THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Some Markov Processes in Finance and Kinetics

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Abstract

This thesis consists of four papers. The first two papers treat extremes for Lévy processes, while papers three and four treat the Kac model with unbounded collision kernel.

The Lévy process papers relate the distribution of the supremum of a Lévy process over a compact time interval to the distribution of the process value at the right endpoint of this interval. Lévy processes are sorted into different classes depending on the tails of their univariate marginal distributions. In the first paper we treat processes with heavier tails, while processes with lighter tails are handled in the second paper. Our results are applicable to many processes recently introduced in mathematical finance. For instance, they may be used to approximate the distribution of the maximum of a stock price over a finite time span.

The papers on the Kac model mainly deal with an approximation of the Kac model with unbounded collsion kernel where small jumps are replaced by a Brownian motion. In the first and more theoretical of these papers we prove convergence of the approximating processes to the process with unbounded collision kernel. We also give results on the spectral gap of the Kac model with unbounded collision kernel. In the second paper on the Kac model we present numerical results which show that our approximation scheme gives a considerable improvement of the standard approximation which uses only a truncated collision kernel and that this improvement is more obvious as the collsion kernel gets more singular. Our numerical investigations are carried out for the Kac model with Gaussian thermostat as well as for a more physically relevant three-dimensional model.

Keywords: CGMY process; Collision kernel; Direct simulation Monte Carlo; Diffusion approximation; Extreme value theory; Feller process; Generalized hyperbolic process; Generalized z-process; Infinitesimal generator; Laplace-Beltrami operator; Lévy Processes; Long-tailed distribution; Kac equation; Kac model; Markov process; Semigroup; Semi-heavy tailed distirbution; Spectral gap; Subexponential distibrution; Superexponential distribution; Tauberian theorem; Thermostat.

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Göteborg, June 2008
Mattias Sundén

This thesis consists of the following papers:

Paper I: On the extremes of Lévy processes, Part I, subexponential and exponential processes, joint work with J.M.P. Albin. To appear in *Stochastic Processes and Their Applications.*

Paper II: On the extremes of Lévy processes, Part II, superexponential processes, joint work with J.M.P. Albin. Submitted to *Stochastic Processes and Their Applications.*

Paper III: The Kac master equation with unbounded collision rate, joint work with Bernt Wennberg. Submitted to *Markov Processes and Related Fields.*

Paper IV: Brownian approximation and Monte Carlo simulation of the non-cutoff Kac equation, joint work with Bernt Wennberg. *Journal of Statistical Physics,* **130** (2) 295-312.

Contents

1 Background

1.1 Markov processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space. A stochastic process $\{X_t\}_{t\geq 0}$ with state space $(\mathbb{S}, \mathcal{S})$, where $\mathcal{S}$ is the Borel σ-algebra over a metric space $\mathbb{S}$, is called a Markov process if for $s \leq t$ and $A \in \mathcal{S}$ it holds that

$$\mathbf{P}\{X_t \in A | \mathcal{F}_s\} = \mathbf{P}\{X_t \in A | X_s\} \text{ a.s.}$$

The intuition here is that our prediction of what happens tomorrow is as good knowing only what happened today as it is knowing the whole history.

All stochastic processes considered in this work are *time-homogeneous* Markov processes, for which the *semigroup* $\{T_t\}_{t\geq 0}$ is given by

$$T_t f(x) = \mathbf{E}\{f(X_{t+s}) | X_s = x\}$$

for $t > 0$ and $s \geq 0$ and $f \in B(\mathbb{S})$, the space of bounded measurable functions equipped with the norm $\|f\| = \sup_{x\in\mathbb{S}} |f(x)|$. Henceforth, by a Markov process we mean a time-homogeneous Markov process. The semigroup is determined by a linear operator $\mathcal{L}$, called the *infinitesimal generator*, with domain $\mathcal{D}(\mathcal{L})$ consisting of all functions $f \in B(\mathbb{S})$ such that the strong limit

$$\mathcal{L}f = \lim_{h\downarrow 0} \frac{T_h f - f}{h}$$

exists. It can be shown that the infinitesimal generator uniquely determines the finite-dimensional distributions of the Markov process. If $\mathcal{L}$ is closed (i.e., if the graph $\{(f, \mathcal{L}f); f \in \mathcal{D}(\mathcal{L})\}$ is a closed subspace of $B(\mathbb{S}) \times B(\mathbb{S})$) a subspace D of $\mathcal{D}(\mathcal{L})$ is a *core* for $\mathcal{L}$ if the closure of the restriction $\mathcal{L}|_D$ of $\mathcal{L}$ to D is $\mathcal{L}$. In this case $\mathcal{L}$ is uniquely determined by $\mathcal{L}|_D$.

