THESIS FOR THE DEGREE OF DOCTOR OF TECHNOLOGY

Continuum Percolation in non-Euclidean Spaces

Johan Tykesson

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematical Sciences
Chalmers University of Technology and Göteborg University
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Johan Tykesson
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Department of Mathematical Sciences
Division of Mathematical Statistics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31 772 1000

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Johan Tykesson

Department of Mathematical Sciences
Chalmers University of Technology
Göteborg University

Abstract

In this thesis we first consider the Poisson Boolean model of continuum percolation in n-dimensional hyperbolic space $\mathbb{H}^n$. Let R be the radius of the balls in the model, and λ the intensity of the underlying Poisson process. We show that if R is large enough, then there is an interval of intensities such that there are infinitely many unbounded components in the covered region. For $n = 2$, more refined results are obtained.

We then consider the model on some more general spaces. For a large class of homogeneous spaces, it is established that if λ is such that there is a.s. a unique unbounded component in the covered region, then this is also the case for any $\lambda_1 > \lambda$. In $\mathbb{H}^2 \times \mathbb{R}$ it is proved that if λ is critical for a.s. having a unique unbounded component in the covered region, then there is a.s. not a unique unbounded component.

Finally, we consider another aspect of continuum percolation in $\mathbb{H}^2$. We show that in the Poisson Boolean model, there are intensities for which infinite geodesics are contained in unbounded components of the covered region. This is also shown for the vacant region, as well as for a larger class of continuum percolation models. We also consider some dynamical models.

Keywords: Bernoulli percolation, continuum percolation, dependent percolation, double phase transition, hyperbolic space, Poisson Boolean model, geodesic percolation

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This thesis consists of the following papers:

Paper I: Tykesson J.H. (2007), The number of unbounded components in the Poisson Boolean model of continuum percolation in hyperbolic space, *Electronic Journal of Probability*, **12** 1379-1401.
(The thesis version includes a correction.)

Paper II: Tykesson, J.H. (2008), Continuum percolation at and above the uniqueness threshold on homogeneous spaces. *Conditionally accepted for publication in Journal of Theoretical Probability*

Paper III: Benjamini I., Jonasson J., Schramm O. and Tykesson, J.H. (2008), Visibility to infinity in the hyperbolic plane despite obstacles. *Preprint.*

Contents

1 Introduction

Continuum percolation is the study of the geometry of connected components in a random subset of a space. Like for most models in statistical mechanics, there are several real world motivations. One example of this is the spread of a fire (or a disease) in a forest. If a tree is set afire, then the fire can spread to all trees within some distance from it. From the trees set afire from the first tree, the fire spreads to trees within some distance from these trees, and so on. The burning trees form a random subset of the forest. One then wonders how far the fire typically spreads. This of course depends on the the density of the trees. If the density of the trees is low, then if a tree starts burning, typically the fire will die out soon. If the density of trees is high, then one would rather expect that the fire will spread far away from the starting point. Therefore, one may guess that there is some *critical density* separating these two scenarios. A possible mathematical model for describing this is given by the Poisson Boolean model of continuum percolation, the properties of which is the main focus of this thesis.

Real-world motivations such as the above tend to be most natural when the percolation process takes place in two- or three-dimensional Euclidean space. When, as in the present thesis, we move on to more exotic spaces such as the hyperbolic plane, such motivations become correspondingly weaker. This, however, is in our opinion amply compensated by the richness of the mathematical phenomena we encounter.

The rest of the introduction is organized as follows. In Section 1.1 we introduce the Poisson Boolean model and give its most basic properties. Since many of our results are for this model in the hyperbolic plane $\mathbb{H}^2$ we give a brief account of hyperbolic geometry in Section 1.2. In the summary of papers in Section 2 we will mention analogs (which are discussed in detail in the papers) of our results in the theory of percolation on graphs. Therefore, it is necessary that we introduce this model also, and we do this in Section 1.3.

1.1 The Poisson Boolean model

Most of the results in this thesis concern the Poisson Boolean model of continuum percolation on some different spaces (sometimes we will refer to this model simply as the Boolean model). Suppose M is a Riemannian manifold with volume measure μ. For concreteness, the reader may keep in mind the case where M is d-dimensional Euclidean space, and μ is Lebesgue measure, although one of the distinguishing features of this thesis is that we go beyond this setting. Before defining the Poisson Boolean model on M, we need to introduce the Poisson point process.

Definition 1.1. *A point process X on M distributed according to the probability measure $\mathbf{P}$ such that for $k \in \mathbb{N}$, $\lambda \geq 0$, and every measurable $A \subset M$ one has*

$$\mathbf{P}[|X(A)| = k] = e^{-\lambda\mu(A)} \frac{(\lambda\mu(A))^k}{k!}$$

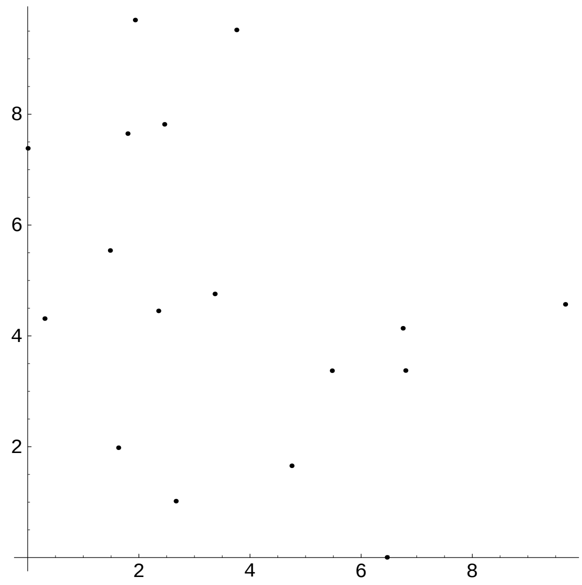

Figure 1.1: A realization of the Poisson process on a portion of $\mathbb{R}^2$.

is called a Poisson process *with intensity* λ *on* M*. Here* $X(A) = X \cap A$ *and* $|\cdot|$ *denotes cardinality.*

Arguably, the Poisson process is the most important and most studied of all point processes. This is partly because of its intrinsic appeal, but also because it tends to serve as a baseline model with respect to which other models are defined. It has many nice properties which makes the analysis of it easier than for other point processes. For example, if A and B are disjoint sets, then the configuration of points in A is independent of the configuration of points in B. Also, if X_1 and X_2 are two independent Poisson processes with intensities λ_1 and λ_2, then $X_1 \cup X_2$ is a Poisson process with intensity $\lambda_1 + \lambda_2$. For more facts about the Poisson process and more general point processes, we refer to [10]. Figure 1.1 shows a realization of a Poisson process on a portion of $\mathbb{R}^2$.

In the *Poisson Boolean model of continuum percolation*, we place, at each point in a Poisson process with intensity λ on the space, a ball of some fixed radius R. The radius may also be a random variable (independent for all points), but in this thesis we consider only the fixed radius version of the model. (However, there are phenomena that only occur if the radius is random, for details see [18].) See figure 1.2 for a realization of the model on a portion of $\mathbb{R}^2$. Let C be the subset of the space which is covered by balls, and let V be its complement. Then C is called the covered region and V the vacant region. Sometimes, instead of studying V, it is for technical reasons preferable to study the closure of V.

The Poisson Boolean model was first introduced in the plane $\mathbb{R}^2$ by Gilbert

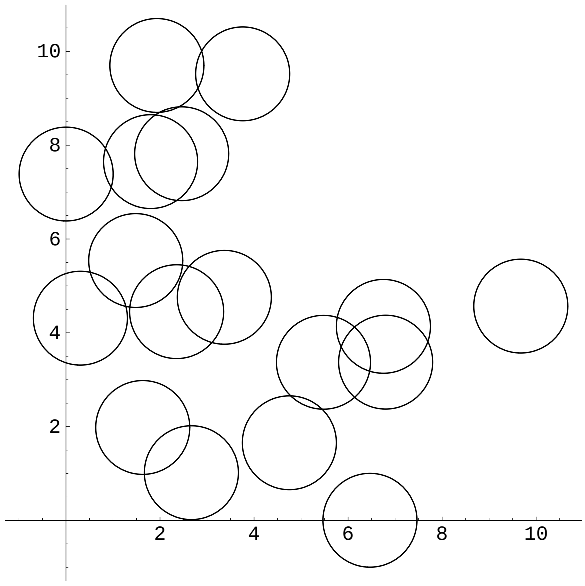

Figure 1.2: A realization of the Poisson Boolean model on a portion of $\mathbb{R}^2$, using the Poisson points in figure 1.1 and balls of radius 1.

[11]. One of his motivations for this was that is could be a simple model for a network of shortrange radio stations spread over a wide area.

One now asks different questions about the geometry of C and V. In particular one is interested in the existence or non-existence of unbounded connected components in C and V, and in case there are such components, how many they are. Obviously, the answer depends on the intensity λ in the underlying Poisson process, the radius R of the balls, and which space we consider. Let us now concentrate on C. Let N_C be the number of unbounded components in C. Then it is known that for the model in any unbounded connected homogeneous[1] Riemannian manifold , N_C is an almost sure constant which is either 0, 1 or ∞ depending on λ. This motivates the introduction of two critical intensities. Define $\lambda_c = \lambda_c(M, R)$ to be the infimum of the set of intensities that produce an unbounded component a.s. and similarly let λ_u be the infimum of the set of intensities that produce a unique unbounded component a.s. Note that $\lambda_c \leq \lambda_u$. For Euclidean space we have

Theorem 1.2. *Consider the Poisson Boolean model on $\mathbb{R}^n$. Then $\lambda_c \in (0, \infty)$ and*

$$N_C = \begin{cases} 0 \ \textit{a.s. if } \lambda < \lambda_c \\ 1 \ \textit{a.s. if } \lambda > \lambda_c \end{cases}$$

[1]A Riemannian manifold M is said to be homogeneous if for any two points $x, y \in M$, there is an isometry mapping x to y.

Moreover it is known that at λ_c we have $N_C = 0$ a.s. or $N_C = 1$ a.s. (On $\mathbb{R}^2$, $N_C = 0$ a.s. at λ_c and it is conjectured that this is the case for any dimension.) Theorem 1.2 is an example of a *phase transition*. At λ_c, the macroscopic behavior of the model changes drastically. In the setting of the forest fire example the trees are represented by the Poisson points, and for the fire to spread directly from one tree to another, they must be within distance $2R$ from each other. If the density of trees is below λ_c the fire will typically die out quite soon, while if the density is above λ_c there risk of having a major forest fire is considerably larger.

So on $\mathbb{R}^n$ we do not have any intensities for which $N_C = \infty$. However, this is not always the case. In [27], which is Paper I in this thesis, we show that for the model in $\mathbb{H}^2$ we have $\lambda_c < \lambda_u$, so that there is an interval of intensities that produce infinitely many unbounded components. Then there is also the question what happens above λ_u. In [28], which is Paper II in this thesis, we show that in the Poisson Boolean model on homogeneous spaces, there are at most three nonempty phases regarding the number of unbounded components in the covered region: for $\lambda \in [0, \lambda_c)$, there are no unbounded components a.s. for $\lambda \in (\lambda_c, \lambda_u)$ there are infinitely many unbounded components a.s. and finally for $\lambda \in (\lambda_u, \infty)$, there is a unique unbounded component a.s. What happens at λ_c and λ_u turns out to depend on the space. For example, in [27] we will see that at λ_u on $\mathbb{H}^2$ there is a unique unbounded component at λ_u, while in [28] we show that this is not the case for the model in $\mathbb{H}^2 \times \mathbb{R}$.

There are also questions about the nature of the unbounded components. One such question we study is the existence of infinite geodesics completely covered by C or V. In [2], which is Paper III in this thesis, we show that there in fact are intensities for which this happens on $\mathbb{H}^2$.

1.2 Hyperbolic Geometry

The aim of this sections is to give a very short introduction to hyperbolic geometry in two dimensions, which is the geometry on which most of the action takes place in this thesis. There are several models of hyperbolic geometry. All of them are equivalent, in the sense that there are isometries between them. Which model one wants to work with very much depends on the nature of the problem of interest. The most common models are the Poincaré unit disc model and the half plane model. These are also the models that appear in this thesis. For an account of other models of hyperbolic geometry, see for example [9]. We now concentrate on the properties of the unit disc model.

Write $z = x + iy$ for a point in the complex plane $\mathbb{C}$. The Poincaré disc model of hyperbolic space is the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ equipped with the metric (which we will refer to as the hyperbolic metric)

$$ds^2 := 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}. \tag{1.1}$$

We write $\mathbb{H}^2$ for this space, which we sometimes will call the hyperbolic plane.

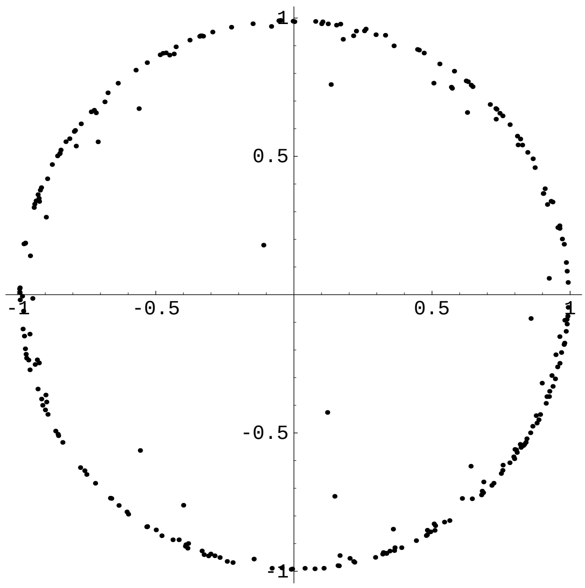

Figure 1.3: A realization of the Poisson process on a portion of $\mathbb{H}^2$.

From 1.1 we see that near the origin, ds^2 behaves like a scaled Euclidean metric, but there is heavy distortion near the boundary of U. The factor 4 in (1.1) is often left out in the definition of the hyperbolic metric. We remark that it is also common to identify points of $\mathbb{H}^2$ with points in the open unit disc in the Euclidean plane rather than in the complex plane. Figures 1.3 and 1.4 show a realization of a Poisson process and the corresponding Poisson Boolean model in $\mathbb{H}^2$.

In the hyperbolic metric, a curve $\{\gamma(t)\}_{t=0}^1$ gets length

$$L(\gamma) = 2\int_0^1 \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}dt$$

and a set A gets area

$$\mu(A) = 4\int_A \frac{dx\,dy}{(1-(x^2+y^2))^2}.$$

If $z_1, z_2 \in \mathbb{H}^2$, then the geodesic (that is, the shortest curve that starts at x and ends at y) between them is either a segment of a Euclidean circle that intersects the boundary of U orthogonally, or a segment of a straight line that passes through the origin. Recall that Euclid's parallel postulate says that given a line and a point not on it, there is exactly one line going through the given point that is parallel to the given line. The space $\mathbb{H}^2$ does not satisfy Euclid's parallel postulate which means $\mathbb{H}^2$ is a non-Euclidean geometry.

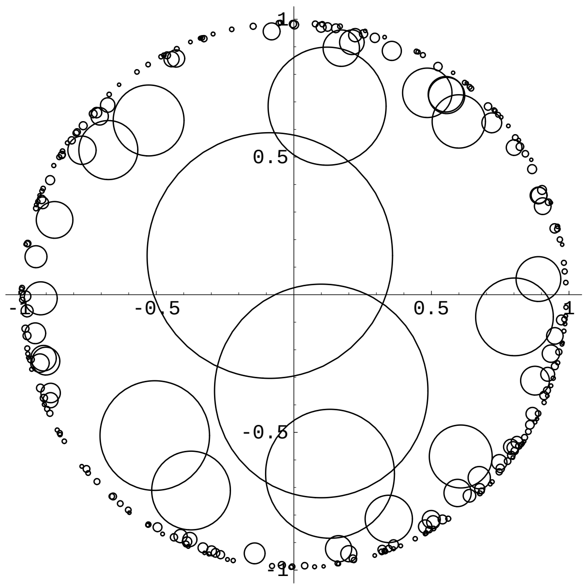

Figure 1.4: A realization of the Poisson Boolean model on a portion of $\mathbb{H}^2$, using the Poisson points in figure 1.2.

Let us consider some areas and lengths in this metric. Let $z_1, z_2 \in \mathbb{H}^2$. The hyperbolic distance between z_1 and z_2 is given by

$$d(z_1, z_2) = 2 \tanh^{-1} \left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right| .$$

Let $S(x, r) := \{y \in \mathbb{H}^2 \, : \, d(x, y) \leq r\}$ be the closed hyperbolic ball of radius r centered at x. The circumference is given by

$$L(\partial S(x, r)) = 2\pi \sinh(r)$$

and the area is given by

$$\mu(S(x, r)) = 2\pi(\cosh(r) - 1). \tag{1.2}$$

Observe that

$$2\pi \sinh(r) = 2\pi r + o(r^2) \tag{1.3}$$

and

$$2\pi(\cosh(r) - 1) = \pi r^2 + o(r^3) \tag{1.4}$$

so the formulas are well approximated with the Euclidean formulas at a small scale. Also, we see that as $r \to \infty$, both the area and circumference grow exponentially with the same rate. Moreover, the ratio between them tends to 1 as $r \to \infty$. In fact, if A is any bounded set with $\mu(A)$ and $L(\partial A)$ well defined, we have

$$L(\partial A) \geq \mu(A). \tag{1.5}$$

This is the so called linear isopermetric inequality for $\mathbb{H}^2$. Such an inequality is not available in the Euclidean plane. The isoperimetric inequality is one of the main tools for our analysis in [27].

In [27] we use the hyperbolic first law of cosines for geodesic triangles. So for an example of hyperbolic trigonometry, let us consider this law. Let A, B, and C be three points in $\mathbb{H}^2$ such that they are not on the same geodesic. Let θ_A, θ_B and θ_C be the corresponding angles. Let a, b and c be the lengths of the geodesics from B to C, from A to C, and from A to B respectively. The the hyperbolic first law of cosines says

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\theta_C)$$

If $\theta = \pi/2$, we get the hyperbolic Pythagorean theorem:
$\cosh(c) = \cosh(a)\cosh(b)$. Note that if a, b and c are small, then the hyperbolic first law of cosines is approximately the same as its Euclidean counterpart, that is $c^2 \approx a^2 + b^2 - 2ab\cos(\theta_C)$.

Next, we consider tilings of $\mathbb{H}^2$. Recall that two sets are said to be congruent if there is an isometry between them. A regular tiling of a space is a collection of congruent polygons that fill the space and overlap only on a set of measure 0, such that the number of polygons that meet at a corner is the same for every corner. For example, there exists exactly three kinds of such tilings of the Euclidean plane. These are made up of equilateral triangles, squares or hexagons (however, given any side length, a regular tiling of any of these types exist). In the hyperbolic plane the situation is different. There exists an infinite number of regular tilings. More precisely, if p and q are positive integers such that $(p-2)(q-2) > 4$, then it is possible to construct a regular tiling of the hyperbolic plane into congruent p-gons, where q of these p-gons meet in each corner. However, given p and q, there is only one side length of the p-gon available. Each regular tiling of $\mathbb{H}^2$ can be identified with graph G. More precisely, each side in the tiling is identified with an edge in G and each corner is identified with a vertex in G. Such a graph is transitive, and it always has a positive isoperimetric constant (see Section 1.3 for definitions). Figure 1.5 shows an example of a tiling.

We mention one more fact about the Poincaré disc model of hyperbolic geometry that comes to use in [2]. Consider the ball $S(x, r)$. This ball actually looks precisely like an Euclidean ball. However, its Euclidean center is closer to origin than its hyperbolic center x, and its Euclidean radius is smaller than its hyperbolic radius r. There are explicit formulas for both these quantities, that are used in [2].

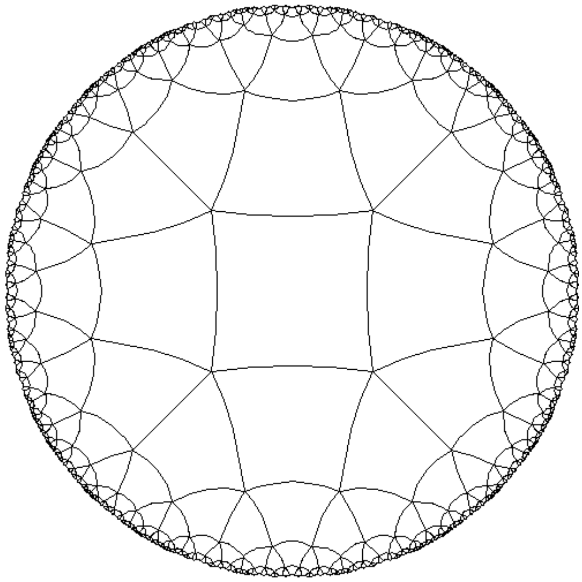

Figure 1.5: A tiling of $\mathbb{H}^2$ into congruent 4-gons, 5 meeting at each corner.

We briefly discuss the Poincaré half plane model. This model is the complex upper half plane $\{x + iy : y > 0\}$ together with the metric

$$ds^2 := 4\frac{dx^2 + dy^2}{y^2}.$$

In this model, the intersection of the upper half plane with Euclidean circles orthogonal to the real line are infinite geodesics. In [2], we use this model to calculate a certain area, which would be more difficult to calculate in the disc model. An isometry from the half plane model to the unit disc model is given by $f(z) = (z - i)/(z + i)$, and the inverse of this isometry is given by $f^{-1}(z) = i(1 + z)/(1 - z)$.

Hyperbolic space is an example of a symmetric space. A symmetric space is a connected Riemannian manifold M, such that for every point $p \in M$, there is an isometry I_p such that $I_p(p) = p$ and I_p reverses all geodesics through p. In $\mathbb{H}^2$, such an isometry is simply given by a rotation of 180 degrees, around the point p. Other symmetric spaces are Euclidean space and the sphere (in any dimensions). Symmetric spaces belong to the class of homogeneous spaces.

1.3 Discrete percolation

Most of the results in this thesis have analogs in the theory of discrete percolation. These analogs will be mentioned in the summaries of the papers in Section 2. Therefore we give here some preliminaries. Grimmett's book [12] is the standard reference for this theory.

Let $G = (V, E)$ be an infinite connected graph with vertex set V and edge set E. A bijective map $f : V \to V$ such that $\{f(u), f(v)\} \in E$ if and only if $\{u, v\} \in E$ is called a graph automorphism on G. The set of graph automorphisms on G forms a group under composition which we denote by $\mathrm{Aut}(G)$. The graph G is said to be transitive if for any pair of vertices $u, v \in V$ there is an $f \in \mathrm{Aut}(G)$ such that $f(u) = v$.

Let ω be a random subgraph of G where all vertices (edges) are kept. We call ω a bond (site) percolation on G. We say that ω is an invariant percolation if the law of ω is invariant under $\mathrm{Aut}(G)$. A connected component of ω will be called a cluster. The most important example is Bernoulli percolation (also called independent percolation), in which each edge (vertex) is kept or deleted inependently with probability p and $1 - p$ respectively.

Bernoulli percolation is the discrete analog to the Poisson Boolean model of continuum percolation. Many of the basic questions in Bernoulli percolation are the same as those in the continuum setting. The model was first studied by Broadbent and Hammersley [7] on the graph whose vertex set is $\mathbb{Z}^d$ and whose edge set consists of all pairs of vertices sitting at Euclidean distance 1 from each other. Write $\mathbb{Z}^d$ for this graph. They showed that there is a critical intensity $p_c = p_c(d) \in (0, 1)$ such that if $p < p_c$, then there is a.s. no infinite clusters, while if $p > p_c$ there are a.s. infinite clusters. Later it was established that in the case $p > p_c$, there is a unique infinite cluster a.s. (for $d = 2$ this is due to Harris [17], while for $d \geq 3$ the result is due to Aizenman et. al [1]). The exact value of $p_c(d)$ is unknown except in the trivial case $d = 1$ where it is 1, and in the case $d = 2$, where Kesten [20] showed that it is $1/2$.

Recent years have witnessed a rapid development of percolation on more exotic graphs than $\mathbb{Z}^d$. Of particular interest is the number of infinite clusters in Bernoulli percolation. It is well known that this number is an almost sure constant that is either 0, 1 or ∞ if the graph is transitive. Let $p_c = p_c(G)$ be the infimum of the set of $p \in [0, 1]$ that produce infinite clusters in Bernoulli percolation on G, and similarly let p_u be the infimum of the set of $p \in [0, 1]$ that produce a unique unbounded component. Then it is known that there is at most three phases regarding the number of infinite clusters: if $p \in [0, p_c)$ then the number is 0 a.s., if $p \in (p_c, p_u)$ then the number is ∞ a.s. and finally when $p \in (p_u, 1]$, then the number is 1 a.s. What happens at the critical values p_c and p_u depends on the graph.

To determine for which graphs one has $p_c < p_u$ the following quantity turns out to be fundamental. Let

$$\kappa_V(G) := \inf_W \frac{|\partial_V W|}{|W|}$$

where the infimum ranges over all finite connected subsets W of V and $\partial_V W$ is the set of vertices of W with at least one neighbor in $V \backslash W$. Then $\kappa_V(G)$ is called the isoperimetric constant or Cheeger constant for the graph. If $\kappa_V(G) > 0$ then G is said to be non-amenable, otherwise it is said to be amenable. Nonamenable, transitive planar graphs are useful in the study of continuum percolation on $\mathbb{H}^2$ since they can be embedded in a proper way in $\mathbb{H}^2$. It is conjectured, and proved for many graphs, that if $\kappa_V(G) > 0$, then $p_c < p_u$. It is known that if $\kappa_V(G) = 0$, then $p_c = p_u$.

2 Summary of papers

In this section we summarize the results of the three papers in this thesis.

2.1 Summary of paper I

In the first paper of this thesis, "The number of unbounded components in the Poisson Boolean model of continuum percolation in hyperbolic space", we study the question of the number of unbounded components in C and V. Denote these numbers by N_C and N_V respectively. First we study the model in $\mathbb{H}^2$. The main result here, is that there are two critical intensities λ_c and λ_u such that $0 < \lambda_c < \lambda_u < \infty$ and

$$(N_C, N_V) = \begin{cases} (0,1), & \lambda \in [0, \lambda_c] \\ (\infty, \infty), & \lambda \in (\lambda_c, \lambda_u) \\ (1,0), & \lambda \in [\lambda_u, \infty) \end{cases}$$

Note that this result includes what happens at the critical intensities λ_c and λ_u. The main inspiration for this is the result by Benjamini and Schramm [5] that says that for any nonamenable planar transitive graph, one has $p_c < p_u$ for Bernoulli bond and site percolation. Techniques that we use include the study of certain random subsets of $\mathbb{H}^2$ that have an isometry invariant law, using the so called mass transport principle.

Then the model is studied in $\mathbb{H}^n$ for any $n \geq 2$. Here the main result is that given n, there is some R_0 such that in the model with balls of radius $R \geq R_0$ we have $\lambda_c(R) < \lambda_u(R)$. There is a discrete analog for this result also: Pak and Smirnova [22] showed that there are (not necessarily planar) Cayley graphs with a large enough set of generators for which $\lambda_c < \lambda_u$ (see [27] for precise definitions). They used a theorem due to Benjamini and Schramm [6], and our proof is inspired by the proof of this theorem.

