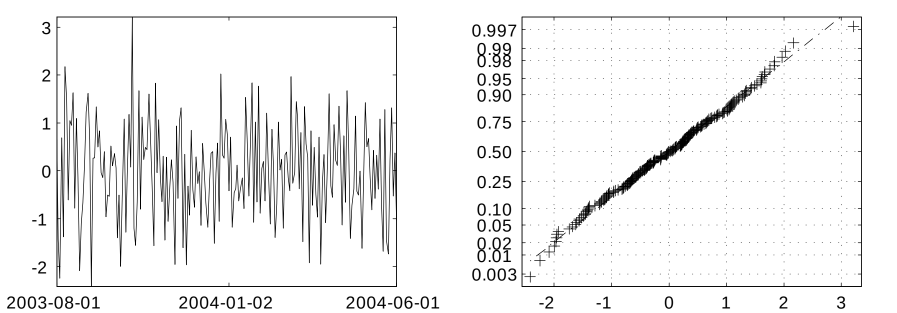

Figure 4.2: *Left:* The normalized returns for AstraZeneca during August 1, 2003, to June 1, 2004. *Right:* The normal probability plot of the normalized returns for AstraZeneca. The theoretical quantiles are on the y-axes.

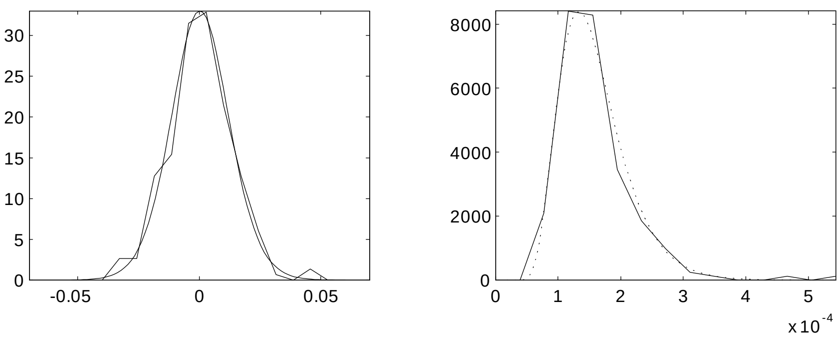

Figure 4.3: *Left:* Histogram of the returns and the implied NIG density obtained from the estimated IG density. *Right:* Histogram of $\hat{\theta}$ ∗ (Number of trades per day) and the estimated IG-density.

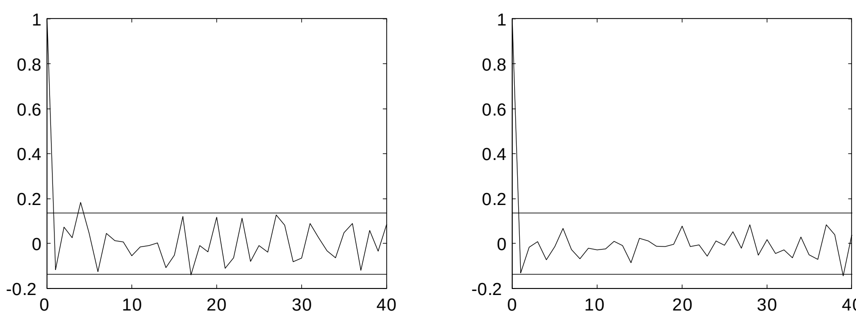

Figure 4.4: *Left:* The autocorrelation function for the absolute normalized returns for AstraZeneca. *Right:* The autocorrelation function for the normalized returns for AstraZeneca. The figures show the first 40 lags, and the straight lines parallel to the x-axes are the asymptotic 95% confidence bands $\pm 1.96/\sqrt{d}$.

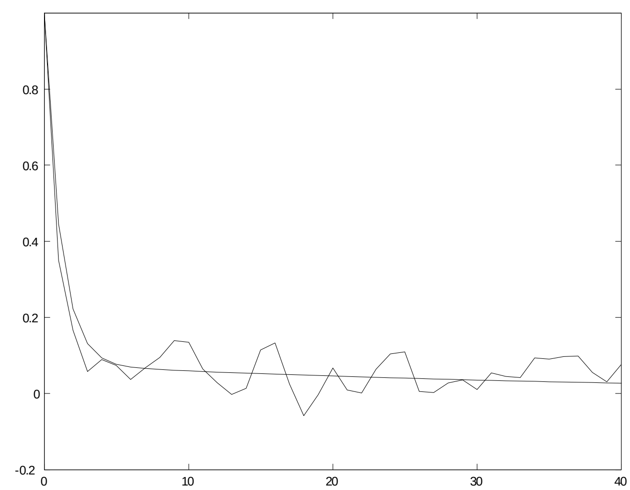

Figure 4.5: The autocorrelation function for the volatility process, and the estimated theoretical autocorrelation for AstraZeneca with two news processes.

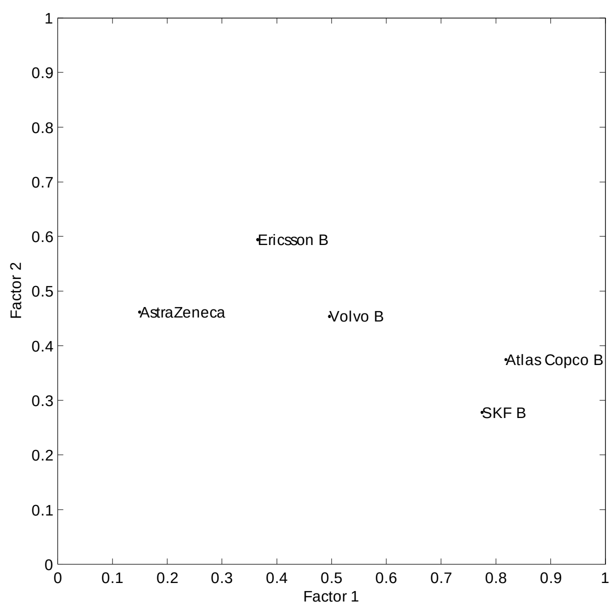

Figure 4.6: The factor loadings. Factor 1 is on the x-axes and factor 2 is on the y-axes.

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