1.2 Feller processes

Here we assume that the metric space $\mathbb{S}$ is locally compact and separable. Let $C_0 = C_0(\mathbb{S})$ be the Banach space of continuous functions vanishing at infinity equipped with the norm $\|f\| = \sup_{x\in\mathbb{S}} |f(x)|$. Under these assumptions a Markov process with semigroup $\{T_t\}_{t\geq 0}$ for which it holds that

$$T_t C_0 \subset C_0 \text{ for } t > 0 \text{ and } T_t f(x) \to f(x) \text{ as } t \downarrow 0 \text{ for } f \in C_0,\ x \in \mathbb{S},$$

is called a *Feller process* and its corresponding semigroup a *Feller semigroup.* All stochastic processes considrerd in this work are Feller processes.

For more on the theory of Markov and Feller processes we refer to the comprehensive textbooks of Ethier and Kurtz [7] and Kallenberg [9].

1.3 Lévy processes

Lévy processes are, simply put, stochastic processes with independent and stationary increments. The feature of independent increments means that Lévy processes constitute a subclass of Markov processes, the canonical examples being the Poisson process and Brownian motion. It can be shown (see e.g., Kallenberg [9], Theorem 19.10) that a

Lévy process is a Feller process.

A formal definition of an $\mathbb{R}$-valued Lévy process is as follows. The process $\xi = \{\xi_t\}_{t\geq 0}$ is a *Lévy process* if

(i) $\xi_0 = 0$,
(ii) $\xi_{t_n} - \xi_{t_{n-1}}, \ldots, \xi_{t_1} - \xi_{t_0}$ are independent for $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n < \infty$,
(iii) the distribution of $\xi_{t+s} - \xi_s$ does not depend on s for $s, t \geq 0$,
(iv) $\mathbf{P}\{|\xi_t - \xi_s| > \varepsilon\} \to 0$ for $\varepsilon > 0$ as $t \to s$ for $s \geq 0$,
(v) every sample path of ξ is right-continuous with left limits.

It is easy to see that the distribution of a Lévy process is uniquely determined by its distribution at $t = 1$. This distribution in turn is specified by the so called characteristic triplet (ν, m, s^2) through the Lévy-Kintchine formula

$$\mathbf{E}\{\mathrm{e}^{i\theta\xi(1)}\} = \exp\left\{i\theta m + \int_{\mathbb{R}} \left(\mathrm{e}^{i\theta x} - 1 - i\theta\kappa(x)\right) d\nu(x) - \frac{\theta^2 s^2}{2}\right\} \tag{1}$$

for $\theta \in \mathbb{R}$, where $\kappa(x) = x/\max\{1, |x|\}$. The quantities $m \in \mathbb{R}$ and $s^2 > 0$ are constants and the measure ν is the (Borel) Lévy measure on $\mathbb{R}$, satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{1, x^2\} d\nu(x) < \infty$. One may note that, discarding the integral with respect to the Lévy measure, equation (1) is just the characteristic function of a Gaussian random variable with mean m and variance s^2. For instance, this means that the standard Brownian motion has the characteristic triplet $(0, 0, 1)$. The Lévy measure describes the jump structure of the process. In many cases $\nu([-1, 1]) = \infty$, which corresponds to sample paths displaying many small jumps. The canonical example of a jump process is the Poisson process with rate λ, which has the characteristic triplet $(\lambda\delta_1, \lambda, 0)$, δ_1 being the unit mass at 1. For more on Lévy processes the reader is referred to Sato [11].

1.4 Extremes for Lévy processes

During the last decade, $\mathbb{R}$-valued Lévy processes have become popular in financial applications, where they are often used to model logarithmic stock returns. Obviously, it could be of interest to quantify the probability of the return process exceeding a very high or a very low level. For $\mathbb{R}$-valued Brownian motion $\{B_t\}_{t\geq 0}$ it is well known that

$$\mathbf{P}\left\{\sup_{t\in[0,h]} B_t > x\right\} = 2\mathbf{P}\{B_h > x\}$$

for $x > 0$. For more general Lévy processes no such formulas are known. However, letting $\mathcal{C}$ denote a suitable class of distributions, results such as

$$\sup_{t\in[0,h]} \xi_t \in \mathcal{C} \Leftrightarrow \xi_h \in \mathcal{C} \tag{2}$$

and

$$\xi_h \in \mathcal{C} \Rightarrow \lim_{x\to\infty} \frac{\mathbf{P}\left\{\sup_{t\in[0,h]} \xi_t > x\right\}}{\mathbf{P}\{\xi_h > x\}} = H \tag{3}$$