2.2 Summary of paper II

In the second paper "Continuum percolation at and above the uniqueness threshold on homogeneous spaces" we study the uniqueness phase of the Poisson Boolean model on some various spaces. In [27] we showed that for all $\lambda > \lambda_u$ there is a unique unbounded connected component with probability 1 on $\mathbb{H}^2$, but left the corresponding problem open in $\mathbb{H}^n$ for $n \geq 3$. So on $\mathbb{H}^2$, if $\lambda_1 < \lambda_2$ and there is a.s. a unique unbounded component at λ_1, then this is also the case at λ_2. This phenomenom is sometimes called uniqueness monotonicity. This was established in the setting of independent bond and site percolation on transitive graphs under the assumption of so called unimodularity by Häggström and Peres [15] and shortly thereafter for all transitive graphs by Schonmann [23].

The first result in this paper resolves the question for $\mathbb{H}^n$, $n \geq 3$, as well as in considerably greater generality: for the Poisson Boolean model on a large class of Riemannian homogeneous space, if $\lambda_1 < \lambda_2$ and $\mathbf{P}_{\lambda_1}[N_C = 1] = 1$ then also $\mathbf{P}_{\lambda_2}[N_C = 1] = 1$. Actually two proofs are given, one is inspired by the proof in the discrete case by Häggström and Peres and the second by the proof of Schonmann. The first proof is valid only for the class of symmetric spaces (which includes for example $\mathbb{H}^n$).

Then we consider the situation at λ_u on $\mathbb{H}^2 \times \mathbb{R}$. In $\mathbb{H}^2$ it is known (see [27] that $N_C = 1$ at λ_u. We show that at λ_u on $\mathbb{H}^2 \times \mathbb{R}$, there is a.s. not a unique unbounded component at λ_u. Hence the number of infinite clusters at λ_u is either 0 or ∞. We expect it to be ∞, a result that would follow immediately if we could prove the natural conjecture that $\lambda_c < \lambda_u$ for such spaces.

On the way to the proof of this result, we obtain a characterization of λ_u in terms of connectivity between large balls.

2.3 Summary of paper III

In the third paper "Visibility to infinity in the hyperbolic plane despite obstacles", coauthored with Benjamini, Jonasson and Schramm, we consider a different aspect of the Poisson Boolean model on $\mathbb{H}^2$. Given a closed random subset Z of $\mathbb{H}^2$ with an isometry invariant law, is it possible to find (bi-infinite) hyperbolic lines that are completely covered by Z? It turns out that there is a useful sufficient criterion for this. Let B be a ball of radius 1 in $\mathbb{H}^2$. Then there is a universal constant p_0 such that if $\mathbf{P}[B \subset Z] > p_0$ then with positive probability there are hyperbolic lines that are fully covered by Z. The first result in this fashion for invariant discrete percolation is due to Häggström [13] for homogeneous trees.

We then introduce the concept of well-behaved percolation on $\mathbb{H}^2$, which is a class of random subsets of $\mathbb{H}^2$ that have an isometry invariant law and satisfy some quite natural conditions. Examples of well-behaved percolations include the covered and the vacant regions in the Poisson Boolean model. For a well-behaved percolation Z, define $f(r)$ to be the probability that a line segment of length r is completely covered by Z. First we show that there is a unique constant α (depending on the law of Z) and a constant $c > 0$ such that $ce^{-\alpha r} \leq f(r) \leq e^{-\alpha r}$ holds for every $r \geq 0$. Then we show that if $\alpha < 1$, then there are a.s. infinite geodesics contained in Z, and if $\alpha \geq 1$, then there are a.s. no infinite geodesics contained in Z. Moreover, if $\alpha < 1$ and we fix $o \in \mathbb{H}^2$, then with positive probability there are (infinite) hyperbolic half-lines starting at o that are contained in Z

Then we consider the cases when Z is the covered and the vacant region in the Poisson Boolean model. For the vacant region V we show that if $\lambda \geq 1/(2\sinh(R))$ then there are a.s. no infinite geodesics contained in V, while if

$\lambda < 1/(2\sinh(R))$ then there are a.s. infinite geodesics contained in V. Thus we know the exact value of the critical intensity λ_{gv} for the occurrence of infinite geodesics in the vacant region. We actually determine this exact value in two quite different ways. The first proof is uses methods from the theory of covering the circle with random arcs (see for example [24]). The second proof amounts to showing that $\alpha = 2\lambda\sinh(R)$ for V.

For the covered region C, we also show that there is a critical intensity λ_{gc} such that if $\lambda \leq \lambda_{gc}$ then there are a.s. no infinite geodesics in C and if $\lambda > \lambda_{gc}$ then there are a.s. infinite geodesics in C. In this case, we do not get an explicit formula for λ_{gc}, but we do get an integral equation for it.

Finally we consider some dynamical versions of the Poisson Boolean model. In one of the models, the balls update their positions according to independent exponential waiting times. The new center for a ball is chosen uniformly at random from a ball of radius 1 around its previous center. We then consider infinite geodesic rays in the vacant region emanating from the origin. We show that if $\lambda < \lambda_{gv}$, then a.s. for all $t \geq 0$ there are infinite rays emanating from the origin contained in that intersect only finitely many balls. We show that if $\lambda \geq \lambda_{gv}$ then a.s. for all $t \geq 0$ there are no infinite geodesics emanating from the origin contained in V. This means that there is no λ for which there are so called exceptional times for which the model behaves differently compared with the static one in this respect.

3 Some open problems

Timar [26] has recently shown the following theorem.

Theorem 3.1. *Consider Bernoulli bond percolation on a transitive unimodular graph. If C_1 and C_2 are two infinite clusters, then the set of vertices in C_1 that are at distance* 1 *from a vertex in C_2 is finite a.s.*

This gives rise to the following natural conjecture:

Conjecture 3.2. *Consider the Poisson Boolean model of continuum percolation in $\mathbb{H}^n$. Fix $r > 0$. If C_1 and C_2 are two unbounded components, then the number of Poisson points in C_1 that are at distance less than or equal to r from C_2 is finite a.s.*

Conjecture 3.3. *Consider the Poisson Boolean model of continuum percolation on a infinite homogeneous space M. Suppose there is a linear isoperimetric inequality. Then*

$$\lambda_c(M) < \lambda_u(M) \tag{3.1}$$

It might also be of interest to consider dynamical versions of the Poisson Boolean model. One such model was described at the end of Section 2.3. A different but at lest as natural, choice would be to let the centers of the balls move according to independent Brownian motions. Meester et. al. [19] considered this model on $\mathbb{R}^d$ and proved that for $\lambda > \lambda_c$ there are no exceptional times when percolation does not occur, and for $\lambda < \lambda_c$ there are no exceptional times when percolation does not occur. The case $\lambda = \lambda_c$ remains open.

Question 3.4. *Consider the Brownian motion dynamical version of the Poisson Boolean model of continuum percolation on $\mathbb{H}^2$. Are there exceptional times when percolation occurs on λ_c? Are there exceptional times when percolation occurs in V on λ_u?*

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Paper I

The number of unbounded components in the Poisson Boolean model of continuum percolation in hyperbolic space

Johan Tykesson*

Abstract

We consider the Poisson Boolean model of continuum percolation with balls of fixed radius R in n-dimensional hyperbolic space $\mathbb{H}^n$. Let λ be the intensity of the underlying Poisson process, and let N_C denote the number of unbounded components in the covered region. For the model in any dimension we show that there are intensities such that $N_C = \infty$ a.s. if R is big enough. In $\mathbb{H}^2$ we show a stronger result: for any R there are two intensities λ_c and λ_u where $0 < \lambda_c < \lambda_u < \infty$, such that $N_C = 0$ for $\lambda \in [0, \lambda_c]$, $N_C = \infty$ for $\lambda \in (\lambda_c, \lambda_u)$ and $N_C = 1$ for $\lambda \in [\lambda_u, \infty)$.

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1 Introduction

We begin by describing the fixed radius version of the so called *Poisson Boolean* model in $\mathbb{R}^n$, arguably the most studied continuum percolation model. For a detailed study of this model, we refer to [18]. Let X be a Poisson point process in $\mathbb{R}^n$ with some intensity λ. At each point of X, place a closed ball of radius R. Let C be the union of all balls, and V be the complement of C. The sets V and C will be referred to as the *vacant* and *covered* regions. We say that *percolation occurs* in C (respectively in V) if C (respectively V) contains unbounded (connected) components. For the Poisson Boolean model in $\mathbb{R}^n$, it is known that there is a *critical intensity* $\lambda_c \in (0, \infty)$ such that for $\lambda < \lambda_c$, percolation does not occur in C, and for $\lambda > \lambda_c$, percolation occurs in C. Also, there is a critical intensity $\lambda_c^* \in (0, \infty)$ such that percolation occurs in V if $\lambda < \lambda_c^*$ and percolation does not occur if $\lambda > \lambda_c^*$. Furthermore, if we denote by N_C and N_V the number of unbounded components of C and V respectively, then it is the case that N_C and N_V are both almost sure constants which are

*Department of Mathematical Sciences, Division of Mathematical Statistics, Chalmers University of Technology, S-41296 Göteborg, Sweden. E-mail: johant@math.chalmers.se.
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either 0 or 1. In $\mathbb{R}^2$ it is also known that $\lambda_c = \lambda_c^*$ and that at λ_c, percolation does not occur in C or V. For $n \geq 3$, Sarkar [21] showed that $\lambda_c < \lambda_c^*$, so that there exists an interval of intensities for which there is an unbounded component in both C and V.

It is possible to consider the Poisson Boolean model in more exotic spaces than $\mathbb{R}^n$, and one might ask if there are spaces for which several unbounded components coexist with positive probability. The main results of this paper is that this is indeed the case for n-dimensional hyperbolic space $\mathbb{H}^n$. We show that there are intensities for which there are almost surely infinitely many unbounded components in the covered region if R is big enough. In $\mathbb{H}^2$ we also show the existence of three distinct phases regarding the number of unbounded components, for any R. It turns out that the main difference between $\mathbb{R}^n$ and $\mathbb{H}^n$ which causes this, is the fact that there is a linear isoperimetric inequality in $\mathbb{H}^n$, which is a consequence of the constant negative curvature of the spaces. In $\mathbb{H}^2$, the linear isoperimetric inequality says that the circumference of a bounded simply connected set is always bigger than the area of the set.

The main result in $\mathbb{H}^2$ is inspired by a theorem due to Benjamini and Schramm. In [6] they show that for a large class of nonamenable planar transitive graphs, there are infinitely many infinite clusters for some parameters in Bernoulli bond percolation. For $\mathbb{H}^2$ we also show that the model does not percolate on λ_c. The discrete analogue of this theorem is due to Benjamini, Lyons, Peres and Schramm and can be found in [4]. It turns out that several techniques from the aforementioned papers are possible to adopt to the continuous setting in $\mathbb{H}^2$.

There is also a discrete analogue to the main result in $\mathbb{H}^n$. In [17], Pak and Smirnova show that for certain Cayley graphs, there is a non-uniqueness phase for the number of unbounded components. In this case, while it is still possible to adopt their main idea to the continuous setting, it is more difficult than for $\mathbb{H}^2$.

The rest of the paper is organized as follows. In section 2 we give a very short review of uniqueness and non-uniqueness results for infinite clusters in Bernoulli percolation on graphs (for a more extensive review, see the survey paper [14]), including the results by Benjamini, Lyons, Peres, Schramm, Pak and Smirnova. In section 3 we review some elementary properties of $\mathbb{H}^n$. In section 4 we introduce the model, and give some basic results. Section 5 is devoted to the proof of the main result in $\mathbb{H}^2$ and section 6 is devoted to the proof of the main theorem for the model in $\mathbb{H}^n$.

2 Non-uniqueness in discrete percolation

Let $G = (V, E)$ be an infinite connected transitive graph with vertex set V and edge set E. In p-Bernoulli bond percolation on G, each edge in E is kept with probability p and deleted with probability $1-p$, independently of all other edges. All vertices are kept. Let $\mathbf{P}_p$ be the probability measure on the subgraphs of G corresponding to p-Bernoulli percolation. (It is also possible to consider p-

Bernoulli site percolation in which it is the vertices that are kept or deleted, and all results we present in this section are valid in this case too.) In this section, ω will denote a random subgraph of G. Connected components of ω will be called *clusters.*

Let I be the event that p-Bernoulli bond percolation contains infinite clusters. One of the most basic facts in the theory of discrete percolation is that there is a critical probability $p_c = p_c(G) \in [0,1]$ such that $\mathbf{P}_p(I) = 0$ for $p < p_c(G)$ and $\mathbf{P}_p(I) = 1$ for $p > p_c(G)$. What happens on p_c depends on the graph. Above p_c it is known that there is 1 or ∞ infinite clusters for transitive graphs. If we let $p_u = p_u(G)$ be the infimum of the set of $p \in [0,1]$ such that p-Bernoulli bond percolation has a unique infinite cluster, Schonmann [22] showed for all transitive graphs, one has uniqueness for all $p > p_u$. Thus there are at most three phases for $p \in [0,1]$ regarding the number of infinite clusters, namely one for which this number is 0, one where the number is ∞ and finally one where uniqueness holds.

A problem which in recent years has attracted much interest is to decide for which graphs $p_c < p_u$. It turns out that whether a graph is *amenable* or not is central in settling this question:

For $K \subset V$, the *inner vertex boundary* of K is defined as $\partial_V K := \{y \in K : \exists x \notin K, [x,y] \in E\}$. The *vertex-isoperimetric* constant for G is defined as $\kappa_V(G) := \inf_W \frac{|\partial_V W|}{|W|}$ where the infimum ranges over all finite connected subsets W of V. A bounded degree graph $G = (V,E)$ is said to be *amenable* if $\kappa_V(G) = 0$.

Benjamini and Schramm [7] have made the following general conjecture:

Conjecture 2.1. *If G is transitive, then $p_u > p_c$ if and only if G is nonamenable.*

Of course, one direction of the conjecture is the well-known theorem by Burton and Keane [8] which says that any transitive, amenable graph G has a unique infinite cluster for all $p > p_c$.

The other direction of Conjecture 2.1 has only been partially solved. Here is one such result that will be of particular interest to us, due to Benjamini and Schramm [6]. This can be considered as the discrete analogue to our main theorem in $\mathbb{H}^2$. First, another definition is needed.

Definition 2.2. *Let $G = (V,E)$ be an infinite connected graph and for $W \subset V$ let N_W be the number of infinite clusters of $G \backslash W$. The number $\sup_W N_W$ where the supremum is taken over all finite W is called the* number of ends *of G.*

Theorem 2.3. *Let G be a nonamenable, planar transitive graph with one end. Then $0 < p_c(G) < p_u(G) < 1$ for Bernoulli bond percolation on G.*

Such a general result is not yet available for non-planar graphs. However, below we present a theorem by Pak and Smirnova [17] which proves non-uniqueness for a certain class of Cayley graphs.

Definition 2.4. *Let Γ be a finitely generated group and let $S = \{g_1^{\pm 1}, ..., g_n^{\pm 1}\}$ be a finite symmetric set of generators for Γ. The (right)* Cayley graph $\Gamma = \Gamma(G, S)$ *is the graph with vertex set Γ and $[g, h]$ is an edge in Γ if and only if $g^{-1}h \in S$.*

Let S^k be the multiset of elements of Γ of the type $g_1 g_2 ... g_k$, $g_1, ..., g_k \in S$ and each such element taken with multiplicity equal to the number of ways to write it in this way. Then S^k generates G.

Theorem 2.5. *Suppose $\Gamma = \Gamma(G, S)$ is a nonamenable Cayley-graph and let $\Gamma_k = \Gamma(G, S^k)$. Then for k large enough,*

$$p_c(\Gamma_k) < p_u(\Gamma_k).$$

Theorem 2.5 is the inspiration for our main result in $\mathbb{H}^n$.

3 Hyperbolic space

We consider the unit ball model of n-dimensional hyperbolic space $\mathbb{H}^n$, that is we consider $\mathbb{H}^n$ as the open unit ball in $\mathbb{R}^n$ equipped with the hyperbolic metric. The hyperbolic metric is the metric which to a curve $\gamma = \{\gamma(t)\}_{t=0}^1$ assigns length

$$L(\gamma) = 2 \int_0^1 \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2} dt,$$

and to a set E assigns volume

$$\mu(E) = 2^n \int_E \frac{dx_1 ... dx_n}{(1 - |x|^2)^n}.$$

The volume measure is invariant under all isometries of $\mathbb{H}^n$, see [20], p.82.

The *linear isoperimetric inequality* for $\mathbb{H}^2$ says that for all measurable $A \subset \mathbb{H}^2$ with $L(\partial A)$ and $\mu(A)$ well defined,

$$\frac{L(\partial A)}{\mu(A)} \geq 1. \tag{3.1}$$

Denote by $d(x, y)$ the hyperbolic distance between the points x and y. Let $S(x, r) := \{y : d(x, y) \leq r\}$ be the closed hyperbolic ball of radius r centered at x. In what follows, area (resp. length) will always mean hyperbolic area (resp. hyperbolic length). The volume of a ball is given by

$$\mu(S(0, r)) = B(n) \int_0^r \sinh(t)^{n-1}\, dt \tag{3.2}$$

where $B(n) > 0$ is a constant depending only on the dimension. We will make use of the fact that for any $\epsilon \in (0, r)$ there is a constant $K(\epsilon, n) > 0$ independent of r such that

$$\mu(S(0, r) \setminus S(0, r - \epsilon)) \geq K(\epsilon, n)\mu(S(0, r)) \tag{3.3}$$

for all r. For more facts about $\mathbb{H}^n$, we refer to [20].

3.1 Mass transport

Next, we present an essential ingredient to our proofs in $\mathbb{H}^2$, the mass transport principle which is due to Benjamini and Schramm [6]. We denote the group of isometries of $\mathbb{H}^2$ by Isom($\mathbb{H}^2$).

Definition 3.1. *A measure ν on $\mathbb{H}^2 \times \mathbb{H}^2$ is said to be* diagonally invariant *if for all measurable A, $B \subset \mathbb{H}^2$ and $g \in$Isom($\mathbb{H}^2$)*

$$\nu(gA \times gB) = \nu(A \times B).$$

Theorem 3.2. *(*Mass Transport Principle in $\mathbb{H}^2$*) If ν is a positive diagonally invariant measure on $\mathbb{H}^2 \times \mathbb{H}^2$ such that $\nu(A \times \mathbb{H}^2) < \infty$ for some open $A \subset \mathbb{H}^2$, then*

$$\nu(B \times \mathbb{H}^2) = \nu(\mathbb{H}^2 \times B)$$

for all measurable $B \subset \mathbb{H}^2$.

In our applications of Theorem 3.2, it is often the case that $\nu(A \times B) = \mathbf{E}[\eta(A, B, \cdot)]$ where the third argument is a random object like C or V, and $\eta(A, B, \cdot) = \eta(gA, gB, g(\cdot))$ for any $g \in$Isom($\mathbb{H}^2$). The function η can often intuitively be interpreted as the amount of mass that is transported from A to B. With this interpretation, the interpretation of $\nu(A \times B)$ becomes the expected amount of mass that goes from A to B. The mass transport principle then says that the expected amount of mass that goes out of B is the same as the expected amount of mass that goes into B.

4 The Poisson Boolean model in hyperbolic space

Definition 4.1. *A point process X on $\mathbb{H}^n$ distributed according to the probability measure $\mathbf{P}$ such that for $k \in \mathbb{N}$, $\lambda \geq 0$, and every measurable $A \subset \mathbb{H}^n$ one has*

$$\mathbf{P}[|X(A)| = k] = e^{-\lambda\mu(A)} \frac{(\lambda\mu(A))^k}{k!}$$

is called a Poisson process *with intensity λ on $\mathbb{H}^n$. Here $X(A) = X \cap A$ and $|\cdot|$ denotes cardinality.*

In the *Poisson Boolean model* in $\mathbb{H}^n$, at every point of a Poisson process X we place a ball with fixed radius R. More precisely, we let $C = \bigcup_{x \in X} S(x, R)$ and $V = C^c$ and refer to C and V as the covered and vacant regions of $\mathbb{H}^n$ respectively. For $A \subset \mathbb{H}^n$ we let $C[A] := \bigcup_{x \in X(A)} S(x, R)$ and $V[A] := C[A]^c$. For x, $y \in \mathbb{H}^n$ we write $x \leftrightarrow y$ if there is some curve connecting x to y which is completely covered by C. Let $d_C(x, y)$ be the length of the shortest curve connecting x and y lying completely in C if there exists such a curve, otherwise let $d_C(x, y) = \infty$. Similarly, let $d_V(x, y)$ be the length of the shortest curve connecting x and y lying completely in V if there is such a curve, otherwise let $d_V(x,y) = \infty$. The collection of all components of C is denoted by $\mathcal{C}$ and the collection of all components of V is denoted by $\mathcal{V}$. Let N_C denote the number of unbounded components in C and N_V denote the number of unbounded components in V. Next we introduce four *critical intensities* as follows. We let

$$\lambda_c := \inf\{\lambda : N_C > 0 \text{ a.s.}\},\ \lambda_u = \inf\{\lambda : N_C = 1 \text{ a.s.}\},$$

$$\lambda_c^* = \sup\{\lambda : N_V > 0 \text{ a.s. }\},\ \lambda_u^* = \sup\{\lambda : N_V = 1 \text{ a.s. }\}.$$

Our main result in $\mathbb{H}^2$ is:

Theorem 4.2. *For the Poisson Boolean model with fixed radius in* $\mathbb{H}^2$

$$0 < \lambda_c < \lambda_u < \infty.$$

Furthermore, with probability 1,

$$(N_C, N_V) = \begin{cases} (0,1), & \lambda \in [0, \lambda_c] \\ (\infty,\infty), & \lambda \in (\lambda_c, \lambda_u) \\ (1,0), & \lambda \in [\lambda_u, \infty) \end{cases}$$

The main result in $\mathbb{H}^n$ for any $n \geq 3$ is:

Theorem 4.3. *For the Poisson Boolean model with big enough fixed radius R in $\mathbb{H}^n$, $\lambda_c < \lambda_u$.*

In what follows, we present several quite basic results. The proofs of the following two lemmas, which give the possible values of N_C and N_V are the same as in the $\mathbb{R}^n$ case, see Propositions 3.3 and 4.2 in [18], and are therefore omitted.

Lemma 4.4. *N_C is an almost sure constant which equals* 0, 1 *or* ∞.

Lemma 4.5. *N_V is an almost sure constant which equals* 0, 1 *or* ∞.

Next we present some results concerning λ_c and λ_c^*.

Lemma 4.6. *For the Poisson Boolean model with balls of radius R in $\mathbb{H}^n$ it is the case that $\lambda_c(R) > \mu(S(0,2R))^{-1}$.*

The proof is identical to the $\mathbb{R}^n$ case, see Theorem 3.2 in [18].

Proposition 4.7. *Consider the Poisson Boolean model with balls of radius R in $\mathbb{H}^n$. There is $R_0 < \infty$ and a constant $K = K(n) > 0$ independent of R such that for all $R \geq R_0$ we have $\lambda_c(R) \leq K\mu(S(0,2R))^{-1}$.*

Proof. We prove the proposition using a supercritical branching process, the individuals of which are points in $\mathbb{H}^n$. The construction of this branching process is done by randomly distorting a regular tree embedded in the space.

Without loss of generality we assume that there is a ball centered at the origin, and the origin is taken to be the 0'th generation. Let a be such that a six-regular tree with edge length a can be embedded in $\mathbb{H}^2$ in such a way that the angles between edges at each vertex all equal $\pi/3$, and $d(u, v) \geq a$ for all vertices u and v in the tree. Suppose R is so large that $2R - 1 > a$.

Next pick three points x_1, x_2, x_3 on $\partial S(0, 2R) \cap \mathbb{H}^2$ such that the angles between the geodesics between the origin and the points is $2\pi/3$. We define a cell associated to x_i as the region in $S(0, 2R) \setminus S(0, 2R-1)$ which can be reached by a geodesic from the origin which diverts from the geodesic from the origin to x_i by an angle of at most $\pi/6$.

For every cell that contains a Poisson point, we pick one of these uniformly at random, and take these points to be the individuals of the first generation. We continue building the branching process in this manner. Given an individual y in the n:th generation, we consider an arbitrary hyperbolic plane containing y and its parent, and pick two points at distance $2R$ from y in this plane such that the angles between the geodesics from y to these two points and the geodesic from y to its parent are all equal to $2\pi/3$. Then to each of the new points, we associate a cell as before, and check if there are any Poisson points in them. If so, one is picked uniformly at random from each cell, and these points are the children of y.

We now verify that all the cells in which the individuals of the branching process were found are disjoint. By construction, if y is an individual in the branching process, the angles between the geodesics from y to its two possible children and its parent are all in the interval $(\pi/3, \pi)$, and therefore greater than the angles in a six-regular tree. Also, the lengths of these geodesics are in the interval $(2R - 1, 2R)$ and therefore larger than a. Thus by the choice of a, if all the individuals were in the same hyperbolic plane, the cells would all be disjoint.

Suppose all individuals are in $\mathbb{H}^2$, with the first individual at the origin. For each child of the origin we may pick two geodesics from the origin to infinity with angle θ less than $\pi/3$ between them that define a sector which contains the child and all of its descendants and no other individuals, and the angle between any of these two geodesics and the geodesic between the origin and the child is $\theta/2$. In the same way, for each child the grandchildren and their corresponding descendants can be divided into sectors with infinite geodesics emanating from the child and so on. Now, such a sector emanating from an individual will contain all the sectors that emanates from descendants in it.

From a sector emanating from an individual, we get a n-dimensional sector by rotating it along the geodesic going through the individual and its corresponding child. Then this n-dimensional sector will contain the corresponding n-dimensional sectors emanating from the child. From this it follows that the cells will always be disjoint.

Now, if the probability that a cell contains a poisson point is greater than $1/2$, then the expected number of children to an individual is greater than 1 and so there is a positive probability that the branching process will never die out, which in turn implies that there is an unbounded connected component in the covered region of $\mathbb{H}^n$.

Let B_R denote a cell. By 3.3 there is $K_1 > 0$ independent of R such that

$\mu(B_R) \geq K_1\mu(S(0,2R))$. By the above it follows that

$$\lambda_c(R) \leq \frac{\log 2}{\mu(B_R)} \leq \frac{\log 2}{K_1\mu(S(0,2R))},$$

completing the proof. □

Lemma 4.8. *For the Poisson Boolean model in $\mathbb{H}^2$, $\lambda_c^* < \infty$.*

Proof. Let Γ be a regular tiling of $\mathbb{H}^2$ into congruent polygons of finite diameter. The polygons of Γ can be identified with the vertices of a planar nonamenable transitive graph $G = (V, E)$. Next, we define a Bernoulli site percolation ω on G. We declare each vertex $v \in V$ to be in ω if and only if its corresponding polygon $\Gamma(v)$ is not completely covered by $C[\Gamma(v)]$. Clearly, the vertices are declared to be in ω or not with the same probability and independently of each other. Now for any v,

$$\lim_{\lambda\to\infty} \mathbf{P}[v \text{ is in } \omega] = 0.$$

Thus, by Theorem 2.3, for λ large enough, there are no infinite clusters in ω. But if there are no infinite clusters in ω, there are no unbounded components of V. Thus $\lambda_c^* < \infty$. □

In $\mathbb{H}^2$, we will need a correlation inequality for *increasing* and *decreasing* events. If ω and ω' are two realizations of a Poisson Boolean model we write $\omega \preceq \omega'$ if any ball present in ω is also present in ω'. An event A measurable with respect to the Poisson process is said to be *increasing* (respectively *decreasing*) if $\omega \preceq \omega'$ implies $1_A(\omega) \leq 1_A(\omega')$ (respectively $1_A(\omega) \geq 1_A(\omega')$).