for some constant $H \geq 1$, are known to some extent. Letting $\mathcal{L}(\alpha)$ denote the class of distribution functions F for which

$$\lim_{u\to\infty} \frac{1 - F(x+u)}{1 - F(x)} = \mathrm{e}^{-\alpha x}$$

for some $\alpha \geq 0$ and $\mathcal{S}(\alpha)$ the subclass of $\mathcal{L}(\alpha)$ for which it also holds that

$$\lim_{u\to\infty} \frac{1 - F \star F(u)}{1 - F(u)} < \infty,$$

Braverman and Samorodnitsky [3] showed that (3) holds for $\mathcal{C} = \mathcal{S}(\alpha)$ and Willekens [13] proved that (2) and (3) hold for $\mathcal{C} = \mathcal{L} = \mathcal{L}(0)$.

1.5 The Kac model

The Kac model, introduced by Kac in 1956 [8], is a toy model or charicature of the Boltzmann equation. It consists of a vector $\mathbf{v} = (v_1, \ldots, v_N)$, which may be considered as the particle velocities of a monoatomic dilute gas. The velocity vector evolves under a random mechanism corresponding to collisions of the atoms. To each particle pair i, j, where $1 \leq i < j \leq N$, is assigned a Poisson clock. When the clock rings the velocities v_i and v_j undergo the transformation

$$v_i \mapsto v_i \cos\theta + v_j \sin\theta \quad \text{and} \quad v_j \mapsto v_j \cos\theta - v_i \sin\theta, \tag{4}$$

where θ is chosen in some random fashion. In Kac's original work [8] the angle θ is uniformly distributed over $[-\pi, \pi]$. Note that the mappings (4) keep the kinetic energy constant. We choose this constant to be N. This means that the Markov process at hand has state space $S^{N-1}(\sqrt{N})$, the sphere of radius $\sqrt{N}$ centered at the origin of $\mathbb{R}^N$. The generator of the process is given by

$$\mathcal{L}f = \frac{2}{N-1} \sum_{1\leq i<j\leq N} \int_{-\pi}^{\pi} \left[f(R_{ij}(\theta)\cdot) - f\right] \rho(\theta) d\theta, \tag{5}$$

where $R_{ij}(\theta)$ is the mapping (4) and ρ determines the distribution of the scattering angle θ. The integral on the right hand side of (5) is to be interpreted as a principal value and the domain $\mathcal{D}(\mathcal{L})$ depends on the choice of ρ.

Letting $\psi_t = \psi_t(\mathbf{v})$, denote the density of the vector $\mathbf{v}$ at time $t \geq 0$, ψ_t is a solution to

$$\frac{\partial}{\partial t} g = \mathcal{L} g,$$

which is also known as the Kac master equation. In the language of probability theory, this is the Kolmogorov backward equation.

An interesting feauture studied by Kac in [8] is that of *propagation of chaos* or the *Boltzmann property*. Intuitively this means that if the particle velocities are initially independent as the particle number N goes to infinity this property propagates in time so that particle velocities are asymptotically independent as $N \to \infty$ for all $t \geq 0$. This feature is not studied in our work and has to our knowledge not been verified to hold for the Kac model with unbounded collision kernels. Thus, verifying if propagation of chaos is a feature of our models may serve as future work.

2 Summary of the papers

2.1 Paper I

In the first paper, "On the asymptotics of Lévy processes, Part I: subexponential and exponential processes", we establish conditions for the existence of a constant H as in equation (3) and tail-behaviour for the very general process classes of generalized hyperbolic, CGMY and generalized z-processes. These classes include most of the processes used in mathematical finance such as NIG, VG and Meixner processes (see e.g., Schoutens [12] for more details). One feature of these processes is semi-heavy tails i.e., the processes have univariate marginal distributions with densities F' for which it holds that

$$F'(u) \sim Cu^{\rho}\mathrm{e}^{-\alpha u} \text{ as } u \to \infty$$

for some constants $C, \alpha > 0$ and ρ.

It turns out that semi-heavy tailed distributions belong to $\mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$ if $\rho \geq -1$ and to $\mathcal{S}(\alpha)$ if $\rho < -1$. Our main results show that for a Lévy process ξ such that the limit

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\xi_t > u\}}{\mathbf{P}\{\xi_h > u\}} \text{ exists for } t \in (0, h), \tag{6}$$

we have that

$$\xi_t \in \mathcal{L}(\alpha) \Leftrightarrow \sup_{t\in[0,h]} \xi_t \in \mathcal{L}(\alpha)$$

and that $H = 1$ if the limit (6) equals zero. This result relies on Tauberian theorems and an extension of Willekens [13] result for $\mathcal{L}$ to the class $\mathcal{OL}$ of distributions F for which

$$0 < \liminf_{u\to\infty} \frac{1 - F(x + u)}{1 - F(u)} \leq \limsup_{u\to\infty} \frac{1 - F(x + u)}{1 - F(u)} < \infty \text{ for } u \geq 0.$$

We also state and prove a result which says that if $H = 1$ and ξ is not a subordinator (an increasing Lévy process that is) then it either holds that

$$\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi_t > u\}}{\mathbf{P}\{\xi_h > u\}} = 0 \text{ for all } t \in (0, h)$$

or $\xi_h \in \mathcal{L}$. This is used to show that $H > 1$ for a non-subordinator ξ such that $\xi_t \in \mathcal{S}(\alpha)$.

2.2 Paper II

The second paper, "On the asymptotics of Lévy processes, Part II: superexponential processes", may be viewed as a continuation of the first paper. It deals with processes with tails lighter than the ones treated in Paper I.

We introduce the notion of *superexponential Lévy processes*, by which we mean Lévy processes ξ such that

$$\mathbf{E}\{\mathrm{e}^{\alpha\xi_1}\} < \infty \quad \text{for } \alpha \geq 0.$$

Also here Tauberian techniques are used to establish the existence of constants H as in (3) and to study tail behaviours. To us it seems that as tails get lighter proving Tauberian theorems becomes more technical.

The paper contains two main results. One gives the existence of a constant H as in (3). A condition for this first result is that the distribution of ξ_t belongs to the Type-I attraction of extremes. This means that, for some positive function w, it holds that

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\xi_t > u + xw(u)\}}{\mathbf{P}\{\xi_t > u\}} = \mathrm{e}^{-x}.$$

Criteria for the Type-I attraction and the determination of the tail behaviour of super-exponential Lévy processes is the other main result of paper II and it is proved under some technical condtitions. We provide a series of lemmas and propositions, some of which allow for verification of the technical conditions starting from the characteristic triplet of the process. Our results are applied to, e.g., Brownian motion with drift, Merton jump-diffusion and totally skewed to the left α-stable processes.

2.3 Paper III

Paper III, "The Kac master equation with unbounded collision rate", treats the Kac model with unbouded collision kernel

$$\rho(\theta) = |\theta|^{-\alpha-1} \text{ for } \alpha \in (0,2) \tag{7}$$

and approximations of this process. We introduce an approximation scheme for simulation of the Kac model with ρ as in (7) which involves a Brownian motion on the sphere $S^{N-1}(\sqrt{N})$. This Brownian approximation scheme was inspired by the works of Asmussen and Rosinski [1] and Cohen and Rosinski [6], where such schemes are presented for Lévy processes. The idea is to replace small jumps by a continuous process. We show that the convergence in terms of infinitesimal generators is faster for the process with Brownian approximation than the convergence of the process given by just truncating the collision kernel.

We also prove that the Kac model with ρ as in (7) is a Feller process and that Brownian motion on $S^{N-1}(\sqrt{N})$ can be obtained as the *grazing collision limit* of the Kac model.

Furthermore, we establish that the rotation inavariant probability measure on the sphere $S^{N-1}(\sqrt{N})$ is an invariant measure for all processes considered in this work. We give lower bounds for the spectral gap k of the generator $\mathcal{L}$. The constant k quantifies the speed of convergence to the distribution given by the invariant measure, here being the uniform distribution and thus having density 1, in the sense

$$\|T_t f - 1\|_2 \leq \mathrm{e}^{-kt}\|f - 1\|_2,$$

where $\|\cdot\|_2$ denotes the L^2-norm, T is the semigroup generated by $\mathcal{L}$ and f is some initial density. The inspiration for this part of our work was found in Carlen, Carvalho and Loss [4], where one of many results is an explicit calculation the spectral gap for the Kac model with $\rho(\theta) = (2\pi)^{-1}$. For more on spectral gaps, we refer to Chen [5].

We also present a diffusion approximation scheme for a physically more relevant model with three-dimensional velocities and constant momentum. The diffusion process used here is generated by the Balescu-Prigogine operator

$$\sum_{1\leq i<j\leq N} (\nabla_j - \nabla_i)|v_i - v_j|^2(\nabla_i - \nabla_j). \tag{8}$$

We give a stochastic differential equation representation for the process generated by (8) which facilitates simulation.