Theorem 4.9. *(*FKG INEQUALITY*) If A and B are both increasing or both decreasing events measurable with respect to the Poisson process X, then $\mathbf{P}[A \cap B] \geq \mathbf{P}[A]\mathbf{P}[B]$.*

The proof is almost identical to the proof in the $\mathbb{R}^n$ case, see Theorem 2.2 in [18]. In particular, we will use the following simple corollary to Theorem 4.9, the proof of which can be found in [12], which says that if A_1, A_2, ..., A_m are increasing events with the same probability, then

$$\mathbf{P}[A_1] \geq 1 - \left(1 - \mathbf{P}[\cup_{i=1}^m A_i]\right)^{1/m}.$$

The same holds when A_1, A_2, ..., A_m are decreasing.

For the proof of Theorem 4.2 we need the following lemma, the proof of which is identical to the discrete case, see [14].

Lemma 4.10. *If $\lim_{d(u,v)\to\infty} \mathbf{P}[u \leftrightarrow v] = 0$ then there is a.s. not a unique unbounded component in C.*

5 The number of unbounded components in $\mathbb{H}^2$

The aim of this section is to prove Theorem 4.2. We perform the proof in the case $R = 1$ but the arguments are the same for any R. We first determine the possible values of (N_C, N_V) for the model in $\mathbb{H}^2$. The first lemma is an application of the mass transport principle. First, some notation is needed.

Definition 5.1. *If H is a random subset of $\mathbb{H}^2$, we say that the distribution of H is* Isom($\mathbb{H}^2$)-invariant *if gH has the same distribution as H for all $g \in$* Isom($\mathbb{H}^2$)*.*

In our applications, H will typically be a union of components from C or V or something similar.

Lemma 5.2. *Suppose H is a random subset of $\mathbb{H}^2$, such that its distribution is Isom($\mathbb{H}^2$)-invariant, and such that it contains only countably many connected components. Also suppose that if $A \subset \mathbb{H}^2$ is measurable and bounded, then $L(A \cap \partial H)$ is well-defined and has finite expectation. If H contains only finite components a.s., then for any measurable $A \subset \mathbb{H}^2$*

$$\mathbf{E}[\mu(A \cap H)] \leq \mathbf{E}[L(A \cap \partial H)].$$

Before the proof we describe the intuition behind it: we place mass of unit density in all of $\mathbb{H}^2$. Then, if h is a component of H, the mass inside h is transported to the boundary of h. Then we use the mass transport principle: the expected amount of mass transported out of a subset A equals the expected amount of mass transported into it. Finally we combine this with the isoperimetric inequality (3.1).

Proof. For A, $B \subset \mathbb{H}^2$, let

$$\eta(A \times B,\, H) := \sum_{h\,:\,L(\partial h)>0} \frac{\mu(B \cap h)\, L(A \cap \partial h)}{L(\partial h)}.$$

and let $\nu(A \times B) := \mathbf{E}[\eta(A \times B,\, H)]$. (Note that only components h that intersect both A and B give a non-zero contribution to the sum above.) Since the distribution of H is Isom($\mathbb{H}^2$)-invariant, we get for each $g \in$Isom($\mathbb{H}^2$)

$$\nu(gA \times gB) = \mathbf{E}[\eta(gA \times gB,\, H)] = \mathbf{E}[\eta(gA \times gB,\, gH)]$$
$$= \mathbf{E}[\eta(A \times B,\, H)] = \nu(A \times B).$$

Thus, ν is a diagonally invariant positive measure on $\mathbb{H}^2 \times \mathbb{H}^2$. We have $\nu(\mathbb{H}^2 \times A) = \mathbf{E}\,[\mu(A \cap H)]$ and

$$\nu(A \times \mathbb{H}^2) = \mathbf{E}\left[\sum_{h\,:\,L(\partial h)>0} \frac{\mu(h)\, L(A \cap \partial h)}{L(\partial h)}\right] \leq \mathbf{E}[L(A \cap \partial H)]$$

where the last inequality follows from the linear isoperimetric inequality. Hence, the claim follows by Theorem 3.2. ☐

In the following lemmas, we exclude certain combinations of N_C and N_V. The first lemma can be considered as a continuous analogue to Lemma 3.3 in [6].

Lemma 5.3. *If H is a union of components from $\mathcal{C}$ and $\mathcal{V}$ such that the distribution of H is Isom($\mathbb{H}^2$)-invariant, then H and/or H^c contains unbounded components almost surely.*

Proof. Suppose H and $D := H^c$ contains only finite components, and let in this proof $\mathcal{H}_0$ and $\mathcal{D}_0$ be the collections of the components of H and D respectively. Then every element h of $\mathcal{H}_0$ is surrounded by a unique element h' of $\mathcal{D}_0$, which in turn is surrounded by a unique element h'' of $\mathcal{H}_0$. In the same way, every element d of $\mathcal{D}_0$ is surrounded by a unique element d' of $\mathcal{H}_0$ which in turn is surrounded by a unique element d'' of $\mathcal{D}_0$. Inductively, for $j \in \mathbb{N}$, let $\mathcal{H}_{j+1} := \{h'' : h \in \mathcal{H}_j\}$ and $\mathcal{D}_{j+1} := \{d'' : d \in \mathcal{D}_j\}$. Next, for $r \in \mathbb{N}$, let

$$A_r := \bigcup_{j=0}^{r} (\{h \in \mathcal{H}_0 : \sup\{i : h \in \mathcal{H}_i\} = j\} \cup \{d \in \mathcal{D}_0 : \sup\{i : d \in \mathcal{D}_i\} = j\}).$$

In words, $\mathcal{H}_j$ and $\mathcal{D}_j$ define layers of components from H and D. Thus A_r is the union of all layers of components from H and D that have at most r layers inside of them. Now let B be some ball in $\mathbb{H}^2$. Note that $L(B \cap \partial A_r) \le L(B \cap \partial C)$ and $\mathbf{E}[L(B \cap \partial C)] < \infty$. Also, almost surely, there is some random r_0 such that B will be completely covered by A_r for all $r \ge r_0$. Thus the dominated convergence theorem gives

$$\lim_{r\to\infty} \mathbf{E}[\mu(B \cap A_r)] = \mu(B) \text{ and } \lim_{r\to\infty} \mathbf{E}[L(B \cap \partial A_r)] = 0.$$

Since the distribution of A_r is Isom($\mathbb{H}^2$)-invariant we get by Lemma 5.2 that there is $r_1 < \infty$ such that for $r \ge r_1$,

$$\mathbf{P}[A_r \text{ has unbounded components}] > 0.$$

But by construction, for any r it is the case that A_r has only finite components. Hence the initial assumption is false. ☐

Lemma 5.4. *The cases $(N_C, N_V) = (\infty, 1)$ and $(N_C, N_V) = (1, \infty)$ have probability* 0.

Proof. Suppose $N_C = \infty$. First we show that it is possible to pick $r > 0$ such that the event

$$A(x,r) := \{S(x,r) \text{ intersects at least 2 disjoint unbounded components of } C[S(x,r)^c]\}$$

has positive probability for $x \in \mathbb{H}^2$. Suppose $S(x, r_0)$ intersects an unbounded component of C for some $r_0 > 0$. Then if $S(x, r_0)$ does not intersect some unbounded component of $C[S(x, r_0)^c]$, there must be some ball centered in $S(x, r_0+2) \backslash S(x, r_0+1)$ being part of an unbounded component of $C[S(x, r_0+1)^c]$, which is to say that $S(x, r_0+1)$ intersects an unbounded component of $C[S(x, r_0+1)^c]$. Clearly we can find $\tilde{r}$ such that

$$B(x, \tilde{r}) := \{S(x, \tilde{r}) \text{ intersects at least 3 disjoint unbounded components of } C\}.$$

By the above discussion it follows that $\mathbf{P}[A(x, \tilde{r}) \cup A(x, \tilde{r}+1)] > 0$, which proves the existence of r such that $A(x, r)$ has positive probability. Pick such an r and let $E(x, r) := \{S(x, r) \subset C[S(x, r)]\}$. E has positive probability and is independent of A so $A \cap E$ has positive probability. By planarity, on $A \cap E$, V contains at least 2 unbounded components. So with positive probability, $N_V > 1$. By Lemma 4.5, $N_V = \infty$ a.s. This finishes the first part of the proof. Now instead suppose $N_V = \infty$ and pick $r > 0$ such that

$$A(x, r) := \{S(x, r) \text{ intersects at least two unbounded components of } V\}$$

has positive probability. Let

$$B(x, r) := \{C[S(x, r+1)^c] \text{ contains at least 2 unbounded components}\}.$$

On A, $C \backslash S(x, r)$ contains at least two unbounded components, which in turn implies that B occurs. Since $\mathbf{P}[A] > 0$ this gives $\mathbf{P}[B] > 0$. Since B is independent of $F(x, r) := \{|X(S(x, r+1))| = 0\}$ which has positive probability, $\mathbf{P}[B \cap F] > 0$. On $B \cap F$, C contains at least two unbounded components. By Lemma 4.4 we get $N_C = \infty$ a.s. □

The proof of the next lemma is very similar to the discrete case, see Lemma 11.12 in [12], but is included for the convenience of the reader.

Lemma 5.5. *The case* $(N_C, N_V) = (1, 1)$ *has probability* 0.

Proof. Assume $(N_C, N_V) = (1, 1)$ a.s. Fix $x \in \mathbb{H}^2$. Denote by $A_C^u(k)$ (respectively $A_C^d(k)$, $A_C^r(k)$, $A_C^l(k)$) the event that the uppermost (respectively lowermost, rightmost, leftmost) quarter of $\partial S(x, k)$ intersects an unbounded component of $C \backslash S(x, k)$. Clearly, these events are increasing. Since $N_C = 1$ a.s.,

$$\lim_{k \to \infty} \mathbf{P}[A_C^u(k) \cup A_C^d(k) \cup A_C^r(k) \cup A_C^l(k)] = 1.$$

Hence by the corollary to the FKG-inequality, $\lim_{k \to \infty} \mathbf{P}[A_C^t(k)] = 1$ for $t \in \{u, d, r, l\}$. Now let $A_V^u(k)$ (respectively $A_V^l(k)$, $A_V^r(k)$, $A_V^l(k)$) be the event that the uppermost (respectively lowermost, rightmost, leftmost) quarter of $\partial S(x, k)$ intersects an unbounded component of $V \backslash S(x, k)$. Since these events are decreasing, we get in the same way as above that $\lim_{k \to \infty} \mathbf{P}[A_V^t(k)] = 1$ for $t \in \{u, d, r, l\}$. Thus we may pick k_0 so big that $\mathbf{P}[A_C^t(k_0)] > 7/8$ and

$\mathbf{P}[A_V^t(k_0)] > 7/8$ for $t \in \{u, d, r, l\}$. Let

$$A := A_C^u(k_0) \cap A_C^d(k_0) \cap A_V^l(k_0) \cap A_V^r(k_0).$$

Bonferroni's inequality implies $\mathbf{P}[A] > 1/2$. On A, $C \setminus S(x, k_0)$ contains two disjoint unbounded components. Since $N_C = 1$ a.s., these two components must almost surely on A be connected. The existence of such a connection implies that there are at least two unbounded components of V, an event with probability 0. This gives $\mathbf{P}[A] = 0$, a contradiction. □

Proposition 5.6. *Almost surely,* $(N_C, N_V) \in \{(1,0), (0,1), (\infty,\infty)\}$.

Proof. By Lemmas 4.4 and 4.5, each of N_C and N_V is in $\{0, 1, \infty\}$. Lemma 5.3 with $H \equiv C$ rules out the case $(0,0)$. Hence Lemmas 5.4 and 5.5 imply that it remains only to rule out the cases $(0,\infty)$ and $(\infty,0)$. But since every two unbounded components of C must be separated by some unbounded component of V, $(\infty,0)$ is impossible. In the same way, $(0,\infty)$ is impossible. □

5.1 The situation at λ_c and λ_c^*

It turns out that to prove the main theorem, it is necessary to investigate what happens regarding N_C and N_V at the intensities λ_c and λ_c^*. Our proofs are inspired by the proof of Theorem 1.1 in [4], which says that critical Bernoulli bond and site percolation on nonamenable Cayley graphs does not contain infinite clusters.

Theorem 5.7. *At* λ_c, $N_C = 0$ *a.s.*

Proof. We begin with ruling out the possibility of a unique unbounded component of C at λ_c. Suppose $\lambda = \lambda_c$ and that $N_C = 1$ a.s. Denote the unique unbounded component of C by U. By Proposition 5.6, V contains only finite components a.s. Let $\epsilon > 0$ be small and remove each point in X with probability ϵ and denote by X_ϵ the remaining points. Furthermore, let $C_\epsilon = \cup_{x \in X_\epsilon} S(x, 1)$. Since X_ϵ is a Poisson process with intensity $\lambda_c(1-\epsilon)$ it follows that C_ϵ will contain only bounded components a.s. Let $\mathcal{C}_\epsilon$ be the collection of all components of C_ϵ. We will now construct H_ϵ as a union of elements from $\mathcal{C}_\epsilon$ and $\mathcal{V}$ such that the distribution of H_ϵ will be Isom($\mathbb{H}^2$)-invariant. For each $z \in \mathbb{H}^2$ we let $U_\epsilon(z)$ be the union of the components of $U \cap C_\epsilon$ being closest to z. We let each h from $\mathcal{C}_\epsilon \cup \mathcal{V}$ be in H_ϵ if and only if $\sup_{z \in h} d(z, U) < 1/\epsilon$ and $U_\epsilon(x) = U_\epsilon(y)$ for all $x, y \in h$. We want to show that for ϵ small enough, H_ϵ contains unbounded components with positive probability. Let B be some ball. It is clear that $L(B \cap \partial H_\epsilon) \to 0$ a.s. and also that $\mu(B \cap H_\epsilon) \to \mu(B)$ a.s. when $\epsilon \to 0$. Also $L(B \cap \partial H_\epsilon) \le L(B \cap (\partial C_\epsilon \cup \partial C))$ and $\mathbf{E}[L(B \cap (\partial C_\epsilon \cup \partial C))] \le K < \infty$ for some constant K independent of ϵ. By the dominated convergence theorem, we have

$$\lim_{\epsilon \to 0} \mathbf{E}[\mu(B \cap H_\epsilon)] = \mu(B) \text{ and } \lim_{\epsilon \to 0} \mathbf{E}[L(B \cap \partial H_\epsilon)] = 0.$$

Therefore we get by Lemma 5.2 that H_ϵ contains unbounded components with positive probability when ϵ is small enough. Suppose $h_1, h_2, ...$ is an infinite sequence of distinct elements from $\mathcal{C}_\epsilon \cup \mathcal{V}$ such that they constitute an unbounded component of H_ϵ. Then $U_\epsilon(x) = U_\epsilon(y)$ for all x, y in this component. Hence $U \cap C_\epsilon$ contains an unbounded component (this particular conclusion could not have been made without the condition $\sup_{z \in h} d(z, U) < 1/\epsilon$ in the definition of $U_\epsilon(z)$). Therefore we conclude that the existence of an unbounded component in H_ϵ implies the existence of an unbounded component in C_ϵ. Hence C_ϵ contains an unbounded component with positive probability, a contradiction.

We move on to rule out the case of infinitely many unbounded components of C at λ_c. Assume $N_C = \infty$ a.s. at λ_c. As in the proof of Lemma 5.4, we choose r such that for $x \in \mathbb{H}^2$ the event

$$A(x,r) := \{S(x,r) \text{ intersects at least 3 disjoint unbounded components of } C[S(x,r)^c]\}$$

has positive probability. Let $B(x,r) := \{S(x,r) \subset C[S(x,r)]\}$ for $x \in \mathbb{H}^2$. Since A and B are independent, it follows that $A \cap B$ has positive probability. On $A \cap B$, x is contained in an unbounded component U of C. Furthermore, $U \backslash S(x, r+1)$ contains at least three disjoint unbounded components. Now let Y be a Poisson process independent of X with some positive intensity. We call a point $y \in \mathbb{H}^2$ a *encounter point* if

- $y \in Y$;
- $A(y,r) \cap B(y,r)$ occurs;
- $S(y, 2(r+1)) \cap Y = \{y\}$.

The third condition above means that if y_1 and y_2 are two encounter points, then $S(y_1, r+1)$ and $S(y_2, r+1)$ are disjoint sets. By the above, it is clear that given $y \in Y$, the probability that y is an encounter point is positive.

We now move on to show that if y is an encounter point and U is the unbounded component of C containing y, then each of the disjoint unbounded components of $U \backslash S(y, r+1)$ contains a further encounter point.

Let $m(s,t) = 1$ if t is the unique encounter point closest to s in C, and $m(s,t) = 0$ otherwise. Then let for measurable sets A, $B \subset \mathbb{H}^2$

$$\eta(A \times B) = \sum_{s \in Y(A)} \sum_{t \in Y(B)} m(s,t)$$

and

$$\nu(A \times B) = \mathbf{E}[\eta(A \times B)].$$

Clearly, ν is a positive diagonally invariant measure on $\mathbb{H}^2 \times \mathbb{H}^2$. Suppose A is some ball in $\mathbb{H}^2$. Since $\sum_{t \in Y} m(s,t) \leq 1$ we get $\nu(A \times \mathbb{H}^2) \leq \mathbf{E}[|Y(A)|] < \infty$. On the other hand, if y is an encounter point lying in A and with positive probability there is no encounter point in some of the unbounded components of $U\backslash S(y,r+1)$ we get $\sum_{s\in Y}\sum_{t\in Y(A)} m(s,t) = \infty$ with positive probability, so $\nu(\mathbb{H}^2 \times A) = \infty$, which contradicts Theorem 3.2.

The proof now continues with the construction of a forest F, that is a graph without loops or cycles. Denote the set of encounter points by T, which is a.s. infinite by the above. We let each $t \in T$ represent a vertex $v(t)$ in F. For a given $t \in T$, let $U(t)$ be the unbounded component of C containing t. Then let k be the number of unbounded components of $U(t)\backslash S(t,r+1)$ and denote these unbounded components by C_1, C_2,..., C_k. For $i = 1, 2, ..., k$ put an edge between $v(t)$ and the vertex corresponding to the encounter point in C_i which is closest to t in C (this encounter point is unique by the nature of the Poisson process).

Next, we verify that F constructed as above is indeed a forest. If v is a vertex in F, denote by $t(v)$ the encounter point corresponding to it. Suppose $v_0, v_1, ..., v_n = v_0$ is a cycle of length ≥ 3, and that $d_C(t(v_0), t(v_1)) < d_C(t(v_1), t(v_2))$. Then by the construction of F it follows that $d_C(t(v_1), t(v_2)) < d_C(t(v_2), t(v_3)) < ... < d_C(t(v_{n-1}), t(v_0)) < d_C(t(v_0), t(v_1))$ which is impossible. Thus we must have that $d_C(t(v_i), t(v_{i+1}))$ is the same for all $i \in \{0, 1, .., n-1\}$. The assumption $d_C(t(v_0), t(v_1)) > d_C(t(v_1), t(v_2))$ obviously leads to the same conclusion. But if $y \in Y$, the probability that there are two other points in Y on the same distance in C to y is 0. Hence, cycles exist with probability 0, and therefore F is almost surely a forest.

Now define a bond percolation $F_\epsilon \subset F$: Define C_ϵ in the same way as above. Let each edge in F be in F_ϵ if and only if both encounter points corresponding to its end-vertices are in the same component of C_ϵ. Since C_ϵ contains only bounded components, F_ϵ contains only finite connected components.

For any vertex v in F we let $K(v)$ denote the connected component of v in F_ϵ and let $\partial_F K(v)$ denote the inner vertex boundary of $K(v)$ in F. Since the degree of each vertex in F is at least 3, and F is a forest, it follows that at least half of the vertices in $K(v)$ are also in $\partial_F K(v)$. Thus we conclude

$$\mathbf{P}[x \in T,\, v(x) \in \partial_F K(v(x)) | x \in Y] \geq \frac{1}{2}\mathbf{P}[x \in T | x \in Y].$$

The right-hand side of the above is positive and independent of ϵ. But the left-hand side tends to 0 as ϵ tends to 0, since when ϵ is small, it is unlikely that an edge in F is not in F_ϵ. This is a contradiction. □

By Proposition 5.6, if $N_C = 0$ a.s., then $N_V = 1$ a.s. Thus we have an immediate corollary to Theorem 5.7.

Corollary 5.8. *At λ_c, $N_V = 1$ a.s.*

Next, we show the corresponding results for λ_c^*. Obviously, the nature of V is quite different from that of C, but still the proof of Theorem 5.9 below differs only in details to that of Theorem 5.7. We include it for the convenience of the reader.

Theorem 5.9. *At λ_c^*, $N_V = 0$ a.s.*

Proof. Suppose $N_V = 1$ a.s. at λ_c^* and denote the unbounded component of V by U. Then C contains only finite components a.s. by Proposition 5.6. Let $\epsilon > 0$ and let Z be a Poisson process independent of X with intensity ϵ. Let $C_\epsilon := \cup_{x \in X \cup Z} S(x,1)$ and $V_\epsilon := C_\epsilon^c$. Since $X \cup Z$ is a Poisson process with intensity $\lambda_c^* + \epsilon$ it follows that C_ϵ has a unique unbounded component a.s. and hence V_ϵ contains only bounded components a.s. Let $\mathcal{V}_\epsilon$ be the collection of all components of V_ϵ. Define H_ϵ in the following way: For each $z \in \mathbb{H}^2$ we let $U_\epsilon(z)$ be the union of the components of $U \cap V_\epsilon$ being closest to z. We let each $h \in \mathcal{C} \cup \mathcal{V}_\epsilon$ be in H_ϵ if and only if $\sup_{z \in h} d(z, U) < 1/\epsilon$ and $U_\epsilon(x) = U_\epsilon(y)$ for all $x, y \in h$. As in the proof of Theorem 5.7, for $\epsilon > 0$ small enough, H_ϵ contains an unbounded component with positive probability, and therefore V_ϵ contains an unbounded component with positive probability, a contradiction.

Now suppose that $N_V = \infty$ a.s. at λ_c^*. Then also $N_C = \infty$ by Proposition 5.6. Therefore, for $x \in \mathbb{H}^2$, we can choose $r > 1$ big such that the intersection of the two independent events

$$A(x,r) := \\ \{S(x,r) \text{ intersects at least 3 disjoint unbounded components of } C[S(x,r)^c]\}$$

and $B(x,r) := \{|X(S(x,r))| = 0\}$ has positive probability. Next, suppose that Y is a Poisson process independent of X with some positive intensity. Now we redefine what an encounter point is: call $y \in \mathbb{H}^2$ an encounter point if

- $y \in Y$;
- $A(y,r) \cap B(y,r)$ ocurrs;
- $S(y,2r) \cap Y = \{y\}$.

By the above discussion,

$$\mathbf{P}[y \text{ is an encounter point} \mid y \in Y] > 0.$$

If y is a encounter point, y is contained in an unbounded component U of V and $U \backslash S(y,r)$ contains at least 3 disjoint unbounded components. Again we construct a forest F using the encounter points and define a bond percolation $F_\epsilon \subset F$. Let V_ϵ be defined as above. Each edge of F is declared to be in F_ϵ if and only if both its end-vertices are in the same component of V_ϵ. The proof is now finished in the same way as Theorem 5.7. ☐

Again, Proposition 5.6 immediately implies the following corollary:

Corollary 5.10. *At λ_c^*, $N_C = 1$ a.s.*

5.2 Proof of Theorem 4.2

Here we combine the results from the previous sections to prove our main theorem in $\mathbb{H}^2$.

Proof of Theorem 4.2. If $\lambda < \lambda_u$ then Proposition 5.6 implies $N_V > 0$ a.s. giving $\lambda \leq \lambda_c^*$. If $\lambda > \lambda_u$ the same proposition gives $N_V = 0$ a.s. giving $\lambda \geq \lambda_c^*$. Thus

$$\lambda_u = \lambda_c^*. \tag{5.1}$$

By Theorem 5.7 $N_C = 0$ a.s. at λ_c, so $N_V > 0$ a.s. at λ_c by Proposition 5.6. Thus by Theorem 5.9

$$\lambda_c < \lambda_c^*. \tag{5.2}$$

Hence the desired conclusion follows by (5.1), (5.2) and Lemma 4.8. □

6 The number of unbounded components in $\mathbb{H}^n$

This section is devoted to the proof of Theorem 4.3. First, we recall a method for dominating the distribution of $C(u)$, the component of C containing $u \in \mathbb{H}^n$. This method is found in for example [15].

Construct a branching process, whose members are points in $\mathbb{H}^n$, as follows. The point in the 0:th generation is taken to be u. Given points $Z_{n1}, Z_{n2}, ..., Z_{nN_n}$ in the n:th generation, the $(n+1)$:th generation is defined as follows. For $l = 1, ..., N_n$ let X_{nl} be a Poisson process with intensity λ, independent of the previous history of the branching process and also of $X_{nl'}$ for $l \neq l'$. At each point of X_{nl} center a ball of radius R. The progeny of Z_{nl} is then taken to be the points of X_{nl} whose associated balls intersect that of Z_{nl}. Observe that the distance between a point and its parent is at most $2R$. The number of descendants of Z_{nl} has a Poisson distribution with expectation $\lambda\mu(S(0, 2R))$. Therefore, by independence, the expected number of points in generation n is given by $\lambda^n \mu(S(0, 2R))^n$.

We next describe how to get a connected component with the same distribution as $C(u)$ using the above branching process.

To each point of the branching process, associate a ball of radius R centered at the point. Then color the balls of the branching process except the ball in the 0:th generation green or red as follows. The balls in the first generation that contain u (the point in the 0:th generation) are colored green, the other balls in the first generation are colored red. Then by induction proceed as follows. For $n \geq 2$, given the colors of the balls in the $n-1$ first generations, for $l = 1, ..., N_n$ the ball corresponding to the point Z_{nl} is colored green if the following conditions are met:

1. The ball corresponding to the parent of Z_{nl} is green.
2. It does not intersect some green ball from a generation earlier than $n-1$.
3. It does not intersect some green ball corresponding to some parent of any of Z_{nk} for $k < l$.

4. It does not contain u.

Otherwise, it is colored red.

Then it is easy to see that the union of green balls has the same distribution as $C(u)$. Thus the distribution of the union of all balls centered at points of the branching process dominates that of $C(u)$.

First part of proof of Theorem 4.3. In view of Lemma 4.10, it is enough to show that $\mathbf{P}[u \leftrightarrow v] \to 0$ as $d(u,v) \to \infty$ for some intensity above λ_c. Fix points u and v and let $d = d(u,v)$. We use a duplication trick. Let $X_1(u)$ and $X_2(u)$ be two independent branching processes as described above, where the point in the 0:th generation is taken to be u for both of them. Let $\tilde{C_1}(u)$ and $\tilde{C_2}(u)$ be the union of balls of radius R placed at every point of $X_1(u)$ and $X_2(u)$ respectively. Let $C_1(u)$ and $C_2(u)$ be the union of the green balls from $\tilde{C_1}(u)$ and $\tilde{C_2}(u)$. If we for some $\epsilon > 0$ can find points u and v on an arbitrarily large distance from each other such that u is connected to v in $C_1(u)$ with probability at least ϵ, then the event

$$B(u,v) := \{u \text{ is connected to } v \text{ in both } C_1(u) \text{ and } C_2(u)\}$$

has probability at least ϵ^2 since $C_1(u)$ and $C_2(u)$ are independent. So it is enough to show that $\mathbf{P}[B(u,v)] \to 0$ as $d(u,v) \to \infty$ for some intensity above λ_c. Let

$$\tilde{B}(u,v) := \{u \text{ is connected to } v \text{ in both } \tilde{C_1}(u) \text{ and } \tilde{C_2}(u)\}.$$

Then clearly

$$\mathbf{P}[B(u,v)] \leq \mathbf{P}[\tilde{B}(u,v)].$$

Let $k = \lceil d/(2R) \rceil$ be the smallest number of balls of radius R needed to connect u and v. Observe that $\tilde{B}(u,v)$ is included in the event that there are integers $l_1, l_2 \geq k-1$ such that there is at least one point x_1 in generation l_1 of $X_1(u)$ and one point x_2 in generation l_2 of $X_2(u)$ such that $d(x_1,x_2) < 2R$ (if two balls intersect, the distance between their corresponding centers is at most $2R$). Below, when we mention (sequences of) points with a certain property, we assume that they are chosen uniformly at random among all (sequences of) points with the property on the event that they exist.