2.4 Paper IV

In the fourth paper, "Brownian approximation and Monte Carlo simulation of the non-cutoff Kac equation", we present numerical results of the approximation schemes presented in paper III. This involves *direct simulation Monte Carlo* (DSMC), which is a probabilistic scheme for solving kinetic equations introduced by Bird [2]. See also Rjasanow and Wagner [10].

The model that we simulate is not the Kac model as described in subsection 1.5, but rather the Kac model with Gaussian thermostat. This means that an extra term is added to the operator $\mathcal{L}$ as in (5) so that the generator of the process is given by

$$\tilde{\mathcal{L}}f = E\sum_{i=1}^{N}\left(1-\frac{J_N}{U_N}v_i\right)\frac{\partial}{\partial v_i}f + \mathcal{L}f \tag{9}$$

for some constant $E > 0$ and f in some space depending on the singularity of ρ and where

$$J_N = \frac{1}{N}\sum_{k=1}^{N} v_k \quad \text{and} \quad U_N = \frac{1}{N}\sum_{k=1}^{N} v_k^2.$$

The first term on the right hand side of (9) corresponds to an external uniform force field and a *Gaussian thermostat.* The force field accelerates the particles between collisions and the energy thus supplied to the system is absorbed by the thermostat, keeping the kinetic energy constant. For more on the Kac model with thermostat, the reader is referred to Wondmagegne [14]. We also make simulations of the three-dimensional model as in paper III but also here modified to incorporate a thermostat.

Our numerical results show that the diffusion approximation scheme is very effective, both in terms of computation time and accuracy, especially for values of α close to 2.

2.5 On the contribution of Mattias Sundén

Papers I and II are joint work of J.M.P. Albin and Mattias Sundén. Mattias and J.M.P. Albin have contributed equally to both papers.

Papers III and IV are joint work of Mattias Sundén and Bernt Wennberg. In paper III most results are due to Mattias, using guidelines and ideas from Bernt. Paper IV is a numerical implementation of some of the ideas from paper III.

References

[1] S. Asmussen and J. Rosinski (2001). Approximations of small jumps of Lévy processes with a view towards simulation. *J. Appl. Probab.* **38**, 482-493.

[2] G.A. Bird (1976). *Molecular Gas Dynamics.* Oxford University Press.

[3] M. Braverman and G. Samorodnitsky (1995). Functionals of infinitely divisible stochastic processes with exponential tails. *Stochastic Process Appl.* **56**, 207-231.

[4] E.A. Carlen, M.C. Carvalho and M. Loss (2003). Determination of the Spectral Gap for Kac's Master Equation and Related Stochastic Evolutions. *Acta Math.* **191**, 1-54.

[5] M-F. Chen (2005). *Eigenvalues, Inequalities, and Ergodic Theory.* Springer.

[6] S. Cohen and J. Rosinski (2007). Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes. *Bernoulli* **13**, 195-210.

[7] S.N. Ethier and T.G. Kurtz (1986). *Markov Processes, Characterization and Convergence.* Wiley.

[8] M. Kac (1956). *Foundations of Kinetic Theory. In Third Berkeley Symposium on Mathematical Statistics and Probability.* Edited by J. Neyman, 171-197.

[9] O. Kallenberg (2002). *Foundations of Modern Probability, Second Edition.* Springer.

[10] S. Rjasanow and W. Wagner (2005). *Stochastic Numerics for the Boltzmann Equation.* Springer.

[11] K. Sato (1999). *Lévy processes and infinitely divisible distributions.* Cambridge.

[12] W. Schoutens (2003). *Lévy processes in Finance.* Wiley.

[13] E. Willekens (1987). On the supremum of an infinitely divisible process. *Stochastic Process Appl.* **26** 173-175.

[14] Y. Wondmagegne (2005). Kinetic Equations with a Gaussian Thermostat, Doctoral Thesis, *Chalmers University of Technology and Göteborg University.*

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On the asymptotic behaviour of Lévy processes, Part I: Subexponential and exponential processes

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Abstract

We study tail probabilities of the suprema of Lévy processes with subexponential or exponential marginal distributions over compact intervals. Several of the processes for which the asymptotics are studied here for the first time have recently become important to model financial time series. Hence our results should be important, for example, in the assessment of financial risk.

MSC: primary 60E07; 60G51; 60G70; secondary 40E05; 44A10; 60F10

Keywords: CGMY process; Esscher transform; Exponential distribution; Extreme value theory; GH process; GZ process; Infinitely divisible distribution; Lévy process; Long-tailed distribution; Semi-heavy-tailed distribution; Subexponential distribution

1. Introduction

In the past decade there has been a great interest to use Lévy processes in mathematical finance, see, e.g., Schoutens [34] for a review. Most of the classes of Lévy processes that feature here, such as generalized z processes (GZ), CGMY processes and generalized hyperbolic processes (GH) have univariate marginal distributions with *semi-heavy tails*.

Recall that a probability distribution is said to have a semi-heavy (upper) tail if it has a probability density function (PDF) f such that

$$f(u) \sim Cu^{\rho}\mathrm{e}^{-\alpha u} \quad \text{as } u \to \infty \text{ for some constants } C, \alpha > 0 \text{ and } \rho \in \mathbb{R}. \tag{1.1}$$

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Owing to the exponential in (1.1) Lévy processes with semi-heavy tails could also be called *exponential processes*. This is a custom that we will adopt later.

We study the (upper) tail behaviour of suprema over compact intervals of Lévy processes $\{\xi(t)\}_{t\geq 0}$ with semi-heavy tails and other related behaviours of the tails of their marginal distributions. More specifically, given a constant $h > 0$ we prove that

$$\sup_{t\in[0,h]} \xi(t) \in \mathcal{C} \Leftrightarrow \xi(h) \in \mathcal{C} \tag{1.2}$$

for different classes of distributions (tail behaviours) $\mathcal{C}$ together with the implication

$$\xi(h) \in \mathcal{C} \Rightarrow \lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{0\leq t\leq h} \xi(t) > u\right\} = H \quad \text{exists.} \tag{1.3}$$

The classes of distributions $\mathcal{C}$ we are most interested in for (1.2) and (1.3) are the exponential classes $\mathcal{L}(\alpha)$ and $\mathcal{S}(\alpha)$ which are well-known from the literature, see Definition 2.5. In particular, $\mathcal{L}(\alpha)$ includes all distributions with semi-heavy tails.

The implication (1.3) is known from Braverman and Samorodnitsky [14] for $\mathcal{C} = \mathcal{S}(\alpha)$. We prove (1.2) and (1.3) for $\mathcal{C} = \mathcal{L}(\alpha)$ under an additional technical condition that always seems to be met in practice. In particular, as $\mathcal{L}(\alpha)$ includes $\mathcal{S}(\alpha)$ and distributions in $\mathcal{S}(\alpha)$ satisfy our technical condition, we complete the result of Braverman and Samorodnitsky [14] with the equivalency (1.2) for $\mathcal{C} = \mathcal{S}(\alpha)$.

It turns out that semi-heavy-tailed Lévy processes with $\rho < -1$ belong to $\mathcal{S}(\alpha)$ while processes with $\rho \geq -1$ belong to $\mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$. Many of the above-mentioned specific Lévy processes have $\rho \geq -1$, so that (1.2) and (1.3) are established here for them for the first time. We also show that $H > 1$ when $\rho < -1$ unless ξ is a subordinator (a result which does not follow from Braverman and Samorodnitsky [14]) and that $H = 1$ for $\rho \geq -1$.

Distributions in the class $\mathcal{L} \equiv \mathcal{L}(0)$ are called *long-tailed*. As these distributions have tails that decay slower than any exponential they can also be called *subexponential*. The relations (1.2) and (1.3) are known from Willekens [39] for $\mathcal{C} = \mathcal{L}$, see Theorem 4.1. We complete his results by establishing a partial converse to (1.3) for $\mathcal{C} = \mathcal{L}$.

As there is no converse to (1.3) for $\mathcal{C} = \mathcal{L}(\alpha)$ when $\alpha > 0$ we consider the larger $\mathcal{O}$-exponential class $\mathcal{OL}$ of Bengtsson [8] and Shimura and Watanabe [37], see Definition 2.5. For $\mathcal{C} = \mathcal{OL}$ we prove (1.2) together with the following version of (1.3):

$$\xi(h) \in \mathcal{OL} \Rightarrow \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{0\leq t\leq h} \xi(t) > u\right\} < \infty. \tag{1.4}$$

In addition, we establish a partial converse to (1.4).

From a practical point of view the implication (1.3) for $\mathcal{C} = \mathcal{L}(\alpha)$ should be the most interesting of our results. For example, an asset price process $\{S(t)\}_{0\leq t\leq h}$ such as a stock price is often modelled by an exponential Lévy model $S(t) = \mathrm{e}^{\xi(t)}$ where ξ is a Lévy process. Then the risk that S falls below a low level ε is given by

$$\mathbf{P}\left\{\inf_{0\leq t\leq h} S(t) < \varepsilon\right\} = \mathbf{P}\left\{\sup_{0\leq t\leq h} -\xi(t) > -\ln(\varepsilon)\right\} \sim H\,\mathbf{P}\{-\xi(h) > \ln(1/\varepsilon)\}$$

as $\varepsilon \downarrow 0$ provided that $-\xi \in \mathcal{C}$ for a class $\mathcal{C}$ such that (1.3) holds.