Suppose that y_n is a point in the n:th generation of $X_1(u)$ and z_m is a point in the m:th generation of $X_2(u)$. By independence of $X_1(u)$ and $X_2(u)$, the expected number of such pairs (y_n, z_m) is given by $(\lambda\mu(S(0,2R))^{n+m}$.

Observe that the distance between y_n and z_m has the same distribution as the distance between u and a point in generation $n+m$ of $X_1(u)$. Let $P_R(n)$ be the probability that a point in generation n of $X_1(u)$ is at distance at most $2R$ from u. Then the expected number of points in generation n of $X_1(u)$ at distance at most $2R$ from u is given by $(\lambda\mu(S(0,2R)))^n P_R(n)$. Consequently, the expected number of pairs (y_n, z_m) with $d(y_n, z_m) < 2R$ is given by $(\lambda\mu(S(0,2R)))^{m+n} P_R(m+n)$.

Therefore we get

$$\mathbf{P}[\tilde{B}(u,v)] \le \sum_{m=k-1}^{\infty} \sum_{n=k-1}^{\infty} (\lambda\mu(S(0,2R)))^{m+n} P_R(n+m).$$

We will now estimate the terms in the sum above.

Lemma 6.1. *There is a sequence of i.i.d. random variables $Y_1, Y_2, ..., Y_{n-1}$ with positive mean such that*

$$P_R(n) \le \mathbf{P}\left[\sum_{i=1}^{n-1} Y_i \le 2R\right].$$

The distribution of Y_i will be defined in the proof.

First part of proof of Lemma 6.1. Suppose $y_0 = u, y_1, ..., y_n$ is a sequence of points in $\tilde{X}_1(u)$ where y_{i-1} is the parent of y_i. Given y_i, the distribution of y_{i+1} is the uniform distribution on $S(y_i, 2R)$ (with respect to the volume measure). Let $d_i := d(y_i, y_{i+1})$.

Then $d_0, d_1 ..., d_{n-1}$ is a sequence of independent random variables with density

$$\frac{d}{dr}\frac{\mu(S(0,r))}{\mu(S(0,2R))} = \frac{\sinh(r)^{n-1}}{\int_0^{2R}\sinh(t)^{n-1}\,dt} \text{ for } r \in [0,2R]. \tag{6.1}$$

Next we write

$$\mathbf{P}[d(y_0,y_n) < 2R] = \mathbf{P}\left[\sum_{i=0}^{n-1}(d(y_0,y_{i+1}) - d(y_0,y_i)) < 2R\right]. \tag{6.2}$$

The terms in the sum 6.2 are neither independent nor identically distributed. However, we will see that the sum is always larger than a sum of i.i.d. random random variables with positive mean. Suppose without loss of generality that $y_0 = u = 0$. Let γ_i be the geodesic between 0 and y_i and let φ_i be the geodesic between y_i and y_{i+1} for $i \ge 1$. Let θ_i be the angle between γ_i and φ_i for $i \ge 1$ and let $\theta_0 = \pi$. Then $\theta_1, \theta_2, ..., \theta_{n-1}$ is a sequence of independent random variables, uniformly distributed on $[0,\pi]$. Since the geodesics γ_i, γ_{i+1} and φ_i are in the same hyperbolic plane, we can express $d(0,y_{i+1})$ in terms of $d(0,y_i)$, d_i and θ_i using the first law of cosines for triangles in hyperbolic space (see [20], Theorem 3.5.3), which gives that

$$
(6.3)\quad \begin{aligned} d(0,y_{i+1}) - d(0,y_i) = \cosh^{-1}\Big(&\cosh(d_i)\cosh(d(0,y_i)) \\ &- \sinh(d_i)\sinh(d(0,y_i))\cos(\theta_i)\Big) - d(0,y_i). \end{aligned}
$$

Next we prove a lemma that states that the random variable above dominates a random variable which is independent of $d(0, y_i)$. Put

$$
f(x,y,\theta) := \cosh^{-1}(\cosh(x)\cosh(y) - \sinh(x)\sinh(y)\cos(\theta)) - y.
$$

Lemma 6.2. *For fixed x and θ, the function $f(x, y, \theta)$ is strictly decreasing in y and $g(x,\theta) := \lim_{y\to\infty} f(x,y,\theta) = \log(\cosh(x) - \sinh(x)\cos(\theta))$.*

Proof. For simplicity write $a = a(x) := \cosh(x)$ and $b = b(x,\theta) := \sinh(x)\cos(\theta)$. Then by rewriting

$$
(6.4)\quad f(x,y,\theta) = \log\left(\frac{a\cosh(y) - b\sinh(y) + \sqrt{(a\cosh(y) - b\sinh(y))^2 - 1}}{\exp(y)}\right)
$$

we get by easy calculations that the limit as $y \to \infty$ is as desired. It remains to show that $f_y^{'}(x,y,\theta) < 0$ for all x, y and θ. We have that

$$
(6.5)\quad f_y^{'}(x,y,\theta) = -1 + \frac{-b\cosh(y) + a\sinh(y)}{\sqrt{-1 + a\cosh(y) - b\sinh(y)}\sqrt{1 + a\cosh(y) - b\sinh(y)}}
$$

which is less than 0 if

$$
(6.6)\quad \sqrt{-1 + a\cosh(y) - b\sinh(y)}\sqrt{1 + a\cosh(y) - b\sinh(y)} > a\sinh(y) - b\cosh(y)
$$

If the right hand side in 6.6 is negative then we are done, otherwise, taking squares and simplifying gives that the inequality 6.6 is equivalent to the simpler inequality

$$
a^2 - b^2 > 1
$$

which holds since $a^2 - b^2 = \cosh^2(x) - \sinh^2(x)\cos^2(\theta) > \cosh^2(x) - \sinh^2(x) = 1$, completing the proof of the lemma. □

Second part of proof of Lemma 6.1. Letting $Y_i := g(d_i, \theta_i)$ we have (since $Y_0 > 0$),

$$
(6.7)\quad \mathbf{P}[d(y_0, y_n) < 2R] \le \mathbf{P}\left[\sum_{i=0}^{n-1} Y_i < 2R\right] \le \mathbf{P}\left[\sum_{i=1}^{n-1} Y_i < 2R\right]
$$

where g is as in Lemma 6.2, which concludes the proof. □

We now want to bound the probability in Lemma 6.1, and for this we have the following technical lemma, which in a slightly different form than below is due to Patrik Albin.

Lemma 6.3. *Let Y_i be defined as above. There is a function $h(R,\epsilon)$ such that for any $\epsilon \in (0,1)$ we have $h(R,\epsilon) \sim Ae^{-R(1-\epsilon)}$ as $R \to \infty$ for some constant $A = A(\epsilon) \in (0,\infty)$ independent of R and such that for any $R > 0$,*

$$\mathbf{P}\left[\sum_{i=1}^{n} Y_i < 2R\right] \le h(R,\epsilon)^n e^R. \tag{6.8}$$

Proof. Let K be the complete elliptic integral of the first kind (see [11], pp. 313-314). Then we have

$$\begin{aligned}\mathbf{E}[e^{-Y_1/2}|d_1] &= \mathbf{E}\left[\frac{1}{\sqrt{\cosh(d_1)-\sinh(d_1)\cos(\theta_1)}}\bigg|d_1\right]\\ &= \mathbf{E}\left[\frac{e^{-d_1/2}}{\sqrt{1-\cos(\theta_1/2)^2(1-e^{-2d_1})}}\bigg|d_1\right]\\ &= \frac{2e^{-d_1/2}K(\sqrt{1-e^{-2d_1}})}{\pi}.\end{aligned}$$

Using the relation $K(x) = \pi\ {}_2F_1(1/2,1/2,1,x)/2$ where ${}_2F_1$ is the hypergeometric function (see [11], Equation 13.8.5), we have

$$\mathbf{E}[e^{-Y_1/2}|d_1] = e^{-d_1/2}\,{}_2F_1(1/2,1/2,1,1-e^{-2d_1}).$$

Since ${}_2F_1(1/2,1/2,1,\cdot)$ is continuous on $\{z \in \mathbb{C} : |z| \le \rho\}$ for any $\rho \in (0,1)$, this gives

$$\mathbf{E}[e^{-Y_1/2}|d_1] \le A_1 e^{-d_1/2} \text{ for } d_1 \le x_0, \tag{6.9}$$

for some constant $A_1(x_0) > 0$, for any $x_0 > 0$. Large values of d_1 makes the argument of ${}_2F_1(1/2,1/2,1,1-e^{-2d_1})$ approach the radius of convergence 1 of ${}_2F_1(1/2,1/2,1,\cdot)$ so we perform the quadratic transformation

$${}_2F_1(a,b,2b,x) = (1-x)^{-a/2}{}_2F_1\left(a, 2b-a, b+1/2, -\frac{(1-\sqrt{1-x})^2}{4\sqrt{1-x}}\right),$$

(see [10], Equation 2.11.30), giving

$$\mathbf{E}[e^{-Y_1/2}|d_1] = {}_2F_1\left(1/2,1/2,1,-e^{d_1}(1-e^{-d_1})^2/4\right).$$

By the asymptotic behaviour of the hypergeometric function (here the analytic continuation of the hypergeometric function is used), we have

$$|{}_2F_1(1/2,1/2,1,x)| \sim A_2\frac{\log|x|}{\sqrt{|x|}}$$

as $|x| \to \infty$ (see [10], Equation 2.3.2.9), for some constant $A_2 > 0$. Combining this with 6.9 we get

$$\mathbf{E}[e^{-Y_1/2}|d_1] \leq A_3(1+d_1)e^{-d_1/2} \leq A_4 e^{-(1-\epsilon)d_1/2}$$

for $d_1 > 0$, for any $\epsilon \in (0,1)$, for some constants $A_3 > 0$ and $A_4(\epsilon) > 0$. Thus

$$\begin{aligned}\mathbf{E}[e^{-Y_1/2}] &\leq \mathbf{E}[A_4 e^{-d_1(1-\epsilon)/2}] \\ &= A_4 \frac{\int_0^{2R} \sinh(t)^{n-1} e^{-t(1-\epsilon)/2}\, dt}{\int_0^{2R} \sinh(t)^{n-1}\, dt}\end{aligned}$$

Clearly $h(R,\epsilon) := A_4 \int_0^{2R} \sinh(t)^{n-1} e^{-t(1-\epsilon)/2}\, dt / \int_0^{2R} \sinh(t)^{n-1}\, dt \sim Ae^{-R(1-\epsilon)}$ as $R \to \infty$ for some constant $A \in (0,\infty)$. Finally we get using Markov's inequality that

$$\begin{aligned}\mathbf{P}\left[\sum_{i=1}^{n} Y_i < 2R\right] &= \mathbf{P}\left[e^{-\frac{1}{2}\sum_{i=1}^{n} Y_i} > e^{-R}\right] \\ &\leq e^R \mathbf{E}\left[e^{-\frac{1}{2}\sum_{i=1}^{n} Y_i}\right] \\ &= e^R \mathbf{E}\left[e^{-Y_1/2}\right]^n \\ &\leq h(R,\epsilon)^n e^R\end{aligned}$$

completing the proof. □

Second part of proof of Theorem 4.3. By the estimates in Proposition 4.7 and Lemma 6.3 we get that

$$\begin{aligned}&\sum_{m=k-1}^{\infty} \sum_{n=k-1}^{\infty} (\lambda_c(R)\mu(S(0,2R)))^{m+n} P_R(m+n) \\ &\qquad\qquad \leq e^R \sum_{m=k-1}^{\infty} \sum_{n=k-1}^{\infty} (Kh(R,\epsilon))^{m+n-1}\end{aligned}$$

for any $\epsilon \in (0,1)$ and some constant $K \in (0,\infty)$. Thus if we take R big enough, the sum goes to 0 as $k \to \infty$. This is also the case if we replace λ_c with $t\lambda_c$ for some $t > 1$, proving that there are intensities above λ_c for which there are infinitely many unbounded connected components in the covered region of $\mathbb{H}^n$ for R big enough. □

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Paper II

Continuum percolation at and above the uniqueness threshold on homogeneous spaces

Johan Tykesson*

Abstract

We consider the Poisson Boolean model of continuum percolation on a homogeneous space M. Let λ be the intensity of the underlying Poisson process. Let λ_u be the infimum of the set of intensities that a.s. produce a unique unbounded component. First we show that if $\lambda > \lambda_u$ then there is a.s. a unique unbounded component at λ. Then we let $M = \mathbb{H}^2 \times \mathbb{R}$ and show that at λ_u there is a.s. not a unique unbounded component. These results are continuum analogs of theorems by Häggström, Peres and Schonmann.

Keywords and phrases: continuum percolation, phase transitions
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1 Introduction and results

In this paper we show continuum analogs to some theorems concerning the uniqueness phase in the theory of independent bond and site percolation on graphs. Before turning to our results, we review these theorems.

Let $G = (V, E)$ be an infinite, connected, transitive graph of bounded degree with vertex set V and edge set E. Keep each edge with probability p and delete it otherwise, independently for all edges. We call this independent bond percolation on G at level p, and let $\mathbf{P}_p$ be the corresponding probability measure on the subgraphs of G. A connected component in the random subgraph obtained in percolation is called a cluster. It is well known that the number of infinite clusters is an a.s. constant which is 0, 1 or ∞. Let

$$p_c(G) := \inf\{p : \mathbf{P}_p - \text{a.s. there is an infinite cluster}\}$$

be the critical probability for percolation.

In what follows we will discuss percolation at different levels, and when we do this, we always use the following coupling. To each $e \in E$ we associate an independent random variable U_e which is uniformly distributed on $[0, 1]$.

*Department of Mathematical Sciences, Division of Mathematical Statistics, Chalmers University of Technology, S-41296 Göteborg, Sweden. E-mail: johant@math.chalmers.se.
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Then say that e is kept at level p if $U_e < p$ and deleted otherwise. Using this construction, we have that if $p_1 < p_2$ then any edge kept at level p_1 is also kept at level p_2. We call this coupling the standard monotone coupling.

Now suppose that $p_c < p_1 < p_2$ and use the monotone coupling. We say that an infinite cluster at level p_2 is p_1-stable if it contains an infinite cluster at level p_1. Schonmann proved the following theorem:

Theorem 1.1. *Suppose G is a transitive graph and that $p_c(G) < p_1 < p_2 \leq 1$. Then any infinite cluster at level p_2 is a.s. p_1-stable.*

Theorem 1.1 was first shown by Häggström and Peres [7], under the technical assumption of so-called unimodularity (for definition see [6]), and shortly thereafter by Schonmann [12] in its full generality.

Theorem 1.1 has the following immediate consequence, where

$$p_u(G) := \inf\{p : \mathbf{P}_p - \text{a.s. there is a unique infinite cluster}\}$$

is the uniqueness threshold for percolation.

Corollary 1.2. *Suppose G is a transitive graph and that $p > p_u(G)$. Then $\mathbf{P}_p[\text{there is a unique infinite cluster}] = 1$.*

So Corollary 1.2 settles what happens above p_u. But there is also the question of what happens at p_u. It turns out that the answer depends on the graph. The following theorem of Peres [11] is of special interest to us:

Theorem 1.3. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two infinite transitive graphs and suppose G is nonamenable and unimodular. Consider Bernoulli percolation on $G \times H$ and let $p_u = p_u(G \times H)$. Then*

$$\mathbf{P}_{p_u}[\exists \text{ a unique infinite cluster}] = 0.$$

In contrast to this result, Benjamini and Schramm [1] showed that on any planar, transitive unimodular graph with one end, there is a.s. a unique infinite cluster at p_u.

In the proof of Theorem 1.7 below we shall need a result concerning *invariant site percolation.* First some definitions are needed. Denote the group of graph automorphisms on G by $\text{Aut}(G)$. A random subgraph of G in which all edges are kept, is called a site percolation on G. We call ω an invariant site percolation on G if the law of ω is invariant under the action of $\text{Aut}(G)$. The graph G is said to be amenable if for every $\epsilon > 0$ there is a finite set of vertices V_0 such that $|\partial V_0| < \epsilon |V_0|$ and nonamenable otherwise. In [2] the following theorem is shown.

Theorem 1.4. *Suppose G is nonamenable and ω is an invariant site percolation on G. There is a constant $c = c(G) < \infty$ such that for $u \in V$,*

$$\mathbf{P}[u \text{ belongs to an infinite cluster of } \omega] \geq \mathbf{P}[u \in \omega](1 + c) - c.$$

Recall that a graph G is said to have one end if for every finite set of vertices $V_0 \subset V$ there is precile one infinite connected component of $G \subset V_0$. A nonamenable, infinite transitive graph with one end is quasi-isometric to $\mathbb{H}^2$ (see for example [1]). We will later embed such a graph $\mathbb{H}^2$.

We will now discuss analogs of Theorems 1.1 and 1.3 in a continuum percolation setting. A Riemannian manifold M is said to be a (Riemannian) homogeneous space if for each $x, y \in M$ there is an isometry that takes x to y. Throughout this paper we assume that M is a simply connected, complete and noncompact homogeneous space, with metric d_M and volume measure μ_M. When it is clear which space we are working with we will write $d = d_M$ and $\mu = \mu_M$. We let 0 denote the origin of the space.

For one of the main results below it is possible to give a somewhat shorter proof under the additional assumption that M is a symmetric space. A connected Riemannian manifold M is said to be a (Riemannian) symmetric space if for each point $p \in M$ there is an isometry I_p such that $I_p(p) = p$ and I_p reverses geodesics through p. While the condition that a space is homogeneous is analog to the condition that a graph is transitive, the condition that a space is symmetric is in some sense the analog to the condition that a graph is unimodular. The most important symmetric spaces to study continuum percolation on are arguably n-dimensional Euclidean space $\mathbb{R}^n$ and n-dimensional hyperbolic space $\mathbb{H}^n$. Also products of symmetric spaces are symmetric spaces, for example $\mathbb{H}^2 \times \mathbb{R}$. Any symmetric space is homogeneous. For examples of a noncompact spaces which are homogeneous but not symmetric, one may consider certain Damek-Ricci spaces, see [3].

We now introduce the Poisson Boolean model of continuum percolation. Let $S(x, r) := \{y \in M : d_M(x, y) \leq r\}$ be the closed ball with radius r centered at x. Let X^λ be a Poisson point process on M with intensity λ relative to volume measure μ_M. Around every point of X^λ we place a ball of unit radius, and denote by C^λ the region of the space that is covered by some ball, that is $C^\lambda := \cup_{x \in X^\lambda} S(x, 1)$. We remark that all proofs below work if we instead consider the model with some arbitrary fixed radius R. Write $\mathbf{P}_\lambda$ for the probability measure corresponding to this model, which is called the Poisson Boolean model with intensity λ.

Next we introduce some additional notation. Let $V^\lambda := (C^\lambda)^c$ be the vacant region. Let $C^\lambda(x)$ be the component of C^λ containing x. $C^\lambda(x)$ is defined to be the empty set if x is not covered. Let $X^\lambda(A)$ be the Poisson points in the set A. Furthermore denote by $C^\lambda[A]$ the union of all balls centered within the set A. Let N_C and N_V denote the number of unbounded connected components of C^λ and V^λ respectively. The number of unbounded components for the Poisson Boolean model on a homogeneous space is an a.s. constant which equals 0, 1 or ∞. The proof of this is very similar to the discrete case, see for example Lemma 2.6 in [6], so we omit it. As in the discrete case, we introduce two critical intensities. Let

$$\lambda_c(M) := \inf\{\lambda : N_C > 0 \text{ a.s.}\}$$

and

$$\lambda_u(M) := \inf\{\lambda : N_C = 1 \text{ a.s.}\}$$

be the critical intensity for percolation and the uniqueness threshold for the Poisson Boolean model.

Remark. Obviously it is only interesting to study what happens at and above λ_u when $\lambda_u < \infty$. For example this is case for $\mathbb{H}^2 \times \mathbb{R}$ and may be proved by adjusting the arguments for the $\mathbb{H}^2$ case, see [13]. Simple modifications (just embed a different graph in the space) of the arguments in Lemma 4.8 in [13] show that for λ large enough there are a.s. unbounded components in C^λ but a.s. no unbounded components in V^λ. Since any two unbounded components in C^λ must be separated by some unbounded component in V^λ it follows that for λ large enough there is a.s. a unique unbounded component in C^λ.

We will often work with the model at several different intensities at the same time. Suppose we do this at the intensities $\lambda_1 < \lambda_2 < ... < \lambda_n$. Then we will always assume that $C^{\lambda_{i+1}}$ is the union of C^{λ_i} and balls centered at the points of a Poisson process, independent of C^{λ_i}, with intensity $\lambda_{i+1} - \lambda_i$. We call this the standard monotone coupling and is obviously the analog of the discrete coupling described earlier.

Now suppose $\lambda_1 < \lambda_2$ and use the monotone coupling. We say that an unbounded component in C^{λ_2} is λ_1-stable if it contains some unbounded component in C^{λ_1}. We now state a continuum version of Theorem 1.1.

Theorem 1.5. *Consider the Poisson Boolean model on a homogeneous space M. Suppose $\lambda_c(M) < \lambda_1 < \lambda_2 < \infty$. Then a.s. any unbounded λ_2-component is λ_1-stable.*

From Theorem 1.5, the following corollary is immediate.

Corollary 1.6. *Consider the Poisson Boolean model on the homogeneous space M. Suppose $\lambda_u(M) < \lambda$. Then $\mathbf{P}_\lambda[N_C = 1] = 1$.*

Remark. Corollary 1.6 is known in the cases $M = \mathbb{R}^n$ for any $n \geq 2$ (see [10]) and $M = \mathbb{H}^2$ (see [13]).

We will present two proofs of Theorem 1.5: one relatively short proof requiring the additional property that M is symmetric, and then a more involved proof for the more general case. The first is inspired by the proof of Theorem 1.1 in [7] and the second is inspired by the proof of Theorem 1.1 in [12].

To get a continuum analog to Theorem 1.3 we consider the Poisson Boolean model on a product space.

Theorem 1.7. *Consider the Poisson-Boolean model on $\mathbb{H}^2 \times \mathbb{R}$. Then*

$$\mathbf{P}_{\lambda_u}[N_C = 1] = 0.$$

Note that on $\mathbb{H}^2$, Corollary 5.10 in [13] says that at λ_u there is a.s. a unique unbounded component. We now move on to the proofs.

2 Uniqueness monotonicity

In this section we first present a relatively short proof of Theorem 1.5 in the symmetric case, and then a proof which only needs the assumption that the space is homogeneous.

First we present an essential ingredient to the first proof, the mass transport principle which is due to Benjamini and Schramm [1]. We denote the group of isometries on the symmetric space M by $\mathrm{Isom}(M)$.

Definition 2.1. *A measure ν on $M \times M$ is said to be* diagonally invariant *if for all measurable A, $B \subset M$ and $g \in$Isom(M)*

$$\nu(gA \times gB) = \nu(A \times B).$$

Theorem 2.2. *(*MASS TRANSPORT PRINCIPLE ON M*) If ν is a positive diagonally invariant measure on $M \times M$ such that $\nu(A \times M) < \infty$ for some open $A \subset M$, then*

$$\nu(B \times M) = \nu(M \times B)$$

for all measurable $B \subset M$.

The intuition behind Theorem (2.2) can be described as follows. In applications for the Poisson Boolean model, it is often the case that $\nu(A \times B) = \mathbf{E}[\eta(A, B, C)]$ where η is a function such that $\eta(A, B, C) = \eta(gA, gB, gC)$ for any isometry g (here C is again the covered region in the model). We think of $\eta(A, B, C)$ as the amount of *mass* that is transported from A to B. Thus Theorem (2.2) says that the expected amount of mass transported into B is the same as the expected amount of mass transported out from B.

Actually the mass transport principle is proved in [1] for the case when $M = \mathbb{H}^2$, but as is remarked there, it holds for any symmetric space.

Proof of Theorem 1.5 in the symmetric case: Suppose $\lambda_c < \lambda_1 < \lambda_2$. We couple C^{λ_1} and C^{λ_2} using the monotone coupling. Since any ball in C^{λ_1} is also present in C^{λ_2}, it is enough to show that any unbounded component of C^{λ_2} intersects an unbounded component of C^{λ_1}. For any point $x \in M$ let

$$D(x) := \inf\{d(x, y) : y \text{ is in an unbounded component of } C^{\lambda_1}\}$$

and let

$$\tilde{D}(x) := \begin{cases} \inf_{y \in C^{\lambda_2}(x)} D(y), & \text{if } x \in C^{\lambda_2} \\ D(x), & \text{otherwise} \end{cases}$$

Define the random set H to be the set of all points x satisfying the conditions

- $C^{\lambda_2}(x)$ is a λ_1-unstable unbounded component
- $D(x) \leq \tilde{D}(x) + 1/2$

and write $B(x)$ for the event that $x \in H$. Suppose that C^{λ_2} contains an unbounded component which does not intersect an unbounded component of C^{λ_1}. Then this unbounded component contains regions of positive volume in H, so it suffices to show that $\mathbf{P}[B(x)] = 0$. Let $H(x)$ be the intersection of H and $C^{\lambda_2}(x)$. Let $B^{\infty}(x) := B(x) \cap \{\mu(H(x)) = \infty\}$ and $B^{f}(x) := B(x) \cap \{\mu(H(x)) < \infty\}$. The events $B^{f}(x)$ and $B^{\infty}(x)$ partition $B(x)$. First we show that $\mathbf{P}[B^{f}(x)] = 0$ using the mass transport principle.

In any unbounded component of C^{λ_2} such that the volume of the intersection between it and H is positive and finite, we put mass of unit density. Then, the mass in any such unbounded C^{λ_2}-component is distributed uniformly on the part of H that intersects it. Suppose A and B are bounded sets and let $\nu(A \times B)$ be the expected amount of mass sent from the set A to the set B. Then ν is easily seen to be a positive diagonally invariant measure on $M \times M$. The expected amount of mass going out from A, that is $\nu(A \times M)$, is at most $\mu(A) < \infty$. However, on $B^{f}(x)$ there are bounded sets of positive finite volume that receive an infinite amount of mass. Hence, by the mass transport principle we must have $\mathbf{P}[B^{f}(x)] = 0$.

Next we show $\mathbf{P}[B^{\infty}(x)] = 0$ by showing $\mathbf{P}[B^{\infty}(x) \cap \{r \leq \tilde{D}(x) \leq r+1/4\}] = 0$ for any $r \geq 0$. To do this we use the following method of building up the process at level λ_2 in three steps. Fix $r \geq 0$. In step one, we add all balls in C^{λ_1}. In the second step, we add all balls that appear in the coupling between level λ_1 and λ_2 and are centered at distance at least $r+1$ from unbounded components in C^{λ_1}. In the third and final step, we add all balls that appear in the coupling between level λ_1 and λ_2 that are centered at distance less than $r+1$ from unbounded components in C^{λ_1}.

For $B^{\infty}(x) \cap \{r \leq \tilde{D}(x) \leq r+1/4\}$ to occur, it is necessary that the following two things happen when building up the process as described above. First, x must belong to an unbounded connected component, using balls only from step one and two, such that it does not intersect any unbounded C^{λ_1}-component but contains infinitely many balls centered at distance between $r+1$ and $r+1+1/2+1/4 = r+7/4$ from unbounded C^{λ_1}-components. Then, this unbounded component must not be connected to any unbounded C^{λ_1}-component by the balls that appear in step three above. However, if the first of these two things happen, then the second will a.s. not happen as can be seen as follows.

Suppose x belongs to an unbounded component using only balls from step one and two, with the properties described above, and call it $C^{\lambda_2}(x)'$. The balls that appear in step three above, are centered at a Poisson process with intensity $\lambda_2 - \lambda_1 > 0$ in the region of the space that is distance less than $r+1$ from some unbounded C^{λ_1} component, and this Poisson process is independent of everything else. By the properties of $C^{\lambda_2}(x)'$, we can find an infinite sequence $\{x_j\}$ of Poisson points (in X^{λ_1}) in unbounded components of C^{λ_1} centered at a distance between $r+1$ and $r+7/4$ from $C^{\lambda_2}(x)'$. From this sequence, we extract an infinite sequence $\{x_j'\}$ such that $S(x_j', r+1)$ and $S(x_i', r+1)$ are disjoint for $i \neq j$. Let A_j be the event that $S(x_j', r+7/4)$ is covered by balls that appear in the coupling between λ_1 and λ_2 and that are centered in the interior of $S(x_j^{'}, r+1)$ (all these balls are added in step three). Obviously A_i and A_j are independent for $i \neq j$ and $\mathbf{P}[A_j]$ is positive and independent of j. Therefore, a.s. infinitely many A_j will happen. But if some A_j happens, then $C^{\lambda_2}(x)^{'}$ will be connected to some unbounded C^{λ_1}-component, which means $C^{\lambda_2}(x)$ is not λ_1-unstable. Thus $\mathbf{P}[B^{\infty}(x) \cap \{r \leq \tilde{D}(x) \leq r + 1/4\}] = 0$ for any r and consequently $\mathbf{P}[B^{\infty}(x)] = 0$. □

For the second proof of Theorem 1.5, we need some preliminary results. First we describe a method for exploring the component of C^{λ} containing a fixed point $x \in M$. This may be considered to be the continuum version of the algorithm described in for example [12] for finding the cluster of a given vertex in discrete percolation.

At a fixed point $x \in M$, we grow a ball with unit speed until it has radius 1, when the growth of the ball stops. Whenever the boundary of this ball hits a Poisson point, a new ball starts to grow with unit speed at this point until it has radius 2. In the same way, every time a new Poisson point (which has not already been found) is hit by the boundary of a growing ball, a ball starts to grow at this point until it has radius 2 (note that if two balls are connected, their corresponding centers are within distance 2 from each other) and so on. Several balls may grow at the same time. Let $L_t^{\lambda}(x)$ denote the set which has been passed by the boundary of some ball at time t. If $C^{\lambda}(x)$ is bounded, then $L_t^{\lambda}(x)$ stops growing at some random time T. In this case $C^{\lambda}[L_T^{\lambda}(x)] = C^{\lambda}(x)$ and $L_T^{\lambda}(x)$ is the 1-neighbourhood of $C^{\lambda}(x)$. (If the first ball does not hit any Poisson point, then $C^{\lambda}(x)$ is the empty set). If $C^{\lambda}(x)$ is unbounded, then $L_t^{\lambda}(x)$ never stops growing. We will refer to this procedure to as "growing the component containing x".

In what follows we will make use of the following lemma, which may be considered intuitively clear. Its proof is inspired by the proof of the corresponding lemma for the discrete situation which is Lemma 1.1 of [12].

Lemma 2.3. *Consider the Poisson Boolean model on a homogeneous space M. Let $R > 0$ and let $\lambda > \lambda_c$. Any unbounded component of C^{λ} a.s. contains balls of radius R.*

For the proof we need to introduce some further notation. For a connected set A containing x we let $C^{\lambda}(x, A)$ be all points in A which can be connected to x by some curve in $C^{\lambda} \cap A$. Let $E_r(x)$ be the union of all balls centered within $S(x, r+1)$ that are connected to x via a chain of balls centered within $S(x, r+1)$. Note that $C^{\lambda}(x, S(x,r)) \subset E_r(x)$.

Let $\delta_r(x) := \sup_{y \in E_r(x) \setminus S(x,r)} d(y, \partial S(x,r))$ where the supremum is defined to be 0 if $E_r(x) \setminus S(x,r)$ is empty. Let $\{A \leftrightarrow B\}$ be the event that there is some continuous curve in C^{λ} which intersects both the set A and the set B. Let A^o be the interior of the set A.

Proof. Since the case $R \leq 1$ is trivial we suppose $R > 1$. Fix a point $x \in M$. For any $r > 0$ let $F_r(x) := \{x \leftrightarrow \partial S(x,r)\}$ and let

$$G_r(x) := \{C^{\lambda}(x, S(x,r)) \text{ does not contain a ball of radius } R\}.$$

Let $D_r(x) := F_r(x) \cap G_r(x)$. Let $D(x)$ be the event that x is an unbounded component that does not contain a ball of radius R. Then $D_r(x) \downarrow D(x)$ so it is enough to show that $\mathbf{P}[D_r(x)] \to 0$ as $r \to \infty$. Note that $D_r(x)$ is independent of the Poisson process outside $S(x, r+1)$. Also note that $\delta_r(x) \in [0, 2]$.

If $D_r(x) \cap \{\delta_r(x) < 1/2\}$ occurs, then there is a ball centered in $S(x, r - 1/2)^o \setminus S(x, r-1)^o$ which is connected to x by a chain of balls centered in $S(x, r-1/2)^o$. All these balls are also included in the set $E_{r-1/2}(x)$, and one of these balls is centered at a distance at most $1/2$ from $\partial S(x, r-1/2)$. This gives

$$D_r(x) \cap \{\delta_r(x) < 1/2\} \subset D_{r-1/2}(x) \cap \{\delta_{r-1/2}(x) \geq 1/2\}. \tag{2.1}$$

We will now proceed by contradiction. Suppose that $\mathbf{P}[D(x)] > 0$ and that $\lim_{r\to\infty} \mathbf{P}[\delta_r(x) < 1/2 | D_r(x)] = 1$. These assumptions imply that

$$\begin{aligned}&\lim_{r\to\infty} \mathbf{P}[D_r(x) \cap \{\delta_r(x) < 1/2\}]\\ &\quad = \lim_{r\to\infty} \mathbf{P}[\delta_r(x) < 1/2 | D_r(x)]\mathbf{P}[D_r(x)] = \lim_{r\to\infty} \mathbf{P}[D_r(x)] = \mathbf{P}[D(x)] > 0.\end{aligned}$$

However, by (2.1) we get that

$$\begin{aligned}\limsup_{r\to\infty} \mathbf{P}[\delta_{r-1/2}(x) \geq 1/2 | D_{r-1/2}(x)] &\geq \limsup_{r\to\infty} \mathbf{P}[D_{r-1/2}(x) \cap \{\delta_{r-1/2}(x) \geq 1/2\}]\\ &\geq \lim_{r\to\infty} \mathbf{P}[D_r(x) \cap \{\delta_r(x) < 1/2\}] > 0,\end{aligned}$$

so that in particular $\mathbf{P}[\delta_r(x) \geq 1/2 | D_r(x)]$ does not go to 0 as $r \to \infty$ which contradicts the assumption $\lim_{r\to\infty} \mathbf{P}[\delta_r(x) < 1/2 | D_r(x)] = 1$. Thus we conclude that $\mathbf{P}[D(x)] = 0$ and/or $\liminf_{r\to\infty} \mathbf{P}[\delta_r(x) < 1/2 | D_r(x)] < 1$. We now assume $\liminf_{r\to\infty} \mathbf{P}[\delta_r(x) < 1/2 | D_r(x)] < 1$ and show that this implies $\mathbf{P}[D(x)] = 0$. By the assumption, we may pick a constant $c_1 > 0$ and a sequence of positive numbers $\{a_k\}_{k=1}^{\infty}$ such that $a_{k+1} - a_k \geq 2R + 1$ and $\mathbf{P}[\delta_{a_k}(x) \geq 1/2 | D_{a_k}(x)] \geq c_1$ for all k. On the event $D_{a_k}(x)$ we may pick a point Y on $\partial S(x, a_k + R + 1)$ such that if $S(Y, R + \max(0, 1 - \delta_{a_k}(x)))$ is completely covered by balls centered within $S(Y, R)$, then $D_{a_{k+1}}(x)^c$ occurs since a ball of radius R has been found in $C(x, S(x, a_{k+1}))$ (this ball is contained in $C(x, S(x, a_{k+1}))$ since $a_{k+1} - a_k \geq 2R + 1$ and $R > 1$). The configuration of balls within $S(Y, R)$ is independent of the Poisson process within $S(x, a_k + 1)$. Now let Δ_k be a random variable with the same distribution as the conditional distribution of $\delta_k(x)$ given the event $D_k(x)$. By the above observations we get that

$$\mathbf{P}[D_{a_{k+1}}(x)^c | D_{a_k}(x)] \geq \mathbf{P}[S(0, R + \max(0, 1 - \Delta_k)) \subset C^{\lambda}[S(0, R)]] \geq c_2$$

for some constant $c_2 > 0$ for all k. Since $\{D_{a_k}(x)\}$ is decreasing, this implies $\lim_{k\to\infty} \mathbf{P}[D_{a_k}(x)] = 0$ and consequently $\mathbf{P}[D(x)] = 0$. $\square$

Proof of Theorem 1.5:

We consider the monotone coupling of the model at intensities $\lambda_1 < \lambda_2$, and we write $C = (C^{\lambda_1}, C^{\lambda_2})$. Let

$$E(x) := \{x \text{ is in an unbounded } C^{\lambda_2}\text{-component which is } \lambda_1\text{-unstable.}\}$$

Let

$$E_1(x) := E(x) \cap \{\tilde{D}(x) \leq 3\} \text{ and } E_2(x) := E(x) \cap \{\tilde{D}(x) > 2\},$$

where $\tilde{D}$ is defined as in the proof of Theorem 1.5.

Finally let E, E_1 and E_2 be the events that $E(x)$, $E_1(x)$ and $E_2(x)$ respectively happen for some x.

We will first show that $\mathbf{P}[E_2(x)] = 0$. Pick a and $R = R(a)$ so that

$$\mathbf{P}[S(x,R) \text{ intersects an unbounded component of } C^{\lambda_1}] \geq 1 - a$$

Let $Z^{'} = (Z^{'\lambda_1}, Z^{'\lambda_2})$ and $Z^{''} = (Z^{''\lambda_1}, Z^{''\lambda_2})$ be two independent copies of C, and let $X^{'} = (X^{'\lambda_1}, X^{'\lambda_2})$ and $X^{''} = (X^{''\lambda_1}, X^{''\lambda_2})$ be their underlying Poisson processes. A prime will be used to denote objects relating to $Z^{'}$ and a double prime will be used to denote objects relating to $Z^{''}$.

Grow the component of $Z^{'\lambda_2}$ containing x as described above, with the exception that if at time t we find that a ball of radius R is contained in $Z^{'\lambda_2}[L_t^{'\lambda_2}(x)]$ we stop the process. Let T denote the random time at which the process stops. Note that $T < \infty$ a.s., since if $Z^{'\lambda_2}(x)$ is unbounded, then $Z^{'\lambda_2}(x)$ contains balls of radius R a.s. by Lemma 2.3. Let F_1 be the event that the process stops when a ball of radius R is found, and note that $Z^{'\lambda_2}(x)$ is a.s. bounded on F_1^c. On F_1, we may (in some way independent of $Z^{''}$) pick a point Y such that $S(Y,R)$ is covered by $Z^{'\lambda_2}[L_T^{'\lambda_2}(x)]$.

For $i = 1, 2$ let

$$X^{\lambda_i} := (X^{'\lambda_i} \cap L_T^{'\lambda_2}(x)) \cup (X^{''\lambda_i} \cap L_T^{'\lambda_2}(x)^c)$$

and $Z^{\lambda_i} := \cup_{x \in X^{\lambda_i}} S(x, 1)$. In this way, Z^{λ_i} is a Poisson Boolean model with intensity λ_i for $i = 1, 2$, and any ball present in Z^{λ_1} is also present in Z^{λ_2}.

Now let

$$F_2 := F_1 \cap \{S(Y,R) \text{ intersects an unbounded component of } Z^{''\lambda_1}\}.$$

On F_2 there is some point in $Z^{\lambda_2}(x)$ which is at distance less than or equal to two from some unbounded Z^{λ_1}-component, that is $\{\tilde{D}(x) \leq 2\}$ occurs for Z so that $E_2(x)$ does not occur for Z. Since $E_2(x)$ is up to a set of measure 0 contained in F_1 we have that

$$\mathbf{P}[E_2(x)] \leq \mathbf{P}[F_1 \cap F_2^c].$$

Since $Z^{'}$ and $Z^{''}$ are independent it follows that

$$\mathbf{P}[F_2 | F_1] = \mathbf{P}[S(Y,R) \text{ intersects an unbounded component of } Z^{''\lambda_1}] \geq 1 - a$$

and consequently

$$\mathbf{P}[F_1 \cap F_2^c] \leq \mathbf{P}[F_2^c | F_1] < a.$$

Since we may choose a arbitrary small it follows that $\mathbf{P}[E_2(x)] = 0$ as desired.

Next we argue that $\mathbf{P}[E_2(x)] = 0$ for all x implies $\mathbf{P}[E_2] = 0$. Let D be a countable dense subset of M. Then $\mathbf{P}\left[\cup_{x \in D} E_2(x)\right] = 0$. But if E_2 occurs then $E_2(x)$ occurs for all x in some unbounded component of C^{λ_2}, in particular for some x in D, so it follows that $\mathbf{P}[E_2(x)] = 0$ implies $\mathbf{P}[E_2] = 0$.

Next we show that $\mathbf{P}[E_1(x)] = 0$. Let $E_1^f(x)$ be the event that $E_1(x)$ occurs and all points in the λ_1-unstable unbounded C^{λ_2}-component of x which are at distance less than or equal to three from some unbounded C^{λ_1}-component are contained in the ball $S(0, N)$ for some random finite N. Let $E_1^\infty(x)$ be the event that $E_1(x)$ occurs but that there is no such finite N. Let E_1^f and E_1^∞ be the events that $E_1^f(x)$ and $E_1^\infty(x)$ respectively happen for some x.

First we show that $\mathbf{P}[E_1^f] = 0$. Let $E_1^{f,M} := E_1^f \cap \{N \le M\}$. We will show that $\mathbf{P}[E_1^f] > 0$ implies that $\mathbf{P}[E_2] > 0$. So suppose $\mathbf{P}[E_1^f] > 0$. Then we may pick M so large that $\mathbf{P}[E_1^{f,M}] > 0$. Again let Z' and Z'' be independent with the same distribution as C. Then for $i = 1, 2$ let Z^{λ_i} be the union of all balls from Z'^{λ_i} centered within $S(0, M+1)$ together with the union of all balls from Z''^{λ_i} centered within $S(0, M+1)^c$. Then if $\{Z'^{\lambda_2}[S(0, M+1)] = \emptyset\}$ occurs and $E_1^{f,M}$ occurs for Z'', then E_2 occurs for Z. So since Z' and Z'' are independent we get

$$\mathbf{P}[E_2] \ge \mathbf{P}\left[Z'^{\lambda_2}[S(0, M+1)] = \emptyset\right] \mathbf{P}\left[E_1^{f,M}\right] > 0$$

which is a contradiction, so $\mathbf{P}[E_1^f] = 0$.

Finally it remains to show that $\mathbf{P}[E_1^\infty] = 0$. However the event $E_1^\infty(x)$ is very similar to the event $B^\infty(x)$ in the first proof of Theorem 1.5, and is shown to have probability 0 in the same way. In the same way it then follows that $\mathbf{P}[E_1^\infty] = 0$. □

3 Connectivity

In this section we show how λ_u can be characterized by the connectivity between large balls. This result will be used when we study the model at λ_u on a product space in the next section. Let

$$\lambda_{BB} := \inf\{\lambda : \lim_{R \to \infty} \inf_{x,y} \mathbf{P}[S(x, R) \leftrightarrow S(y, R) \text{ in } C^\lambda] = 1\}.$$

Note that obviously $\lambda_{BB} \ge \lambda_c$. We will show the following:

Theorem 3.1. *For the Poisson Boolean model on a homogeneous space with $\lambda_u < \infty$ we have $\lambda_u = \lambda_{BB}$.*

The discrete counterpart of this result is Theorem 3.2 of [12], and the proof is similar. The proof is also similar to the second proof of Theorem 1.5 above.

Proof. First we show that $\lambda_u \le \lambda_{BB}$. Suppose that $\lambda_{BB} < \lambda_1 < \lambda_2$ and again use the monotone coupling of the model. We will show that at λ_2 there is a.s.

a unique unbounded component. For $i = 1, 2$ let

$$A_i(x, y) := \{\mu(C^{\lambda_1}(x)) = \infty,\, \mu(C^{\lambda_1}(y)) = \infty,\, C^{\lambda_i}(x) \neq C^{\lambda_i}(y)\},$$

and let

$$A_i := \bigcup_{x,y} A_i(x, y).$$

Since $\lambda_{BB} \geq \lambda_c$ we have by Theorem 1.5 that any unbounded λ_2 component a.s. intersects some unbounded λ_1 component. Therefore

$$\bigcup_{x,y} \{\mu(C^{\lambda_2}(x)) = \infty,\, \mu(C^{\lambda_2}(y)) = \infty,\, C^{\lambda_2}(x) \neq C^{\lambda_2}(y)\} \subset A_2 \cup N \tag{3.1}$$

where N is a set of measure 0. In the same way as in the second proof of Theorem 1.5 we have that $\mathbf{P}[A_i(x, y)] = 0$ for all x and y implies $\mathbf{P}[A_i] = 0$. By (3.1), $\mathbf{P}[A_2] = 0$ implies $\mathbf{P}$[there is a unique unbounded component at level λ_2] $= 1$. Hence it suffices to show that $\mathbf{P}[A_2(x, y)] = 0$ for all x and y. To show this, we need the following definition.

Definition 3.2. *Suppose C_1 and C_2 are two distinct components in the Poisson Boolean model. A pair of Poisson points $x_1 \in C_1$ and $x_2 \in C_2$ is called a* boundary-connection *between C_1 and C_2 if $d(x_1, x_2) < 6$ (so that the distance between their corresponding balls is less than 4) or there is a sequence of Poisson-points $y_1, \ldots, y_n$ such that*

- *the unit ball centered around y_i intersects the unit ball centered around y_{i+1} for all i.*
- *y_i is outside C_1 and C_2 for all i.*
- *$d(x_1, y_1) < 4$ and $d(x_2, y_n) < 4$.*

Note that if there is a boundary connection between two components, then at most two more balls are needed to merge them into one component.

If $x, y \in C^{\lambda_1}$ and $C^{\lambda_1}(x) \neq C^{\lambda_1}(y)$, let $B(x, y)$ be the number of boundary connections between $C^{\lambda_1}(x)$ and $C^{\lambda_1}(y)$. Let

$$A_1^0(x, y) := A_1(x, y) \cap \{B(x, y) = 0\},$$

$$A_1^f(x, y) := A_1(x, y) \cap \{B(x, y) < \infty\},$$

$$A_1^\infty(x, y) := A_1(x, y) \cap \{B(x, y) = \infty\},$$

and for $t \in \{0, f, \infty\}$ let A_1^t be the event that $A_1^t(x, y)$ happens for some x and y. In the same way as before it is seen that $\mathbf{P}[A_1^t(x, y)] = 0$ for all x and y implies $\mathbf{P}[A_1^t] = 0$.

Next we will argue that

$$\mathbf{P}[A_1^0(x, y)] = 0 \text{ for all } x \text{ and } y. \tag{3.2}$$

Let Z'^{λ_1} and Z''^{λ_1} be two independent copies of the Poisson Boolean model with intensity λ_1 and let X'^{λ_1} and X''^{λ_1} be their underlying Poisson processes. Since $\lambda_1 > \lambda_{BB}$ we may for any $a > 0$ pick $R = R(a)$ such that

$$\inf_{z_1, z_2} \mathbf{P}_{\lambda_1}[S(z_1, R) \leftrightarrow S(z_2, R)] > 1 - a.$$

Fix x and y and grow the component of x in Z'^{λ_1} (as described earlier) but stop if a ball of radius R is found. Do the same for y. Let F_1 be the event that the processes are stopped when balls of radius R are found, and note that $A_1^0(x, y)$ is up to a set of measure 0 included in F_1. Let T_x and T_y denote the random times at which the processes are stopped. On F_1 we pick X and Y in some way independent of Z''^{λ_1} such that $S(X, R) \subset Z'^{\lambda_1}[L_{T_x}'^{\lambda_1}(x)]$ and $S(Y, R) \subset Z'^{\lambda_1}[L_{T_y}'^{\lambda_1}(y)]$. Let

$$X^{\lambda_1} := (X'^{\lambda_1} \cap (L_{T_x}'^{\lambda_1}(x) \cup L_{T_y}'^{\lambda_1}(y))) \cup (X''^{\lambda_1} \cap (L_{T_x}'^{\lambda_1}(x) \cup L_{T_y}'^{\lambda_1}(y))^c)$$

and $Z^{\lambda_1} := \cup_{x \in X^{\lambda_1}} S(x, 1)$. The distribution of Z^{λ_1} is by construction the distribution of the Poisson Boolean model with intensity λ_1. Let

$$F_2 := F_1 \cap \{S(X, R) \leftrightarrow S(Y, R) \text{ in } Z''^{\lambda_1}\}.$$

If we are on F_2 then either $\{Z^{\lambda_1}(x) = Z^{\lambda_1}(y)\}$ occurs or $\{B(x, y) \geq 1\}$ occurs for Z^{λ_1} and in neither case we are on $A_1^0(x, y)$. Since

$$\mathbf{P}[F_2 | F_1] = \mathbf{P}[S(X, R) \leftrightarrow S(Y, R) \text{ in } Z''^{\lambda_1}] > 1 - a$$

it therefore follows that

$$\mathbf{P}[A_1^0(x, y)] \leq \mathbf{P}[F_1 \cap F_2^c] \leq \mathbf{P}[F_2^c | F_1] < a$$

proving (3.2).

Next we show that

$$\mathbf{P}[A_1^f] = 0. \tag{3.3}$$

Let $A_1^{f,N}$ be the event there are two distinct unbounded components in C^{λ_1} such there is a finite number of boundary connections between them and they are all contained in the ball $S(0, N)$. Suppose $\mathbf{P}[A_1^f] > 0$ and pick N so large that $\mathbf{P}[A_1^{f,N}] > 0$. Let Z^{λ_1} be the union of the balls from Z'^{λ_1} centered outside $S(0, N)$ and the balls from Z''^{λ_1} centered inside $S(0, N)$. Now suppose that $A_1^{f,N}$ happens for Z'^{λ_1} and that $\{Z''^{\lambda_1}[S(0, N)] = \emptyset\}$ happens. Then we can find two separate unbounded components of Z^{λ_1} such that there are no boundary connections between them. It follows by the independence of Z' and Z'' that

$$\mathbf{P}[A_1^0] \geq \mathbf{P}\left[A_1^{f,N}\right] \mathbf{P}\left[Z''^{\lambda_1}[S(0, N)] = \emptyset\right] > 0,$$

a contradiction which proves (3.3).

The event $A_1^\infty(x,y)$ may happen in three different ways. Either there are infinitely many Poisson points in $C^{\lambda_1}(x)$ that are part of boundary connections between $C^{\lambda_1}(x)$ and $C^{\lambda_1}(y)$ and only finitely many in $C^{\lambda_1}(y)$, or there are infinitely many in $C^{\lambda_1}(y)$ and only finitely many in $C^{\lambda_1}(x)$, or there are infinitely many in both of $C^{\lambda_1}(x)$ and $C^{\lambda_1}(y)$. If any of the first two cases happen, then we get a configuration in A_1^0 by deleting all Poisson points within some bounded region in the same way as it was proved that $\mathbf{P}[A_1^f] = 0$, which shows that these cases have probability 0. Now suppose the third case happens, that is, there are infinitely many Poisson points in both $C^{\lambda_1}(x)$ and $C^{\lambda_1}(y)$ that are part of boundary connections between $C^{\lambda_1}(x)$ and $C^{\lambda_1}(y)$. Then it is a.s. not possible to delete all boundary connections between $C^{\lambda_1}(x)$ and $C^{\lambda_1}(y)$ by deleting all Poisson points within any bounded region. Therefore $C^{\lambda_1}(x)$ and $C^{\lambda_1}(y)$ will almost surely be merged into one unbounded component at level λ_2 by balls that appear in the coupling between level λ_1 and λ_2. That is, $\mathbf{P}[A_2(x,y) \cap A_1^\infty(x,y)] = 0$. Thus, since $A_2(x,y) \subset A_1(x,y)$ and $A_1(x,y)$ is partitioned by $A_1^f(x,y)$ and $A_1^\infty(x,y)$ we conclude

$$\mathbf{P}[A_2(x,y)] = \mathbf{P}[A_2(x,y) \cap A_1^f(x,y)] + \mathbf{P}[A_2(x,y) \cap A_1^\infty(x,y)] = 0.$$

for all x and y and so $\lambda_u \leq \lambda_{BB}$.

Next we show the easier result that $\lambda_u \geq \lambda_{BB}$. Suppose $\lambda > \lambda_u$. By Theorem 1.5 there is a.s. a unique unbounded component in C^λ which we denote by C_∞^λ. By the continuum version of the FKG inequality (see [10]) and the fact that there is an isometry mapping x to y it follows that

$$\begin{aligned}\mathbf{P}_\lambda[S(x,R) \leftrightarrow S(y,R)] &\geq \mathbf{P}_\lambda[S(x,R) \text{ and } S(y,R) \text{ intersect } C_\infty^\lambda] \\ &\geq \mathbf{P}_\lambda[S(x,R) \text{ intersects } C_\infty^\lambda]^2.\end{aligned}$$

Since $\lim_{R\to\infty} \mathbf{P}_\lambda[S(x,R) \text{ intersects } C_\infty^\lambda] = 1$ it follows that $\lambda > \lambda_{BB}$ and thus $\lambda_u \geq \lambda_{BB}$. □

4 The situation at λ_u on $\mathbb{H}^2 \times \mathbb{R}$

This section is devoted to the proof of Theorem 1.7. We introduce some new notation: if the points $x, y \in \mathbb{H}^2 \times \mathbb{R}$ are in the same component of C^λ then $d_{X^\lambda}(x,y)$ is the smallest number of balls in that component forming a sequence that connects x to y. For a set A we let $C^\lambda(A)$ be the union of all components of C^λ that intersect A. In this proof $\mu = \mu_{\mathbb{H}^2}$ and $d = d_{\mathbb{H}^2\times\mathbb{R}}$.