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To establish (1.3) for the class $\mathcal{L}(\alpha)$ we develop Tauberian results for infinitely divisible distributions which should be of substantial interest in their own.

The paper is organized as follows: In Section 2 we review various classes of subexponential and exponential distributions that feature in the paper.

In Section 3 we develop the mentioned Tauberian results. In particular, they show that $\xi(h) \in \mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$ when ξ has a semi-heavy-tailed Lévy measure such that $\rho \geq -1$ in (1.1). We also express the tail behaviour of $\xi(h)$ in terms of the Lévy characteristic triple of the process. Note that on a less precise level than our Tauberian results, it is well-known that a Lévy measure with an exponential tail more or less corresponds to an infinitely divisible distribution with an exponential tail with the same exponent, see, e.g., Sato [33], Theorem 25.3.

In Section 4 we prove a partial converse to (1.3) for $\mathcal{C} = \mathcal{L}$. This converse is crucial for our proof that $H > 1$ in (1.3) for $\mathcal{C} = \mathcal{S}(\alpha)$.

In Section 5 we prove (1.2) for $\mathcal{C} = \mathcal{OL}$ as well as the implication (1.4) together with a partial converse to that implication. The equivalency (1.2) for $\mathcal{C} = \mathcal{OL}$ is crucial for our proof of (1.2) for $\mathcal{C} = \mathcal{L}(\alpha)$ in Section 6.

In Section 6 we prove (1.2) and (1.3) for $\mathcal{C} = \mathcal{L}(\alpha)$. The results from Section 3 are crucial to verify the hypothesis of these results.

In Section 7 we give applications to the process classes GZ, CGMY and GH.

In the companion article Albin and Sundén [2] we study the tail behaviour of *superexponential* Lévy processes with lighter than exponential tails. This rich class of processes includes many processes for which the limit H in (1.3) does not exist.

2. Subexponential and exponential distributions

In this section we review classes of probability distributions that feature in our work.

2.1. Subexponential distributions

Here we discuss distributions with tails that are heavier than exponential ones.

The following classes of distributions $\mathcal{L}$ and $\mathcal{S}$ are well-known from the literature:

Definition 2.1. A cumulative distribution function (CDF) F belongs to the class of *long-tailed* distributions $\mathcal{L}$ if

$$\lim_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} = 1 \quad \text{for } x \in \mathbb{R}.$$

A CDF F belongs to the class of *subexponential* distributions $\mathcal{S}$ if

$$\lim_{u\to\infty} \frac{1-F\star F(u)}{1-F(u)} = 2.$$

In Definition 2.1 $\star$ means convolution, that is, $F\star F(u) = \int_{\mathbb{R}} F(u-x)\mathrm{d}F(x)$.

It turns out that $\mathcal{S} \subsetneq \mathcal{L}$ (see Embrechts and Goldie [16], Section 3). It is easy to see that $F \in \mathcal{S}$ if $1-F$ is regularly varying at infinity with a non-positive index:

Definition 2.2. A measurable function $g > 0$ is *regularly varying* at infinity with *index* $\varrho \in \mathbb{R}$, $g \in \mathcal{R}(\varrho)$, if

$$\lim_{u\to\infty} \frac{g(ux)}{g(u)} = x^{\varrho} \quad \text{for } x \in (0,\infty).$$

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A measurable function $g > 0$ is $\mathcal{O}$*-regularly varying* at infinity, $g \in \mathcal{OR}$, if

$$0 < \liminf_{u\to\infty} \frac{g(ux)}{g(u)} \le \limsup_{u\to\infty} \frac{g(ux)}{g(u)} < \infty \quad \text{for } x \in (0,\infty).$$

Example 2.3. Given constants (parameters) $x_0, \varrho > 0$, the Pareto distribution $F(x) = 1 - (x/x_0)^{-\varrho}$ for $x \ge x_0$ satisfies $1 - F \in \mathcal{R}(-\varrho)$, so that $F \in \mathcal{S} \subseteq \mathcal{L}$.

For the class $\mathcal{L}$ we will need the following lemma, the proof of which is elementary:

Lemma 2.4. *A CDF F belongs to the class $\mathcal{L}$ if and only if*

$$\liminf_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} \ge 1 \quad \textit{for some } x > 0.$$

2.2. *Exponential distributions*

Here we discuss distributions with exponential tails.