Proof. As noted earlier, $\lambda_u(\mathbb{H}^2 \times \mathbb{R}) < \infty$. Suppose that λ_* is such that there is a.s. a unique unbounded component in the Poisson Boolean model with intensity λ_* on $\mathbb{H}^2 \times \mathbb{R}$. We consider the monotone coupling of the model for all intensities below λ_*. It suffices to show that there is some intensity below λ_* that also a.s. produces a unique unbounded component, so this is what we set out to do. Denote the unbounded component at λ_* with $C_\infty^{\lambda_*}$. For any $r > 0$, any positive integer n, and any $\lambda \in (0, \lambda_*)$ we define the following three random sets:

$$A_1(r) := \{z \in \mathbb{H}^2 \times \mathbb{R} \,:\, S(z,r) \cap C_\infty^{\lambda_*} \neq \emptyset\}$$

$$A_2(r,n) := \{z \in \mathbb{H}^2 \times \mathbb{R} \,:\, \sup\{d_{X^{\lambda_*}}(s,t) \,:\, s,t \in S(z, r+3/2) \cap C_\infty^{\lambda_*}\} < n\}$$

$$A_3(r,n,\lambda) := \{z \in \mathbb{H}^2 \times \mathbb{R} \,:\, S(z, r+2n) \cap (X^{\lambda_*} \setminus X^{\lambda}) = \emptyset\}.$$

Then let

$$A(r,n,\lambda) := A_1(r) \cap A_2(r,n) \cap A_3(r,n,\lambda).$$

Pick y_1, $y_2 \in \mathbb{R}$ and let for $i = 1, 2$

$$D_i = D_i(y_i, r, n, \lambda) := \{x \in \mathbb{H}^2 \,:\, (x, y_i) \in A(r,n,\lambda)\}.$$

Let $D = D(y_1, y_2, r, n, \lambda) := D_1 \cap D_2$. Then D, D_1 and D_2 are random sets in $\mathbb{H}^2$ such that their laws are $\mathrm{Isom}(\mathbb{H}^2)$-invariant. Note that the laws of D_1 and D_2 are the same and independent of the choices of y_1 and y_2. Next we will show that given a, we can choose the parameters r, n and λ in such a way that the probability a given point in $\mathbb{H}^2$ belongs to an unbounded component of D is larger than $1 - a/2$ for any choice of y_1 and y_2.

To do this, we embed an infinite nonamenable transitive graph G with one end in $\mathbb{H}^2$, in such a way that the faces are congruent. Identify each face of G with a vertex of the dual graph $G^\dagger$. For $i = 1, 2$ let $\Gamma_i = \Gamma_i(y_i, r, n, \lambda)$ be all vertices of $G^\dagger$ for which the corresponding face is completely covered by D_i, and let $\Gamma = \Gamma(y_1, y_2, r, n, \lambda) := \Gamma_1 \cap \Gamma_2$. Then Γ, Γ_1 and Γ_2 are site percolations on $G^\dagger$ such that their laws are invariant. Note that the laws of Γ_1 and Γ_2 are the same and independent of the choices of y_1 and y_2.

Suppose E is some bounded set in $\mathbb{H}^2 \times \mathbb{R}$. It is clear that

$$\lim_{r \to \infty} \mathbf{P}[E \subset A_1(r)] = 1, \tag{4.1}$$

and that for fixed r,

$$\lim_{n \to \infty} \mathbf{P}[E \subset A_2(r,n)] = 1, \tag{4.2}$$

and that for fixed r and n,

$$\lim_{\lambda \uparrow \lambda_0} \mathbf{P}[E \subset A_3(r,n,\lambda)] = 1. \tag{4.3}$$

Since the law of Γ is invariant and $G^\dagger$ is nonamenable, we get by Theorem 1.4 that there is a constant $c = c(G^\dagger) < \infty$ such that for any $u \in V(G^\dagger)$ we have $\mathbf{P}[u \text{ is in an infinite cluster of } \Gamma] \geq \mathbf{P}[u \in \Gamma](1+c) - c$. By (4.1), (4.2) and (4.3) we get that we can find first r_1 big enough, and then n_1 big enough, and finally λ_1 close enough to λ_* so that $\mathbf{P}[u \in \Gamma_i(y_i, r_1, n_1, \lambda_1)] > 1 - a/(4(1+c))$ for $i = 1, 2$ and any u. With these choices of parameters, we get $\mathbf{P}[u \in \Gamma(y_1, y_2, r_1, n_1, \lambda_1)] \geq 1 - a/(2(1+c))$ for all u and any choice of y_1 and y_2 and consequently $\mathbf{P}[u \text{ is in an infinite cluster of } \Gamma] \geq 1 - a/2$ for any u and any choice of y_1 and y_2. If the vertex u is in an unbounded component of

Γ, then all points in $\mathbb{H}^2$ in the corresponding face of $G^\dagger$ are in an unbounded component of D which implies

$$\mathbf{P}[x \text{ belongs to an unbounded component of } D(y_1, y_2, r_1, n_1, \lambda_1)] \geq 1 - a/2$$

for all $x \in \mathbb{H}^2$ and any choice of y_1 and y_2.

Since the event that D contains unbounded components is Isom($\mathbb{H}^2$)-invariant and determined by the underlying Poisson processes in the model, D contains unbounded components with probability 1.

Suppose that $u_1, u_2, \ldots \in \mathbb{H}^2$ is an infinite sequence of points such that they are all in the same component of D, $d_{\mathbb{H}^2}(u_i, u_{i+1}) < 1/2$ for all i and $d_{\mathbb{H}^2}(u_1, u_i) \to \infty$ as $i \to \infty$. Since $(u_i, y_1) \in A_1$ there is some ball s_i in $C_\infty^{\lambda_*}$ centered within distance $r_1 + 1$ from (u_i, y_1). Since $d((u_i, y_1), (u_{i+1}, y_1)) < 1/2$ and $(u_i, y_1) \in A_2$ for all i there is a sequence of at most n balls in $C_\infty^{\lambda_*}$ connecting the center of s_i to the center of s_{i+1} . Since the distance between the center of any ball in this sequence and (u_i, y_1) is at most $r_1 + 2n$ and $(u_i, y_1) \in A_3$, all balls in the sequence is present also at level λ_1. Thus there is an unbounded component in C^{λ_1} that comes within distance r_1 from (u_i, y_1) for all i. In the same way there is an unbounded component in C^{λ_1} that comes within distance r_1 from (u_i, y_2) for all i.

Now choose λ_2 and λ_3 so that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_*$. For $x \in \mathbb{H}^2$ let $D(x)$ be the component of D containing x. Then we have from the above that

$$\mathbf{P}[S((x, y_1), r_1) \leftrightarrow S((x, y_2), r_1) \text{ in } C^{\lambda_2} | \mu(D(x)) = \infty] = 1. \tag{4.4}$$

This follows from the fact that the two unbounded components at level λ_1 above will a.s. be connected by balls appearing in the coupling between level λ_1 and λ_2. Since the probability that x belongs to an unbounded component of D is at least $1 - a/2$ it follows by (4.4) that

$$\mathbf{P}[S((x, y_1), r_1) \leftrightarrow S((x, y_2), r_1) \text{ in } C^{\lambda_2}] \geq 1 - a/2 \text{ for all } y_1 \text{ and } y_2. \tag{4.5}$$

Fix two points $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ of $\mathbb{H}^2 \times \mathbb{R}$. For $y \in \mathbb{R}$ let

$$F_y := \{S(z_1, r_1) \leftrightarrow S((u_1, y), r_1) \text{ in } C^{\lambda_2}\} \cap \{S(z_2, r_1) \leftrightarrow S((u_2, y), r_1) \text{ in } C^{\lambda_2}\}$$

By (4.5) we get $\mathbf{P}[F_y] \geq 1 - a$ for all y. In particular it follows that with probability at least $1 - a$ the set $\{y \in \mathbb{R} : F_y \text{ occurs}\}$ is unbounded. But then the set of points in $C^{\lambda_2}(S(z_1, r_1))$ that come within distance $2r_1 + d_{\mathbb{H}^2}(u_1, u_2)$ from $C^{\lambda_2}(S(z_2, r_1))$ is unbounded. But if this occurs then some component in C^{λ_2} intersecting $S(z_1, r_1)$ will a.s. be connected to some component in C^{λ_2} intersecting $S(z_2, r_1)$ by balls occurring in the coupling between level λ_2 and λ_3. That is,

$$\mathbf{P}[S(z_1, r_1) \leftrightarrow S(z_2, r_1) \text{ in } C^{\lambda_3}] \geq 1 - a.$$

Since a is arbitrary small it follows by Theorem 3.1 there is a.s. a unique unbounded component in C^{λ_3}. □

Remark. The proof works without substantial modifications if $\mathbb{H}^2 \times \mathbb{R}$ is replaced by $\mathbb{H}^n \times M$ for any $n \geq 2$ and any noncompact homogeneous space M such that $\lambda_u(\mathbb{H}^n \times M) < \infty$.

5 Further problems

In this section we list some open problems.

1. For which manifolds is $\lambda_u < \infty$?
2. In [13] it is shown that $\lambda_c(\mathbb{H}^n) < \lambda_u(\mathbb{H}^n)$ for any $n \geq 2$ if the radius of the percolating balls is big enough (for $n = 2$ this is shown for any radius). For which manifolds is $\lambda_c < \lambda_u$?
3. For which manifolds with $\lambda_u < \infty$ is there a.s. a unique unbounded component at λ_u? For which manifolds is there a.s. not a unique unbounded component at λ_u?

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Paper III

Visibility to infinity in the hyperbolic plane, despite obstacles

Itai Benjamini* Johan Jonasson† Oded Schramm‡
Johan Tykesson§

Abstract

Suppose that $\mathcal{Z}$ is a random closed subset of the hyperbolic plane $\mathbb{H}^2$, whose law is invariant under isometries of $\mathbb{H}^2$. We prove that if the probability that $\mathcal{Z}$ contains a fixed ball of radius 1 is larger than some universal constant $p_0 < 1$, then there is positive probability that $\mathcal{Z}$ contains lines.

We additionally consider the Poisson Boolean model of continuum percolation in the hyperbolic plane $\mathbb{H}^2$. Let λ be the intensity of the underlying Poisson process and let R be the radius of the balls. We show that there is a critical value $\lambda_{\mathrm{gv}} = \lambda_{\mathrm{gv}}(R)$ such that if $\lambda < \lambda_{\mathrm{gv}}$, then there are a.s. hyperbolic lines contained in the complement of the covered region, but not if $\lambda \geq \lambda_{\mathrm{gv}}$. A similar result is proved for the covered region itself, instead of the complement. We also find the exact value of the critical λ in both cases. We also consider dynamical versions of the Poisson Boolean model, and rule out the existence of certain types of exceptional times.

Keywords and phrases: continuum percolation, phase transitions, hyperbolic space
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1 Introduction and main results

In this paper, we are interested in the existence of hyperbolic half-lines and lines (that is, infinite geodesic rays and bi-infinite geodesics respectively) contained

*Departments of Mathematics, The Weizmann Institute, Rehovot, Israel 76100. E-mail: itai.benjamini@weizmann.ac.il

†Department of Mathematical Sciences, Division of Mathematical Statistics, Chalmers University of Technology, S-41296 Göteborg, Sweden. E-mail: jonasson@math.chalmers.se.
Research partially supported by the Swedish Natural Science Research Council.

‡Microsoft Research, One Microsoft Way, Redmond WA 98052, USA. E-mail: schramm@microsoft.com

§Department of Mathematical Sciences, Division of Mathematical Statistics, Chalmers University of Technology, S-41296 Göteborg, Sweden. E-mail: johant@math.chalmers.se.
Research supported by the Swedish Natural Science Research Council.

in unbounded connected components of some continuum percolation models. Our first result is quite general:

Theorem 1.1. *Let $\mathcal{Z}$ be a random closed subset of $\mathbb{H}^2$, whose law is invariant under isometries of $\mathbb{H}^2$, and let B denote some fixed ball of radius 1 in $\mathbb{H}^2$. There is a universal constant $p_o < 1$ such that if $\mathbf{P}\big[B \subset \mathcal{Z}\big] > p_o$, then with positive probability $\mathcal{Z}$ contains hyperbolic lines.*

The first result of this type was proven by Olle Häggström [5] for regular trees of degree at least 3. That paper shows that for automorphism invariant site percolation on such trees, when the probability that a site is open is sufficiently close to 1, there are infinite open clusters with positive probability. This was subsequently generalized to transitive nonamenable graphs [1]. The new observation here is that in $\mathbb{H}^2$, one can actually find lines contained in unbounded components when the marginal is sufficiently close to 1. The proof of Theorem 1.1 is not too difficult, and is based on a reduction to the tree case.

We also obtain more refined results in some standard continuum percolation models. Consider a Poisson point process with intensity λ on a manifold M. In the *Poisson Boolean model of continuum percolation* with parameters λ and R, balls of radius R are centered around the points of the Poisson process. One then studies the geometry of the connected components of the union of balls, or the connected components of the complement. In particular, one asks for which values of the parameters there are unbounded connected components or a unique unbounded component. In this note, we ask when the union of the balls or its complement contains half-lines or lines. It is easy to see that this can never happen on $\mathbb{R}^n$. Here, we deal mostly with the hyperbolic plane $\mathbb{H}^2$, though we raise questions regarding other spaces.

Other aspects of the Poisson Boolean model in $\mathbb{H}^2$ have previously been studied in [14]. For further studies of percolation in the hyperbolic plane, the reader may consult [2, 9]. In [4], an introduction to hyperbolic geometry is found, and for an introduction to the theory of percolation on infinite graphs see for example [3, 10, 6].

Let $X = X_\lambda$ be the set of points in a Poisson process of intensity λ in $\mathbb{H}^2$. Let

$$\mathcal{B} := \bigcup_{x \in X} \overline{B(x,R)}$$

denote the *occupied set*, where $B(x,r)$ denotes the open ball of radius r centered at x. The closure of the complement

$$\mathcal{W} := \overline{\mathbb{H}^2 \setminus \mathcal{B}}$$

will be reffered to as the *vacant set.* Let $\lambda_{\mathrm{gv}} = \lambda_{\mathrm{gv}}(R)$ denote the infimum of the set of $\lambda \geq 0$ such that for the parameter values (R, λ) a.s. $\mathcal{W}$ does not contain a hyperbolic line. Let $\bar{\lambda}_{\mathrm{gv}}$ denote the infimum of the set of $\lambda \geq 0$ such that with positive probability, a fixed point $x \in \mathbb{H}^2$ belongs to a half-line contained in $\mathcal{W}$. Later, we shall see that $\lambda_{\mathrm{gv}} = \bar{\lambda}_{\mathrm{gv}}$. Clearly, if $\lambda > \lambda_{\mathrm{gv}}$, there are a.s. no hyperbolic lines in $\mathcal{W}$. Let $f(r) = f_{R,\lambda}(r)$ denote the probability that a fixed line segment of length r in $\mathbb{H}^2$ is contained in $\mathcal{W}$.

Theorem 1.2. *For every $R > 0$, we have $0 < \lambda_{\mathrm{gv}}(R) = \bar{\lambda}_{\mathrm{gv}}(R) < \infty$, and the following statements hold at $\lambda_{\mathrm{gv}}(R)$.*

1. *A.s. there are no hyperbolic lines within $\mathcal{W}$.*
2. *Moreover, $\mathcal{W}$ a.s. does not contain any hyperbolic ray (half-line).*
3. *There is a constant $c = c_R > 0$, depending only on R, such that*

$$c\,e^{-r} \le f(r) \le e^{-r}, \qquad \forall r > 0\,. \tag{1.1}$$

Furtheremore, the analogous statements hold with $\mathcal{B}$ in place of $\mathcal{W}$ (with possibly a different critical intensity).

We also show that the critical value λ_{gv} is given by

$$\lambda_{\mathrm{gv}}(R) = \frac{1}{2\sinh(R)}\,. \tag{1.2}$$

An equation characterising the corresponding critical λ for $\mathcal{B}$ follows from our results (i.e., (4.1) with $\alpha = 1$), but in this case, we do not have a closed form for the critical λ.

The key geometric property allowing for geodesic percolation to occur for some λ is the exponential divergence of geodesics. This does not hold in Euclidean space. It is of interest to determine which homogeneous spaces admit a regime of intensities with geodesics percolating.

We then consider two dynamical continuum percolation models. In the first model, each ball in the Poisson Boolean model independently of all other balls update its position as follows. First it waits a random amount of time, which is exponentially distributed with parameter 1. Then, the ball moves to a center which is chosen uniformly at random within the ball of radius 1 around the original center. This procedure is then repeated.

In this setting, we consider half-lines contained in $\mathcal{W}$, emanating from a fixed point $x \in \mathbb{H}^2$. We show that if $\lambda < \lambda_{\mathrm{gv}}$, then a.s. for all $t \ge 0$ there are half-lines containing x intersecting only finitely many balls in $\mathcal{W}$. Then we show that for $\lambda \ge \lambda_{\mathrm{gv}}$ a.s. for all $t \ge 0$ there are no half-lines containing x contained in $\mathcal{W}$. In other words, in this model, there are no *exceptional times* for which the model behaves differently from the stationary one in this sense.

In the second model, we start with a Poisson process X in $\mathbb{H}^2 \times \mathbb{R}$ with some intensity λ. Around each point, place an open ball of radius R. First suppose that λ is supercritical for the a.s. existence of hyperbolic lines contained the vacant region of $\mathbb{H}^2 \times t$ for a fixed $t \in \mathbb{R}$. We show that the set of $t \in \mathbb{R}$ such that the vacant region of $\mathbb{H}^2 \times t$ contains hyperbolic lines is empty a.s. Then we suppose that λ is subcritical for the a.s. existence of hyperbolic lines contained in the vacant region of $\mathbb{H}^2 \times t$ for fixed $t \in \mathbb{R}$, and show that the set of $t \in \mathbb{R}$ such that the covered region of $\mathbb{H}^2 \times t$ does not contain hyperbolic lines is a.s. empty. The analogous results are obtained for the covered region.

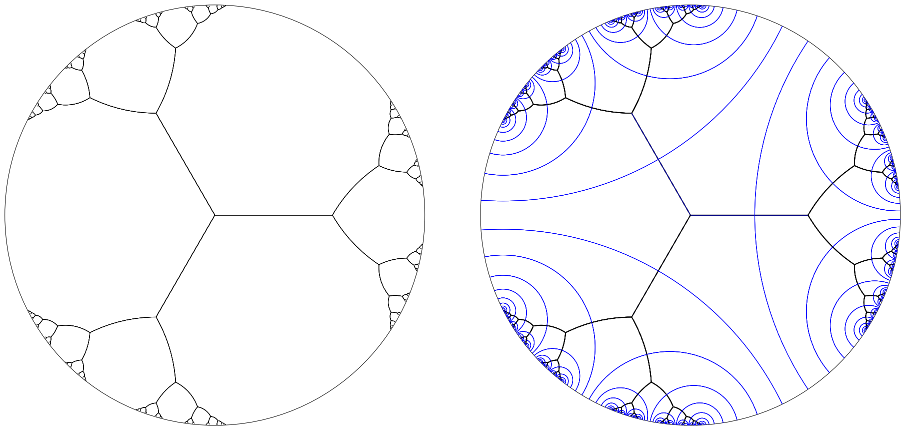

Figure 2.1: A tree embedded in the hyperbolic plane, in the Poincaré disk model. On the right appears the tree together with some of its lines of symmetry.

2 Lines appearing when the marginal is large

The proof of Theorem 1.1 is based on a reduction to the tree case. We will need the following construction of a tree embedded in $\mathbb{H}^2$, which is illustrated in Figure 2.1. (This construction should be rather obvious to the readers who are proficient in hyperbolic geometry.) Consider the hyperbolic plane in the Poincaré disk model. Let $o \in \mathbb{H}^2$ correspond to the center of the disk. Let A_0 be an arc on the unit circle of length smaller than $2\pi/3$. Let A_j denote the rotation of A_0 by $2\pi j/3$; that is $A_j := e^{2\pi j/3} A_0$, $j = 1, 2$. Let L_j, $j = 0, 1, 2$, denote the hyperbolic line whose endpoints on the ideal boundary $\partial\mathbb{H}^2$ are the endpoints of A_j. Let Γ denote the group of hyperbolic isometries that is generated by the reflections γ_0, γ_1 and γ_2 in the lines L_0, L_1 and L_2, respectively. If $w = (w_1, w_2, \dots, w_n) \in \{0, 1, 2\}^n$, then let γ_w denote the composition $\gamma_{w_1} \circ \gamma_{w_2} \circ \cdots \circ \gamma_{w_n}$. We will say that w is *reduced* if $w_{j+1} \neq w_j$ for $j = 1, 2, \dots, n-1$. A simple induction on n then shows that $\gamma_w(o)$ is separated from o by L_{w_1} when w is reduced and $n > 0$. In particular, for reduced $w \neq ()$, we have $\gamma_w(o) \neq o$ and $\gamma_w \neq \gamma_{()}$. Clearly, every γ_w where w has $w_j = w_{j+1}$ for some j is equal to $\gamma_{w'}$ where w' has these two consecutive elements of w dropped. It follows that Γ acts simply and transitively on the orbit Γo. ("Simply" means that $\gamma v = v$ where $\gamma \in \Gamma$ and $v \in \Gamma o$ implies that γ is the identity.) Now define a graph T on the vertex set Γo by letting each $\gamma(o)$ be connected by edges to the three points $\gamma \circ \gamma_j(o)$, $j = 0, 1, 2$. Then T is just the 3-regular tree embedded in the hyperbolic plane. In fact, this is a Cayley graph of the group Γ, since we may identify Γ with the orbit Γo. (One easily verifies that Γ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.)

We will need a few simple properties of this embedding of the 3-regular tree in $\mathbb{H}^2$. It is easy to see that every simple path $v_0, v_1, \dots$ in T has a unique limit point on the ideal boundary $\partial\mathbb{H}^2$. (Figure 2.1 does not lie.) Moreover, if $v_0 = o$

and $v_1 = \gamma_j(o)$, then the limit point will be in the arc A_j. If $(v_j : j \in \mathbb{Z})$ is a bi-infinite simple path in T with $v_0 = o$, then its two limit points on the ideal boundary will be in two different arcs A_j. Hence, the distance from o to the line in $\mathbb{H}^2$ with the same pair of limit points on $\partial\mathbb{H}^2$ is bounded by some constant R, which does not depend on the path $(v_j : j \in \mathbb{Z})$. Invariance under the group Γ now shows that for every bi-infinite simple path β in T, the hyperbolic line L_β joining its limit points passes within distance R from each of the vertices of β. It follows that there is some constant $R' > 0$ such that L_β is contained in the R'-neighborhood of the set of vertices of β.

We are now ready to prove our first theorem.

Proof of Theorem 1.1. We use the above construction of T, Γ and the constant R'. Given $\mathcal{Z}$, let $\omega \subset V(T)$ denote the set of vertices $v \in V(T)$ such that the ball $B(v, R')$ is contained in $\mathcal{Z}$. Then ω is a (generally dependent) site percolation on T and its law is invariant under Γ. Set $q := \mathbf{P}\big[o \in \omega\big]$. By [1], there is some $p_0 \in (0,1)$ such that if $q \geq p_0$, then ω has infinite connected components with positive probability. (We need to use [1], rather than [5], since the group Γ is not the full automorphism group of T.) Set $p_1 := (p_0+1)/2$. Let N be the number of balls of radius 1 that are sufficient to cover $B(o, R')$. Now suppose that $\mathbf{P}\big[B(o,1) \subset \mathcal{Z}\big] > 1 - (1-p_0)/(2N)$. Then a sum bound implies that $q > (p_0+1)/2$. Therefore, if we intersect ω with an independent Bernoulli site percolation with marginal $p > (p_0+1)/2$, the resulting percolation will still have infinite components with positive probability, by the same argument as above. Thus, we conclude that with positive probability ω has infinite components with more than one end and therefore also bi-infinite simple paths. The line determined by the endpoints on $\partial\mathbb{H}^2$ of such a path will be contained in $\mathcal{Z}$, by the definition of R'. The proof is thus complete. □

3 Lines in well-behaved percolation

The proofs of the statements in Theorem 1.2 concerning $\mathcal{B}$ are essentially the same as the proofs concerning $\mathcal{W}$. We therefore find it worthwhile to employ an axiomatic approach, which will cover both cases.

Definition 3.1. In the following, we fix a closed disk $B \subset \mathbb{H}^2$ of radius 1. A *well-behaved percolation* on $\mathbb{H}^2$ is a random closed subset $\mathcal{Z} \subset \mathbb{H}^2$ satisfying the following assumptions.

1. The law of $\mathcal{Z}$ is invariant under isometries of $\mathbb{H}^2$.
2. The set $\mathcal{Z}$ satisfies positive correlations; that is, for every pair g and h of bounded increasing measurable functions of $\mathcal{Z}$, we have
$$\mathbf{E}\big[g(\mathcal{Z})\, h(\mathcal{Z})\big] \geq \mathbf{E}\big[g(\mathcal{Z})\big]\, \mathbf{E}\big[h(\mathcal{Z})\big].$$
3. There is some $R_0 < \infty$ such that $\mathcal{Z}$ satisfies independence at distance R_0, namely, for every pair of subsets $A, A' \subset \mathbb{H}^2$ satisfying $\inf\{d(a,a') : a \in A, a' \in A'\} \geq R_0$, the intersections $\mathcal{Z} \cap A$ and $\mathcal{Z} \cap A'$ are independent.

4. The expected number m of connected components of $B \setminus \mathcal{Z}$ is finite.

5. The expected length ℓ of $B \cap \partial\mathcal{Z}$ is finite.

6. $p_0 := \mathbf{P}\big[B \subset \mathcal{Z}\big] > 0$.

Invariance under isometries implies that m, ℓ and p_0 do not depend on the choice of B. We say that $\mathcal{Z}$ is Λ-well behaved, if it is well-behaved and $p_0, m^{-1}, \ell^{-1}, R_0^{-1} > \Lambda$. Many of our estimates below can be made to depend only on Λ. In the following, we assume that $\mathcal{Z}$ is Λ-well behaved, where $\Lambda > 0$, and use $O(g)$ to denote any quantity bounded by $c\,g$, where c is an arbitrary constant that may depend only on Λ.

If $x, y \in \mathbb{H}^2$, let $[x, y]_s$ denote the union of all line segments $[x', y']$ where $d(x, x') < s$ and $d(y, y') < s$. Let $A(x, y, s)$ be the event that there is some connected component of $\mathcal{Z} \cap [x, y]_s$ that intersects $B(x, s)$ as well as $B(y, s)$, and let $Q(x, y, s)$ be the event that $[x, y]_s \subset \mathcal{Z}$.

Lemma 3.2. *There is a constant $c = c(\Lambda) < \infty$, which depends only on Λ, such that for all $x, y \in \mathbb{H}^2$ satisfying $d(x, y) \geq 4$ and for all $\epsilon > 0$*

$$\mathbf{P}\big[Q(x, y, \epsilon)\big] > (1 - c\,\epsilon)\,\mathbf{P}\big[A(x, y, \epsilon)\big]. \tag{3.1}$$

Proof. Observe that the expected minimal number of disks of small radius ϵ that are needed to cover $\partial\mathcal{Z} \cap B$ is $O(\ell/\epsilon)$. It follows by invariance that

$$\mathbf{P}\big[B(x, \epsilon) \cap \partial\mathcal{Z} \neq \emptyset\big] = O(\epsilon)\,\ell = O(\epsilon) \tag{3.2}$$

holds for $x \in \mathbb{H}^2$.

Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ denote a hyperbolic line parameterized by arclength, and let L_t denote the hyperbolic line through $\gamma(t)$ which is orthogonal to γ. Set

$$g(r, s) := \mathbf{P}\big[A\big(\gamma(0), \gamma(r), s\big) \setminus Q\big(\gamma(0), \gamma(r), s\big)\big].$$

By invariance, we have $\mathbf{P}\big[A(x, y, s) \setminus Q(x, y, s)\big] = g\big(d(x, y), s\big)$.

Set $B := B\big(\gamma(0), 1\big)$. Fix some $\epsilon \in (0, 1/10)$. Let S_j denote the intersection of B with the open strip between $L_{2j\epsilon}$ and $L_{2(j+1)\epsilon}$, where $j \in J := \mathbb{N} \cap [0, \epsilon^{-1}/10]$. Let x_j and y_j denote the two points in $L_{(2j+1)\epsilon} \cap \partial B$. Let J_1 denote the set of $j \in J$ such that S_j is not contained in $\mathcal{Z}$ but there is a connected component of $\mathcal{Z} \cap S_j$ that joins the two connected components of $S_j \cap \partial B$. Observe that the number of connected components of $B \setminus \mathcal{Z}$ is at least $|J_1| - 1$. Hence $\mathbf{E}\big[|J_1|\big] \leq m + 1$. Let J_2 denote the set of $j \in J$ such that $A(x_j, y_j, \epsilon) \setminus Q(x_j, y_j, \epsilon)$ holds. Note that if $j \in J_2 \setminus J_1$, then $\partial\mathcal{Z}$ is within distance $O(\epsilon)$ from $x_j \cup y_j$. Therefore, $\mathbf{P}\big[j \in J_2 \setminus J_1\big] = O(\epsilon)\,\ell$ holds for every $j \in J$, by (3.2). Consequently,

$$\mathbf{E}\big[|J_2|\big] \leq \mathbf{E}\big[|J_2 \setminus J_1|\big] + \mathbf{E}\big[|J_1|\big] \leq O(\epsilon)\,\ell\,|J| + m + 1 = O(1)\,.$$

Thus, there is at least one $j = j_\epsilon \in J$ satisfying

$$\begin{aligned} \mathbf{P}\big[A(x_j, y_j, \epsilon) \setminus Q(x_j, y_j, \epsilon)\big] &= \mathbf{P}\big[j \in J_2\big] \\ &\leq O(1)/|J| = O(\epsilon)\,. \end{aligned} \tag{3.3}$$

Set $r_\epsilon := d(x_{j_\epsilon}, y_{j_\epsilon})$, and note that $r_\epsilon \in (1,2]$. Now suppose that $x, y \in \mathbb{H}^2$ satisfy $d(x,y) = 2$. Let x_0 be the point in $[x,y]$ at distance r_ϵ from y, and let y_0 be the point in $[x,y]$ at distance r_ϵ from x. Observe that $A(x,y,\epsilon) \subset A(x_0,y,\epsilon) \cap A(x,y_0,\epsilon)$. Moreover, since $[x,y]_\epsilon \subset [x,y_0]_\epsilon \cup [x_0,y]_\epsilon$, we have $Q(x,y,\epsilon) \supset Q(x,y_0,\epsilon) \cap Q(x_0,y,\epsilon)$. Thus,

$$A(x,y,\epsilon) \setminus Q(x,y,\epsilon) \subset \big(A(x_0,y,\epsilon) \setminus Q(x_0,y,\epsilon)\big) \cup \big(A(x,y_0,\epsilon) \setminus Q(x,y_0,\epsilon)\big)$$

and therefore (3.3) and invariance gives

$$g(2,\epsilon) \le 2\,\mathbf{P}\big[A(x_{j_\epsilon}, y_{j_\epsilon}, \epsilon) \setminus Q(x_{j_\epsilon}, y_{j_\epsilon}, \epsilon)\big] = O(\epsilon)\,. \tag{3.4}$$

The same argument shows that

$$g(r',\epsilon) \le 2\,g(r,\epsilon), \qquad \text{if } 2 \le r < r' \le 2\,r\,. \tag{3.5}$$

We will now get a bound on $g(2\,k,\epsilon)$ for large $k \in \mathbb{N}$. For $j \in [k] := \mathbb{N} \cap [0,k]$, let r_j be the distance from $\gamma(2\,j)$ to the complement of $[\gamma(0),\gamma(2\,k)]_\epsilon$. Let $A_j := A\big(\gamma(2\,j), \gamma(2\,j+2), r_j \vee r_{j+1}\big)$, $Q_j := Q\big(\gamma(2\,j), \gamma(2\,j+2), r_j \vee r_{j+1}\big)$, where $j \in [k-1]$. Also set $\bar{A} := A\big(\gamma(0),\gamma(2\,k),\epsilon\big)$. Then

$$Q\big(\gamma(0),\gamma(2\,k),\epsilon\big) \supset \bigcap_{j=0}^{k-1} Q_j\,.$$

Hence,

$$g(2\,k,\epsilon) \le \sum_{j=0}^{k-1} \mathbf{P}\big[\bar{A} \setminus Q_j\big]. \tag{3.6}$$

We now claim that

$$\mathbf{P}\big[\bar{A} \setminus Q_j\big] = O(1)\,\mathbf{P}\big[\bar{A}\big]\,\mathbf{P}\big[A_j \setminus Q_j\big], \tag{3.7}$$

where the implied constant depends only on p_0 and R_0. Let $j' := \lfloor j - R_0/2 - 2 \rfloor$ and $j'' := \lceil j + R_0/2 + 3 \rceil$. Suppose first that $j' > 0$ and $j'' < k$. Let $\bar{A}'(j)$ denote the event that $\mathcal{Z} \cap [\gamma(0),\gamma(2\,k)]_\epsilon$ contains a connected component that intersects both $B\big(\gamma(0),\epsilon\big)$ and $B\big(\gamma(2\,j'),\epsilon\big)$, and let $\bar{A}''(j)$ denote the event that $\mathcal{Z} \cap [\gamma(0),\gamma(2\,k)]_\epsilon$ contains a connected component that intersects both $B\big(\gamma(2\,j''),\epsilon\big)$ and $B\big(\gamma(2\,k),\epsilon\big)$. Then $\bar{A} \subset \bar{A}'(j) \cap \bar{A}''(j) \cap A_j$. Indepedence at distance R_0 therefore gives

$$\mathbf{P}\big[\bar{A} \setminus Q_j\big] \le \mathbf{P}\big[\bar{A}'(j) \cap \bar{A}''(j)\big]\,\mathbf{P}\big[A_j \setminus Q_j\big].$$

Now note that the fact that $\mathcal{Z}$ satisfies positive correlations shows that

$$\mathbf{P}\big[\bar{A}'(j) \cap \bar{A}''(j)\big] \le O(1)\,\mathbf{P}\big[\bar{A}\big],$$

where the implied constant depends only on R_0 and p_0. Thus, we get (3.7) in the case that $j' > 0$ and $j'' < k$. The general case is easy to obtain (one just needs to drop $\bar{A}'(j)$ or $\bar{A}''(j)$ from consideration). Now, (3.6) and (3.7) give

$$g(2\,k,\epsilon) \le O(1)\,\mathbf{P}\big[\bar{A}\big] \sum_{j=0}^{k-1} g(2, r_j \vee r_{j+1})\,. \tag{3.8}$$

Note that there is a universal constant $a \in (0,1)$ such that $r_j \le a^{|j|\wedge|k-j|}\,\epsilon$. (This is where hyperbolic geometry comes into play.) Hence, we get by (3.4) and (3.8) that $g(2\,k,\epsilon) \le O(1)\mathbf{P}\big[\bar{A}\big]\epsilon$, where the implied constant may depend on ℓ, m, R_0 and p_0. This proves (3.1) in the case where $d(x,y)$ is divisible by 2. The general case follows using (3.5) with $r' = d(x,y)$ and $r = 2\,\lfloor r'/2 \rfloor$. □

Let $f(r)$ denote the probability that a fixed line segment of length r is contained in $\mathcal{Z}$. Clearly,

$$\mathbf{P}\big[Q(x,y,s)\big] \le f(\text{length}[x,y]) \le \mathbf{P}\big[A(x,y,s)\big],$$

and Lemma 3.2 shows that for s sufficiently small the upper and lower bounds are comparable.

Lemma 3.3. *There is a unique $\alpha \ge 0$ (which depends on the law of $\mathcal{Z}$) and some $c(\Lambda) > 0$ (depending only on Λ) such that*

$$c\,e^{-\alpha r} \le f(r) \le e^{-\alpha r} \tag{3.9}$$

holds for every $r \ge 0$.

Proof. Since the uniqueness statement is clear, we proceed to prove existence. Positive correlations imply that

$$f(r_1 + r_2) \ge f(r_1)\,f(r_2)\,, \tag{3.10}$$

that is, f is supermultiplicative. Therefore, $-\log f(r)$ is subadditive, and Fekete's Lemma says that we must have

$$\alpha := \lim_{r\to\infty} \frac{-\log f(r)}{r} = \inf_{r>0} \frac{-\log f(r)}{r}\,.$$

Since for every r we have $\alpha \le -\log\big(f(r)\big)/r$, the right inequality in (3.9) follows.

On the other hand, if we fix some $R > R_0$, then independence at distance larger than R_0 gives

$$f(r_1)\,f(r_2) \ge f(r_1 + R + r_2) \overset{(3.10)}{\ge} f(r_1 + r_2)\,f(R)\,.$$

Dividing by $f(R)^2$, we find that the function $r \mapsto f(r)/f(R)$ is submultiplicative. Thus, by Fekete's lemma again,

$$\lim_{r\to\infty} \frac{\log\big(f(r)/f(R)\big)}{r} = \inf_{r>0} \frac{\log\big(f(r)/f(R)\big)}{r}\,.$$

The left hand side is equal to $-\alpha$, and we get for every $r > 0$

$$-\alpha \leq \frac{\log\big(f(r)/f(R)\big)}{r}\,.$$

By positive correlations, there is some $c = c(\Lambda)$ such that $f(R) \geq c$, which implies the left inequality in (3.9). □

Lemma 3.4. *If $\alpha \geq 1$ (where α is defined in Lemma 3.3), then a.s. there are no half-lines contained in $\mathcal{Z}$.*

Proof. Fix a basepoint $o \in \mathbb{H}^2$. Let $s = (2c)^{-1}$, where c is the constant in (3.1). Then

$$\mathbf{P}\big[A(x,y,s)\big]/2 \leq \mathbf{P}\big[Q(x,y,s)\big] \leq f\big(d(x,y)\big) \leq e^{-d(x,y)} \tag{3.11}$$

holds for every $x, y \in \mathbb{H}^2$ satisfying $d(x,y) \geq 4$. For every integer $r \geq 4$ let $V(r)$ be a minimal collection of points on the circle $\partial B(o,r)$ such that the disks $B(z,s)$ with $z \in V$ cover that circle. Let X_r be the set of points $z \in V(r)$ such that $A(o,z,s)$ holds. By (3.11)

$$\mathbf{E}\big[|X_r|\big] \leq 2\,|V(r)|\,f(r) = O(1)\,s^{-1}\,\mathrm{length}\big(\partial B(o,r)\big)\,e^{-r} = O(1)\,, \tag{3.12}$$

since we are treating s as a constant and the length of $\partial B(o,r)$ is $O(e^r)$.

The rest of the argument is quite standard, and so we will be brief. By (3.12) and Fatou's lemma, we have $\limsup_{r\to\infty} |X_r| < \infty$ a.s. Thus $\sup_r |X_r| < \infty$ a.s. Now fix some large r and let $r' \in \mathbb{N}$ satisfy $r' > r + R_0 + 2$. Since $\mathcal{Z} \setminus B(o, r+R_0+1)$ is independent from $\mathcal{Z} \cap B(o,r)$, positive correlations implies that

$$\mathbf{P}\big[X_{r'} = \emptyset \bigm| \mathcal{Z} \cap B(o,r)\big] \geq p^{-|X_r|}, \tag{3.13}$$

where $p > 0$ is a constant (which we allow to depend on the law of $\mathcal{Z}$). Since $\sup_r |X_r| < \infty$ a.s., it follows by (3.13) that $\inf_r |X_r| = 0$ a.s., which means that $\max\{r : X_r \neq \emptyset\} < \infty$ a.s. Therefore, a.s. there is no half-line that intersects $B(o,s)$. Since $\mathbb{H}^2$ can be covered by a countable collection of balls of radius s, the lemma follows. □

Lemma 3.5. *Suppose that $\alpha < 1$. Then (i) a.s. $\mathcal{Z}$ contains hyperbolic lines, (ii) for every fixed $x \in \mathbb{H}^2$, there is a positive probability that $\mathcal{Z}$ contains a half-line containing x, and (iii) for every fixed point x in the ideal boundary $\partial\mathbb{H}^2$ there is a.s. a geodesic line passing through x whose intersection with $\mathcal{Z}$ containes a half-line.*

Proof. We first prove (ii). Fix some point $o \in \mathbb{H}^2$. Let A denote a closed half-plane with $o \in \partial A$, and let $I := A \cap \partial B(o,1)$. For $r > 1$ and $x \in \partial B(o,1)$, let $L_r(x)$ denote the line segment which contains x, has length r and has o as an endpoint. Set $Y_r := \{x \in I : L_r(x) \subset \mathcal{Z}\}$, and let y_r denote the length of Y_r. Then we have

$$\mathbf{E}\big[y_r\big] = \mathrm{length}(I)\, f(r)\,.$$

The second moment is given by

$$\mathbf{E}\big[y_r^2\big] = \int_I \int_I \mathbf{P}\big[x, x' \in Y_r\big]\, dx\, dx'\,.$$

Now note that if $r_2 > r_1 > 0$, then the distance from $L_{r_2}(x') \setminus L_{r_1}(x')$ to $L_{r_2}(x)$ is at least $d(x,x')\, e^{r_1}/O(1)$ as $x \to x'$. Consequently, by independence on sets at distance larger than R_0, we have

$$\mathbf{P}\big[x, x' \in Y_r\big] \le f(r)\, f\big(r + \log d(x,x') + O(1)\big).$$

Now applying the above and (3.9) gives

$$\frac{\mathbf{E}\big[y_r^2\big]}{\mathbf{E}\big[y_r\big]^2} \le O(1) \int_I \int_I \exp\big(-\alpha \,\log d(x,x')\big)\, dx\, dx'$$

$$= O(1) \int_I \int_I d(x,x')^{-\alpha}\, dx\, dx' = O(1)\,,$$

since $\alpha < 1$. Therefore, the Paley-Zygmund inequality implies that

$$\inf_{r>1} \mathbf{P}\big[y_r > 0\big] > 0\,.$$

Since y_r is monotone non-increasing, it follows that

$$\mathbf{P}\big[\forall_{r>1}\ \ y_r > 0\big] > 0\,.$$

By compactness, on the event that $y_r > 0$ for all $r > 1$ we have $\bigcap_{r>1} Y_r \ne \emptyset$. If $x \in \bigcap Y_r$, then the half-line with endpoint o passing through x is contained in $\mathcal{Z} \cap A$. This proves (ii).

We now prove (i). Fix $s = 1/(2\,c)$, where c is given by Lemma 3.2. For $x \in \partial B(o,1)$ let $z_r(x)$ denote the endpoint of $L_r(x)$ that is different from o and let Y_r' be the set of points $x \in I$ such that $[z, z_r(x)] \subset \mathcal{Z}$ holds for every $z \in B(o,s)$. Let y_r' denote the length of Y_r'. Then $Y_r' \subset Y_r$ and therefore $y_r' \le y_r$. By the choice of s, we have $\mathbf{E}\big[y_r'\big] \ge \mathbf{E}\big[y_r\big]/2$. On the other hand, $\mathbf{E}\big[(y_r')^2\big] \le \mathbf{E}\big[y_r^2\big] = O(1)\,\mathbf{E}\big[y_r\big]^2$. As above, this implies that with positive probability $Y_\infty' := \bigcap_{r>1} Y_r' \ne \emptyset$. Suppose that $x \in Y_\infty'$. Let $\tilde{x}$ denote the endpoint on the ideal boundary $\partial\mathbb{H}^2$ of the half-line starting at o and passing through x. Then for every $z \in B(o,s)$ the half-line $[z, \tilde{x})$ is contained in $\mathcal{Z}$. By invariance and positive correlations, for every $\epsilon > 0$ there is positive probability that Y_∞' is within distance ϵ from each of the two points in $\partial A \cap I$. If x' and x'' are two points in Y_∞' that are sufficiently close to the two points in $\partial A \cap I$, then the hyperbolic line joining the two endpoints at infinity of the corresponding half-lines through o intersects $B(o,s)$. In such a case, this line will be contained in $\mathcal{Z}$. Thus, we see that for every line L (in this case ∂A) for every point $o \in L$ and for every $\epsilon > 0$, there is positive probability that $\mathcal{Z}$ contains a line passing within distance ϵ of the two points in $\partial B(o,1) \cap L$. Now (i) follows by invariance and by independence at a distance.

The proof of (iii) is similar to the above, and will be omitted. □

Remark 3.6. Let $o \in \mathbb{H}^2$. Let Y denote the set of points z in in the ideal boundary $\partial\mathbb{H}^2$ such that the half-line $[o,z)$ is contained in $\mathcal{Z}$. It can be concluded from the first and second moments computed in the proof of Lemma 3.5 and a standard Frostman measure argument that the essential supremum of the Hausdorff dimension of Y is given by

$$\| \dim_H(Y) \|_\infty = 1 - \alpha \, .$$

It would probably not be too hard to show that $\dim_H(Y) = 1 - \alpha$ a.s. on the event that $Y \neq \emptyset$.

A modification of the above arguments shows that there is positive probability that $\mathcal{Z}$ contains a line through o if and only if $\alpha < 1/2$. In case $\alpha < 1/2$, the essential supremum of the Hausdorff dimension of the set of lines in $\mathcal{Z}$ through o is $1 - 2\,\alpha$.

It should be possible to show that the Hausdorff dimension of the union of the lines in $\mathcal{Z}$ is a.s. $3 - 2\,\alpha$ when $\alpha \in [1/2, 1)$.

4 Boolean occupied and vacant percolation

Recall the definition of $\mathcal{B}$ and $\mathcal{W}$. First, we claim that $\mathcal{B}$ and $\mathcal{W}$ are well-behaved.

Proposition 4.1. *Fix a compact interval $I \subset (0,\infty)$. Then there is some $\Lambda = \Lambda(I) > 0$ such that if $\lambda, R \in I$, then $\mathcal{B}$ and $\mathcal{W}$ are Λ-well behaved.*

Proof. It is well known that $\mathcal{B}$ and $\mathcal{W}$ (and therefore also $\mathcal{W}$ satisfy positive correlations. For $\mathcal{W}$, m is bounded by the expected number of points in X that fall in the R-neighborhood of B. Observe that each connected component of $\mathcal{W} \cap B$, with the possible exception of one, has on its boundary an intersection point of two circles of radius R centered at points in X. Since the second moment of the number of points in X that fall inside the R-neighborhood of B is finite, it follows that m is also bounded for $\mathcal{B}$. The remaining conditions are easily verified and left to the reader. □

We are now ready to prove our main theorem.

Proof of Theorem 1.2. We start by considering $\mathcal{W}$. Fix some $R \in (0,\infty)$. If we let $\lambda \searrow 0$, then $f(1) \nearrow 1$ and by (3.9) $\alpha \searrow 0$. Thus, Lemma 3.5 implies that $\lambda_{\mathrm{gv}} > 0$. (We could alternatively prove this from Theorem 1.1.) It is also clear that $\lambda_{\mathrm{gv}} < \infty$, since for λ sufficiently large a.s. $\mathcal{W}$ has no unbounded connected component.

Since the constant c in Lemma 3.3 depends only on Λ, that lemma implies that α is continuous in $(\lambda, R) \in (0,\infty)^2$. In particular, Lemmas 3.4 and 3.5

show that when $\lambda = \lambda_{\text{gv}}(R)$, we have $\alpha = 1$ and that there are a.s. no lines or half-lines in $\mathcal{W}$. Also, we get (1.1) from (3.9). Finally, it follows from Lemma 3.4 and Lemma 3.5 (ii) that $\lambda_{\text{gv}} = \bar{\lambda}_{gv}$. The proof for $\mathcal{B}$ is similar. □

Next, we calculate α for $\mathcal{B}$ and $\mathcal{W}$.

Lemma 4.2. *The value of α for line percolation in $\mathcal{W}$ is given by*

$$\alpha = 2\,\lambda \sinh R\,.$$

and consequently

$$\lambda_{gv}(R) = 1/(2\sinh R)\,.$$

Proof. Consider a line $\gamma : \mathbb{R} \to \mathbb{H}^2$, parameterized by arclength, and let $r > 0$. A.s. the interval $\gamma[0, r]$ is contained in $\mathcal{W}$ if and only if the R-neighborhood of the interval does not contain any points of X. Let N denote this neighborhood, and let A denote its area. Then $f(r) = e^{-\lambda A}$. For each point $z \in \mathbb{H}^2$, let t_z denote the t minimizing the distance from z to $\gamma(t)$. Then $N = N_0 \cup N_1 \cup N_2$, where $N_0 := \{z \in \mathbb{H}^2 : d(z, \gamma(t_z)) < R, t_z \in [0, r]\}$, $N_1 := \{z \in B(\gamma(0), R) : t_z < 0\}$ and $N_2 := \{z \in B(\gamma(r), R) : t_z > r\}$. Observe that N_1 and N_2 are two half-disks of radius R, so that their areas are independent of r. We can conveniently calculate the area of N_0 explicitly in the upper half-plane model for $\mathbb{H}^2$, for which the hyperbolic length element is given by $|ds|/y$, where $|ds|$ is the Euclidean length element. We choose $\gamma(t) = (0, e^t)$. Recall that the intersection of the upper half-plane with the Euclidean circles orthogonal to the real line are lines in this model. It is easy to see that for $z = (\rho\,\cos\theta, \rho\,\sin\theta)$, we have $\gamma(t_z) = (0, \rho)$. Moreover, the distance from z to γ is

$$\Big|\int_\theta^{\pi/2} \frac{\rho\,d\psi}{\rho\sin\psi}\Big| = \big|\log\tan(\theta/2)\big|.$$

Thus, if we choose $\theta \in (0, \pi/2)$ such that $\tan(\theta/2) = e^{-R}$, then N_0 consists of the set $\big\{(\rho\,\cos\psi, \rho\,\sin\psi) : \rho \in [1, e^r], \psi \in (\theta, \pi - \theta)\big\}$. Thus,

$$\text{area}(N_0) = \int_\theta^{\pi-\theta}\int_1^{e^r} \frac{\rho\,d\rho\,d\psi}{\rho^2\sin^2\psi} = 2\,r\cot\theta = r(\cot\tfrac{\theta}{2} - \tan\tfrac{\theta}{2}) = 2\,r\sinh R\,.$$

The result follows. □

Remark 4.3. Let $\lambda_c(R)$ be the infimum of the set of intensities $\lambda \geq 0$ such that $\mathcal{B}$ contains unbounded components a.s. and let $\lambda_u = \lambda_u(R)$ be the infimum of the set of intensities $\lambda \geq 0$ such that $\mathcal{B}$ contains a unique unbounded component a.s. Proposition 4.7 in [14] says that $\lambda_c(R)/e^{2R} = O(1)$ as $R \to \infty$. Since obviously $\lambda_u(R) \geq \lambda_{\text{gv}}(R)$ we get by Theorem 4.1 in [14] that for R big enough, there are intensities for which there are lines in the vacant component, but also infinitely many unbounded components in both the covered and vacant regions. On the other hand, since $\lambda_c(R) \geq 1/(2\pi(\cosh(2R) - 1))$, it follows that for R small enough, we have $\lambda_c(R) > \lambda_{\text{gv}}(R)$. So for R small enough, there are no intensities for lines in the vacant region coexist with unbounded components in the covered region.

Lemma 4.4. *In the setting of line percolation in $\mathcal{B}$, α is the unique solution of the equation*

$$1 = \int_0^{2R} e^{\alpha t}\, H'_{R,\lambda}(t)\, dt\,, \tag{4.1}$$

where

$$H_{R,\lambda}(t) := -\exp\left(-4\,\lambda \int_0^{t/2} \sinh\left(\cosh^{-1}\left(\frac{\cosh R}{\cosh s}\right)\right) ds\right).$$

Proof. Consider a line $\gamma : \mathbb{R} \to \mathbb{H}^2$, parameterised by arclength. Recall that X is the underlying Poisson process. We now derive an integral equation satisfied by

$$f(r) = \mathbf{P}\big[\gamma[0,r] \subset \mathcal{B}\big].$$

For a point x in the R-neighborhood of γ, let $u_+(x) := \sup\{s : \gamma(s) \in B(x,R)\}$ and $u_-(x) := \inf\{s : \gamma(s) \in B(x,R)\}$. Let $X_0 := \{x \in X : u_-(x) < 0 < u_+(x)\}$. This is the set of $x \in X$ such that $\gamma(0) \in B(x,R)$. Also set

$$S := \begin{cases} \inf\{u_+(x) : x \in X_0\} & X_0 \neq \emptyset, \\ -\infty & X_0 = \emptyset\,. \end{cases}$$

Assume that $r \geq 2\,R$. A.s., if $S = -\infty$, then $\gamma[0,r]$ is not contained in $\mathcal{B}$. On the other hand, if we condition on $S = s$, where $s \in (0, 2R)$ is fixed, then $\gamma[0,s) \subset \mathcal{B}$ and the conditional distribution of $\gamma[s,r] \cap \mathcal{B}$ is the same as the unconditional distribution. (Of course, $S = s$ has probability zero, and so this conditioning should be understood as a limit.) Therefore, we get

$$\mathbf{P}\big[\gamma[0,r] \subset \mathcal{B} \bigm| S\big] = f(r - S), \tag{4.2}$$

where, of course, $f(\infty) = 0$.

Let $G(t) := \mathbf{P}\big[S \in (0,t)\big]$. Shortly, we will show that $G(t) = H_{R,\lambda}(t) + 1$. But presently, we just assume that $G'(t)$ is continuous and derive (4.1) with G in place of H. Since the probability density for S in $(0, 2\,R)$ is given by $G'(t)$, we get from (4.2)

$$f(r) = \int_0^{2R} f(r-s)\, G'(s)\, ds\,. \tag{4.3}$$

Suppose that $\beta > 0$ satisfies

$$1 = \int_0^{2R} e^{\beta s}\, G'(s)\, ds\,. \tag{4.4}$$

Since $\int_0^{2R} G'(s)\, ds = \mathbf{P}\big[S > 0\big] < 1$, continuity implies that there is some such β. Suppose that there is some $r > 0$ such that $f(r) \leq e^{-\beta r} f(2R)$, then let r_0 be the infimum of all such r. Clearly, $r_0 \geq 2\,R$. By the definition of r_0 and (4.3), we get

$$f(r_0) > \int_0^{2R} e^{-\beta(r_0-s)}\, f(2R)\, G'(s)\, ds \overset{(4.4)}{=} e^{-\beta r_0}\, f(2\,R)\,.$$

Since $f(r)$ is continuous on $(0,\infty)$, this contradicts the definition of r_0. A similar contradiction is obtained if one assumes that there is some $r > 0$ satisfying $f(r) \geq e^{-\beta(r-2R)}$. Hence $e^{-\beta r}\, f(2R) \leq f(r) \leq e^{-\beta(r-2R)}$, which gives $\alpha = \beta$.

It remains to prove that $G(t) = H_{R,\lambda}(t) + 1$. Let $Q_t := B\big(\gamma(0), R\big) \setminus B\big(\gamma(t), R\big)$. Observe that

$$G(t) = \mathbf{P}\big[X \cap Q_t \neq \emptyset\big] = 1 - \mathbf{P}\big[X \cap Q_t = \emptyset\big] = 1 - e^{-\lambda\ \mathrm{area}(Q_t)}. \tag{4.5}$$

Hence, we want to calculate area(Q_t). For $z \in \mathbb{H}^2$ let $u(z)$ denote the $t \in \mathbb{R}$ that minimizes $d\big(z, \gamma(t)\big)$, and let $\phi(t, y)$ denote the point in $\mathbb{H}^2$ satisfying $u(z) = t$ which is at distance y to the left of γ if $y \geq 0$, or $-y$ to the right of γ otherwise. Observe that $\big\{z \in B\big(\gamma(0), R\big) : u(z) < -t/2\big\}$ is isometric to (see Figure 4.1)

$$\big\{z \in B\big(\gamma(t), R\big) : u(z) < t/2\big\} = \big\{z \in B\big(\gamma(0), R\big) : u(z) < t/2\big\} \setminus \overline{Q_t}\,.$$

Therefore,

$$\mathrm{area}(Q_t) = \mathrm{area}\big\{z \in B\big(\gamma(0), R\big) : u(z) \in [-t/2, t/2]\big\}. \tag{4.6}$$

By the hyperbolic Pythagorian theorem, we have

$$\cosh d\big(\gamma(0), \phi(s, y)\big) = \cosh s\, \cosh y\,.$$

Hence, the set on the right hand side of (4.6) is

$$\big\{\phi(s, y) : s \in [-t/2, t/2], \cosh y \leq \cosh R / \cosh s\big\}. \tag{4.7}$$

At the end of the proof of Lemma 4.2, we saw that the area of a set of the form $\big\{\phi(s, y) : s \in [0, r], |y| \leq R\big\}$ is $2\,r\, \sinh R$. Hence, the area of (4.7) (and also the area of Q_t) is given by

$$\int_{-t/2}^{t/2} 2\, \sinh\big(\cosh^{-1}(\cosh R / \cosh s)\big)\, ds\,.$$

The result follows by (4.4) and (4.5), since $\alpha = \beta$. □

4.1 Circle covering approach

In this section we give a different and more basic proof of the fact that $\bar{\lambda}_{gv} = 1/(2 \sinh R)$, which does not use any of the results that were needed to prove Theorem 1.2. We will use methods from the theory of circle covering, so we first give some preliminaries about this.

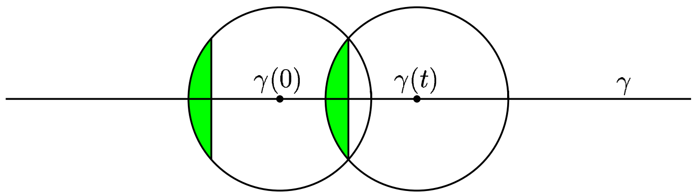

Figure 4.1: Calculating the area of Q_t. The set Q_t is the left ball minus the right ball. The area is calculated by first exchanging the left cap by its "shift".

Let C be a circle with circumference 1 and let $(l_n)_{n\geq 1}$ be a decreasing sequence of positive numbers approaching 0 as $n \to \infty$. Let I_n be the open interval of length l_n centered at centered at a point chosen uniformly at random from C independently of all other intervals. Let $E := \limsup_n I_n$ be the set of points on C which are covered by infinitely many intervals, and let $F := E^c$. By the Borel-Cantelli lemma, $\mathbf{P}[x \in E] = 1$ for any $x \in C$ if and only if $\sum_{n=1}^{\infty} l_n = \infty$, and in this case the Lebesque measure of F is a.s. 0. Shepp [13] proved

Theorem 4.5. $\mathbf{P}[F = \emptyset] = 1$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{l_1 + \ldots + l_n} = \infty.$$

In particular if $l_n = c/n$ *for all* n*, then* $\mathbf{P}[F = \emptyset] = 1$ *if and only if* $c \geq 1$.

Dynamical versions of this model have been studied in [8] and [7]. In one such model, one associates to each interval an independent jump process which is a Poisson process with intensity 1. At each jump time of the Poisson process associated to the nth interval, I_n is given a new center chosen uniformly on C. For every t, let F_t be the set of points on C that are not covered by infinitely many intervals at time t. One then wonders if there are times for which $F_t \neq \emptyset$ even if the condition in Theorem 4.5 holds. Let $u_n = \prod_{i=1}^{n}(1 - l_i)$ be the probability that a point is not covered by any of the n first intervals.

The following theorem from [7] will be of use to us.

Theorem 4.6. *If* $\limsup_n nu_n < \infty$*, then* $\mathbf{P}[\exists t \geq 0 \,:\, F_t \neq \emptyset] = 0$.

In particular, Theorem 4.6 covers the case $l_n = 1/n$. In fact, the proof of Theorem 4.6 does not use the fact that the new center of an interval is chosen uniformly at random from C, only that the process studied is an reversible Markov chain.

Circle covering proof of $\bar{\lambda}_{\mathrm{gv}}(R) = 1/(2\sinh R)$. Let X_n be the n:th closest Poisson point from the origin, let $d_n := d(o, X_n)$ and let $V_n := \mathrm{area}(B(o, d_n))$.

Then $\{V_n - V_{n-1}\}_{n=1}^{\infty}$ is a sequence of independent exponential random variables with parameter λ (here $V_0 = 0$). The relation between V_n and d_n is given by

$$d_n = \cosh^{-1}\left(\frac{V_n}{2\pi} + 1\right) \tag{4.8}$$

Now let $c_R := \tanh(R/2)$ and consider the ball of radius R centered at $X_{(n)}$. Such a ball is also an Euclidean ball. The Euclidean distance from the origin to its Euclidean center is given by

$$d_n^e := \frac{(1-c_R^2)\tanh(d_n/2)}{1-c_R^2\tanh(d_n/2)^2} \tag{4.9}$$

and its Euclidean radius is given by

$$R_n^e := \frac{c_R(1-\tanh(d_n/2)^2)}{1-c_R^2\tanh(d_n/2)^2}. \tag{4.10}$$

A derivation of formulas (4.10) and (4.9) can be found in [12]. Let θ_n denote the angle between the two geodesics starting at the origin and touching the closed ball of radius R centered at X_n at exactly one point (if the origin is covered by this ball, we let $\theta_n = 2\pi$). Then

$$\theta_n = 2\arcsin(R_n^e/d_n^e) = 2\arcsin\left(\frac{\sinh(R)}{\sinh(d_n)}\right) \tag{4.11}$$

where the second equality follows from elementary calculations.

Let $l_n = \theta_n/(2\pi)$ be the proportion of the ideal boundary $\partial\mathbb{H}^2$ that can not be reached by a geodesic starting from the origin and not intersecting $B(X_{(n)}, R)$. Inserting 4.8 in 4.11 and performing a long simplification gives that l_n can be expressed in terms of V_n as

$$l_n = \frac{1}{\pi}\arcsin\left(\frac{2\pi\sinh(R)}{\sqrt{V_n^2+4\pi V_n}}\right). \tag{4.12}$$

Fix $\epsilon \in (0,1/2)$. Since V_n is a sum of independent exponential random variables with parameter λ it follows that $\mathbf{P}[|V_n - n/\lambda| > n^{1/2+\epsilon} \text{ i.o.}] = 0$ which implies

$$\mathbf{P}[|1/V_n - \lambda/n| > n^{-3/2+\epsilon} \text{ i.o.}] = 0. \tag{4.13}$$

Since

$$\arcsin(2\pi\sinh R/\sqrt{x^2+4\pi x})/\pi = 2\sinh R/x + O(1/x^2)$$

as $x \to \infty$ it follows from 4.12 and 4.13 that the limit

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} l_i - 2\lambda \sinh R \log n\right) \tag{4.14}$$

a.s. exists.

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{l_1+\ldots+l_n} < \infty \text{ a.s.}$$

if $\lambda < 1/(2\sinh R)$ and the sum is infinite a.s. if $\lambda \geq 1/(2\sinh R)$. Recall that the ideal boundary $\partial\mathbb{H}^2$ can be identified with a circle of radius 1. By Theorem 4.5, there are a.s. half-lines starting at the origin that intersect only finitely many balls if and only if $\lambda < 1/(2\sinh R)$. Therefore, the probability that $\mathcal{W}$ contains half-lines starting at the origin is positive if and only if $\lambda < 1/(2\sinh R)$. □

5 Dynamical versions

In this section, we consider two dynamical models based on the stationary Poisson Boolean model.

In the *space-time Poisson model*, let X be a Poisson process with intensity λ in $\mathbb{H}^2 \times \mathbb{R}$. Then let $\mathcal{B}^\dagger := \bigcup_{x\in X} \overline{B(x,R)}$, and for $t \in \mathbb{R}$ let

$$\mathcal{B}_t := \{x \in \mathbb{H}^2 \,:\, (x,t) \in \mathcal{B}^\dagger\}.$$

Here, $B(x,R)$ is the open ball of radius R centered at x in $\mathbb{H}^2 \times \mathbb{R}$. Also let $\mathcal{W}_t := \overline{\mathbb{H}^2 \setminus \mathcal{B}_t}$. The parameter t is interpreted as time, and $\mathcal{B}_t$ is the covered set at time t and $\mathcal{W}_t$ is the vacant set at time t. In this model, as time increases, balls appear in $\mathbb{H}^2$ as points, grow to radius R, and then shrink to a point before they disappear.

Observe that both $\mathcal{B}_t$ and $\mathcal{W}_t$ are well-behaved percolations on $\mathbb{H}^2$, and their laws are invariant in t. Let λ_{rgv} be the infimum of the set of $\lambda \geq 0$ such that $\mathcal{W}_t$ a.s. contains hyperbolic lines. Then, as in the proof of Theorem 1.2, we get that $0 < \lambda_{rgv} < \infty$. Let G_t be the set of lines contained in $\mathcal{W}_t$. If $\lambda < \lambda_{rgv}$, we say that t is an *exceptional time* for $\mathcal{W}_t$ if G_t is empty, and if $\lambda \geq \lambda_{rgv}$ we say that t is exceptional if G_t is non-empty. Note that the Lebesgue measure of the set of exceptional times is a.s. 0.

Proposition 5.1. *In the space-time Poisson model, the set of exceptional times for $\mathcal{W}_t$ is empty a.s. for $\lambda > \lambda_{rgv}$ and $\lambda < \lambda_{rgv}$. The analogous statement hold for $\mathcal{B}_t$.*

Proof. Suppose $\lambda < \lambda_{rgv}$. For $t \geq 0$ let $\tilde{\mathcal{B}}_t := \bigcup_{0\leq s\leq t} \mathcal{B}_s$ and $\tilde{\mathcal{W}}_t := \overline{\mathbb{H}^2 \setminus \tilde{\mathcal{B}}_t}$, and note that $\tilde{\mathcal{W}}_t$ is a well-behaved percolation for every t. Put $\alpha_t := \alpha(\tilde{\mathcal{W}}_t)$. Lemma 3.3 implies that α_t is continuous for $t \in [0,\infty)$. In particular, $\alpha_t \searrow \alpha_0$ as

$t \searrow 0$. Since $\lambda < \lambda_{rgv}$ we have $\alpha_0 < 1$, so that for t_0 small enough, we also have $\alpha_{t_0} < 1$. By Lemma 3.5, $\tilde{\mathcal{W}}_{t_0}$ contains lines a.s. If $\tilde{\mathcal{W}}_{t_0}$ contains hyperbolic lines, then $\mathcal{W}_s$ contains lines for every $s \in [0, t_0]$. Therefore, by countable additivity, a.s. there are no exceptional times for which $\mathcal{W}_t$ does not contain hyperbolic lines.

The proof for the case $\lambda > \lambda_{rgv}$ is similar. One just redefines $\tilde{\mathcal{B}}_t := \bigcap_{0 \le s \le t} \mathcal{B}_s$ and observes that $\alpha_{t_0} > 1$ for t_0 small enough.

The proof for $\mathcal{B}_t$ is similar to the above, so we omit it. □

We do not know what happens at the critical intensity $\lambda = \lambda_{rgv}$, but conjecture that there are no exceptional times in this case as well.

Next, we consider a different model. To each ball in the Poisson Boolean model, we associate a jump process which is an independent Poisson process with intensity 1. At each jump time in the Poisson process, the center of the corresponding ball moves to a point chosen uniformly at random within distance 1 from the original center. Let F_t be the set of half-lines from the origin to infinity that intersect finitely many balls at time t.

Theorem 5.2. *If* $\lambda \ge \lambda_{\mathrm{gv}}$ *then* $\mathbf{P}[\exists t \ge 0 : F_t \ne \emptyset] = 0$. *If* $\lambda < \lambda_{\mathrm{gv}}$ *then* $\mathbf{P}[\exists t \ge 0 : F_t = \emptyset] = 0$.

Proof. First we remark that we cannot use the same technique for this model as in the proof of Proposition 5.1, since the part of $\mathbb{H}^2$ that has been covered by some ball at some time $s \in [0, t]$ is not a well-behaved percolation for any t. More precisely, for any $t > 0$, there is not independence at any distance R_0.

First consider the case $\lambda \ge \lambda_{\mathrm{gv}}$. Since the process is a reversible Markov chain, it follows by Theorem 4.6 that it is enough to check that

$$\limsup_n n u_n < \infty \text{ a.s.} \tag{5.1}$$

where $u_n = \prod_{i=1}^n (1 - l_n)$. Since $\lambda \ge \lambda_{\mathrm{gv}}$ it follows by (4.13) that there is a constant $c_\lambda \ge 1$ such that the event $J := \{|l_n - c_\lambda/n| \ge 1/n^{4/3} \text{ i.o. }\}$ has probability 0. On J^c, we get by straightforward calculations that $\limsup_n n u_n < \infty$ and therefore $\limsup_n n u_n < \infty$ a.s.

Next suppose that $\lambda < \lambda_{\mathrm{gv}}$. Let $Z_n(t)$ be the Poisson process associated to the n:th closest ball of the origin. Let $X_n(t)$ be the center at time t of the ball which at time 0 was the n:th closest ball of the origin. Let $B(t) := \cup_{n=1}^{\infty} B(X_n(t), R)$. Then let $\tilde{B}(t) := \cup_{n=1}^{\infty} B(X_n, R + Z_n(t))$. Then for each $t \ge 0$ we have

$$\cup_{s \le t} B(s) \subset \tilde{B}(t). \tag{5.2}$$

We will show that it is possible to find t so small that with positive probability, there are half-lines emanating from the origin that do not intersect $\tilde{B}(t)$. By (5.2), it then follows that with positive probability, there are halflines from the origin that do not intersect any ball during the interval $[0, t]$, which implies that a.s. $F_s \neq \emptyset$ for all $s \in [0,t]$. Countable additivity then implies that a.s. $F_t \neq \emptyset$ for all $t \in [0,\infty)$.

Let

$$l_n = l_n(t) := \frac{1}{\pi} \arcsin\left(\frac{2\pi \sinh(R + Z_n(t))}{\sqrt{V_n^2 + 4\pi V_n}}\right)$$

if $B(X_n, R + Z_n(t))$ does not contain the origin, and let $l_n(t) = 1$ otherwise. Then l_n is the fraction of the ideal boundary $\partial\mathbb{H}^2$ that can not be reached by a half-line starting from the origin not intersecting $B(X_n, R + Z_n(t))$.

Next pick a larger than but close enough to 1 so that $a\lambda < \lambda_{\text{gv}}$. By (4.13), it follows that the event

$$E := \left\{ \frac{2\pi \sinh(R)}{\sqrt{V_n^2 + 4\pi V_n}} \leq \frac{2\pi a\lambda \sinh(R)}{n + 2\pi a\lambda \sinh(R)} \text{ for all } n \right\}$$

has positive probability. Note that on E, we have $l_n(0) \leq 1/2$ for all $n \geq 1$. If we let

$$\hat{l}_n = \hat{l}_n(t) := \frac{1}{\pi} \arcsin\left(\frac{2\pi a\lambda \sinh(R + Z_n(t))}{n + 2\pi a\lambda \sinh(R)}\right)$$

if the argument in the arcsine function is less than one, and $\hat{l}_n(t) = 1$ otherwise, then

$$\{\hat{l}_n(t) \geq l_n(t) \text{ for all } n\}$$

occurs on E for all $t \geq 0$. Note that $\hat{l}_n(t) \leq 1/2$ if $\hat{l}_n(t) \neq 1$ and that there is positive probability that for fixed t,

$$E_t := \{\hat{l}_n(t) \leq 1/2 \text{ for all } n\}$$

occurs. Write $\mathbf{P}_Z$ for the conditional probability given the random variables $(Z_n(t))_{n=1}^{\infty}$.

Now consider the circle covering model, where interval I_n has length $l_n(t)$. Let μ denote Lebesgue measure on C. Let $U_n = U_n(t) := \{x \in C \,:\, x \in \cap_{k=1}^n I_k^c\}$ and $U := \cap_{n=1}^{\infty} U_n$. Define $\hat{U}_n = \hat{U}_n(t)$ in the same way as U_n but with intervals of length $\hat{l}_n(t)$ instead. Then $\mathbf{P}[F \neq \emptyset] = 1$ if and only if

$$\mathbf{P}[U \neq \emptyset] = \lim_{n\to\infty} \mathbf{P}[U_n \neq \emptyset] = \lim_{n\to\infty} \mathbf{P}[\mu(U_n) > 0] > 0.$$

We have

$$\mathbf{P}[\mu(U_n) > 0] \geq \mathbf{P}[\mu(U_n) > 0|E]\mathbf{P}[E].$$

Since $l_n \leq \hat{l}_n$ for all n on E, we get (since $\hat{l}_n$ is independent of E for all n)

$$\mathbf{P}[\mu(U_n) > 0|E] \geq \mathbf{P}[\mu(\hat{U}_n) > 0] = \mathbf{E}[\mathbf{P}_Z(\mu(\hat{U}_n) > 0)] \geq$$
$$\mathbf{E}[\mathbf{P}_Z(\mu(\hat{U}_n) > 0)|E_t]\mathbf{P}[E_t].$$

By the Paley-Zygmund inequality we get

$$\mathbf{P}_Z[\mu(\hat{U}_n) > 0] \geq \frac{\mathbf{E}_Z[\mu(\hat{U}_n)]^2}{\mathbf{E}_Z[\mu(\hat{U}_n)^2]}.$$

Let $A_x = A_x(t) := \{x \in \hat{U}_n(t)\}$. By Fubinis theorem, $\mathbf{E}_Z[\mu(\hat{U}_n)]^2 = \mathbf{P}_Z[A_x]^2$ and

$$\mathbf{E}_Z[\mu(\hat{U}_n)^2] = \int_C \int_C P_Z[A_x \cap A_y]dx\,dy \leq 2\int_C P_Z[A_x \cap A_0]dx.$$

Since $\hat{l}_n \leq 1/2$ for all $n \geq 1$ on E_t we get

$$\mathbf{P}_Z[A_x \cap A_0] = \prod_{k=1}^{n}(1 - 2\hat{l}_k + (\hat{l}_k - x)^+) \leq \prod_{k=1}^{n}(1 - \hat{l}_k)^2(1 + (\hat{l}_k - x)^+)(1 + 5\hat{l}_k^2)$$

$$= \mathbf{P}_Z[A_x]^2 \prod_{k=1}^{n}(1 + (\hat{l}_k - x)^+)(1 + 5\hat{l}_k^2)$$

on E_t.

Thus we get

$$\mathbf{P}[\mu(\hat{U}_n) > 0] \geq \mathbf{E}\left[\frac{1}{2\int_0^1 \prod_{k=1}^n (1 + (\hat{l}_k - x)^+)(1 + 5\hat{l}_k^2)dx}|E_t\right]\mathbf{P}[E_t]$$

$$\geq \frac{\mathbf{P}[E_t]}{2\mathbf{E}[\int_0^1 \prod_{k=1}^n (1 + (\hat{l}_k - x)^+)(1 + 5\hat{l}_k^2)dx|E_t]}$$

where the second inequality follows from the conditional version of Jensen's inequality. It now remains to show that we can find t so small that the expectation in the denominator does not tend to infinity with n.

The conditional distribution of $\hat{l}_k$ given E_t is stochastically dominated by its unconditional distribution. So, using $\mathbf{E}[\sum_{k=1}^\infty \hat{l}_k^2] < \infty$ and independence we get

$$\mathbf{E}\left[\int_0^1 \prod_{k=1}^{n}(1 + (\hat{l}_k - x)^+)(1 + 5\hat{l}_k^2)dx|E_t\right] \leq C_1 \int_0^1 e^{\sum_{k=1}^n \mathbf{E}[(\hat{l}_k - x)^+]}dx$$

Next let

$$\tilde{l}_k = \tilde{l}_k(t) := \frac{2a\lambda\sinh(R + Z_k(t))}{k + 2\pi a\lambda\sinh(R)}.$$

It is easy to see that $\mathbf{E}[\sum_{k=1}^\infty |\hat{l}_k - \tilde{l}_k|] < \infty$. So for any $x \in [0,1]$ we have

$$\sum_{k=1}^{n}\mathbf{E}[(\hat{l}_k - x)^+] \leq \sum_{k=1}^{n}\mathbf{E}[(\tilde{l}_k - x)^+] + C_2$$

for some constant C_2 independent of x and n. Furthermore for $x \in [0,1]$ we get that

$$\begin{aligned}\sum_{k=1}^{n} \mathbf{E}[(\tilde{l}_k(t) - x)^+] &= e^{-t} \sum_{k=1}^{n} \sum_{j=0}^{\infty} \frac{t^j}{j!} \left(\frac{2a\lambda \sinh(R+j)}{k + 2\pi a\lambda \sinh(R)} - x \right)^+ \\ &\leq \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{k=1}^{\lfloor 2a\lambda \sinh(R+j)/x \rfloor} \frac{2a\lambda \sinh(R+j)}{k} \\ &= C_3 + \sum_{j=0}^{\infty} \frac{t^j}{j!} 2a\lambda \sinh(R+j) \log(2a\lambda \sinh(R+j)/x) \\ &= C_4 - 2a\lambda \left(\frac{e^{R+te} - e^{-R+te^{-1}}}{2} \right) \log(x)\end{aligned}$$

Let $g(R,t) := (\exp(R+te) - \exp(-R+te^{-1}))/2$. Since $a\lambda < \lambda_{\mathrm{gv}}$, we can pick t_0 small enough so that $2a\lambda g(R,t_0) < 1$. Putting this together we get that

$$\int_0^1 e^{\sum_{k=1}^{n} \mathbf{E}[(\hat{l}_k(t_0)-x)^+]} dx \leq C_5 \int_0^1 x^{-2a\lambda g(R,t_0)} dx < \infty.$$

This implies the theorem. □

6 No planes in higher dimensions

It is natural to ask for high dimensional variants. Fix some $d \in \mathbb{N}$, $d > 2$. Let $\lambda, R > 0$. Let $\mathcal{B} := \bigcup_{x \in X} B(x,R)$, where X is a Poisson point process of intensity λ in $\mathbb{H}^d$. Let $\mathcal{W}$ be the closure of $\mathbb{H}^d \setminus \mathcal{B}$.

Proposition 6.1. *For every $d \in \mathbb{N} \cap [3,\infty)$, $\lambda, R > 0$, a.s. there are no 2-dimensional planes in $\mathbb{H}^d$ that are contained in $\mathcal{B}$. Similarly, there are no 2-dimensional planes in $\mathbb{H}^d$ that are contained in $\mathcal{W}$.*

Proof. Let $\mathcal{Z}$ be $\mathcal{W}$ or $\mathcal{B}$. Fix some $o \in \mathbb{H}^d$, and let $r > 0$ be large. Let Y_r be the set of planes L intersecting the ball $B(o,2)$ such that $L \cap B(o,r)$ is contained in the 1-neighborhood of $\mathcal{Z}$. If there is a plane L intersecting $B(o,1)$ such that $L \cap B(o,r) \subset \mathcal{Z}$, then Y_r contains the set of planes L' such that the Hausdorff distance between $L \cap B(o,r)$ and $L' \cap B(o,r)$ is less than 1. It therefore follows that if there is such an L, then the measure of Y_r (with respect to the invariant measure on the Grassmannian) is at least $\exp\bigl(-O(r)\bigr)$. However, if we fix a plane L that intersects $B(o,2)$, then $\mathbf{P}\bigl[L \in Y_r\bigr] \leq \exp(-c\,e^r)$ for some $c = c(d,R,\lambda) > 0$, because there are order e^r points in $L \cap B(o,r)$ such that the distance between any two is larger than $R+3$. This means that the expected measure of Y_r is at most $\exp(-c\,e^r)$. Consequently, the probability that there is some plane L intersecting $B(o,1)$ such that $B(o,r) \cap L \subset \mathcal{Z}$ goes to zero as $r \to \infty$. □

7 Further Problems

We first consider quantitative aspects of Theorem 1.1.

Conjecture 7.1. *Fix some $o \in \mathbb{H}^2$. For every $r > 0$ let p_r be the least $p \in [0,1]$ such that for every random closed $\mathcal{Z} \subset \mathbb{H}^2$ with an isometry-invariant law and $\mathbf{P}\big[B(o,r) \subset \mathcal{Z}\big] > p$ there is positive probability that $\mathcal{Z}$ contains a hyperbolic line. Theorem 1.1 implies that $p_r < 1$ for every $r > 0$. We conjecture that $\limsup_{r\searrow 0}(1-p_r)/r < \infty$.*

It is easy to see that $\liminf_{r\searrow 0}(1-p_r)/r > 0$; for example, take a Poisson point process $X \subset \mathbb{H}^2$ with intensity λ sufficiently large and let $\mathcal{Z}$ be the complement of the ϵ-neighborhood of $\bigcup_{x\in X} \partial B(x,1)$, where $0 < \epsilon < r$.

Problem 7.2. *What is $\lim_{r\searrow 0}(1-p_r)/r$?*

The behaviour of p_r as $r \to \infty$ seems to be an easier problem, though potentially of some interest as well. We now move on to problems related to Theorem 1.2 and its proof.

Question 7.3. For either $\mathcal{W}$ or $\mathcal{B}$, is there some pair (λ, R) for which there is with positive probability a percolating ray such that every other percolating ray with the same endpoint at infinity is contained in it? (Note, such a ray must be exceptional among the percolating rays.)

Question 7.4. Is it true that whenever $\mathcal{Z}$ has an unique infinite connected component, the union of the lines in $\mathcal{Z}$ is connected as well?

Question 7.5. For which homogenous spaces $\mathcal{W}$ or $\mathcal{B}$ a.s. contain infinite geodesics for some parameters (λ, R)?

Note that since $\mathbb{H}^2 \times \mathbb{R}$ contains $\mathbb{H}^2$, it follows that for every R there is some λ such that $\mathcal{W}$ on $\mathbb{H}^2 \times \mathbb{R}$ contains lines within an $\mathbb{H}^2$ slice, and the same holds for $\mathcal{B}$.

Question 7.6. Let V be the orbit of a point $x \in \mathbb{H}^2$ under a group of isometries Γ. Suppose that V is discrete and $\mathbb{H}^2/\Gamma$ is compact. (E.g., V is a co-compact lattice in $\mathbb{H}^2$.) Let $\mathcal{W}_V(R) := \mathbb{H}^2 \setminus \bigcup_{v\in V} B(v,R)$, and let R_c^V denote the supremum of the set of R such that $\mathcal{W}_V(R)$ contains uncountably many lines. Does $\mathcal{W}_V(R_c^V)$ contain uncountably many lines?

Problem 7.7. *It is not difficult to adapt our proof to show that in $\mathbb{H}^d$, $d \geq 2$, for every $R > 0$ when λ is critical for the existence of lines in $\mathcal{W}$, there are a.s. no lines inside $\mathcal{W}$. This should also be true for $\mathcal{B}$, but we presently do not know a proof. It seems that what is missing is an analog of Lemma 3.2.*

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