The following classes $\mathcal{L}(\alpha)$ and $\mathcal{S}(\alpha)$ are well-known from the literature. The class $\mathcal{OL}$ of Bengtsson [8] and Shimura and Watanabe [37] is an exponential analogue of $\mathcal{OR}$.

Definition 2.5. Given a constant $\alpha \ge 0$, a CDF F belongs to the class $\mathcal{L}(\alpha)$ if

$$\lim_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} = \mathrm{e}^{-\alpha x} \quad \text{for } x \in \mathbb{R}. \tag{2.1}$$

A CDF F belongs to the *exponential* class $\mathcal{S}(\alpha)$ if $F \in \mathcal{L}(\alpha)$ and

$$\lim_{u\to\infty} \frac{1-F\star F(u)}{1-F(u)} \quad \text{exists (and is finite).} \tag{2.2}$$

A CDF F belongs to the class of $\mathcal{O}$*-exponential* distributions $\mathcal{OL}$ if

$$0 < \liminf_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} \le \limsup_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} < \infty \quad \text{for } x \in \mathbb{R}.$$

Pitman [31], p. 338, argued that the class $\mathcal{L}$ should be called subexponential rather than $\mathcal{S}$. By his logic the class $\mathcal{L}(\alpha)$ should be called exponential instead of $\mathcal{S}(\alpha)$. In fact, the exponential distribution itself belongs to $\mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$ by Example 2.11. This is the reason that we talk about exponential distributions when dealing with $\mathcal{L}(\alpha)$.

Note that $\mathcal{L}(0) = \mathcal{L}, \mathcal{S}(0) = \mathcal{S}$ [as the limit (2.2) is 2 for $\mathcal{S}(0)$, see Embrechts and Goldie [17], Section 2], and $\mathcal{S}(\alpha) \subseteq \mathcal{L}(\alpha) \subseteq \mathcal{OL}$. To illustrate how $\mathcal{OL}$ differs from $\cup_{\alpha\ge 0}\mathcal{L}(\alpha)$ we give the following simple result which is proved in Appendix A.1:

Proposition 2.6. *An absolutely continuous CDF F belongs to $\mathcal{L}(\alpha)$ if and only if*

$$F(u) = 1 - \exp\left\{-\int_{-\infty}^{u} (a(x)+b(x))\mathrm{d}x\right\} \quad \textit{for } u \in \mathbb{R}, \tag{2.3}$$

for some measurable functions a and b with $a + b \ge 0$ such that

$$\lim_{x\to\infty} a(x) = \alpha, \qquad \lim_{u\to\infty} \int_{-\infty}^{u} a(x)\mathrm{d}x = \infty \quad \textit{and} \quad \lim_{u\to\infty} \int_{-\infty}^{u} b(x)\mathrm{d}x \quad \textit{exists.} \tag{2.4}$$

An absolutely continuous CDF F belongs to $\mathcal{OL}$ *if and only if* (2.3) *holds for some measurable functions a and b with* $a + b \geq 0$ *such that*

$$\limsup_{x\to\infty} |a(x)| < \infty, \qquad \liminf_{u\to\infty} \int_{-\infty}^{u} a(x)\mathrm{d}x = \infty \quad \textit{and}$$

$$\limsup_{u\to\infty} \left| \int_{-\infty}^{u} b(x)\mathrm{d}x \right| < \infty. \tag{2.5}$$

Example 2.7. Let $a(x) = \alpha\, \mathbf{1}_{[0,\infty)}(x)$ and $b(x) = \beta \cos(\mathrm{e}^x - 1)\mathbf{1}_{[0,\infty)}(x)$ in (2.3) where $|\beta| \leq \alpha$ and $\alpha > 0$ are constants. Then we have $F \in \mathcal{L}(\alpha)$ since $\lim_{u\to\infty} \int_0^u \cos(\mathrm{e}^x - 1)\mathrm{d}x$ exists. Instead, if we take $b(x) = \beta \cos(x)\, \mathbf{1}_{[0,\infty)}(x)$, then (2.5) holds but (2.4) does not unless $\beta = 0$, so that $F \in \mathcal{OL} \setminus \mathcal{L}(\alpha)$ for $\beta \neq 0$.

The following elementary result for the class $\mathcal{OL}$ corresponds to Lemma 2.4 for $\mathcal{L}$:

Lemma 2.8. *A CDF F belongs to the class* $\mathcal{OL}$ *if and only if*

$$\liminf_{u\to\infty} \frac{1 - F(u + x)}{1 - F(u)} > 0 \quad \textit{for some } x > 0.$$

2.3. Distributions with semi-heavy tails

In mathematical finance log increments of asset prices are often modelled to have *semi-heavy tails*, see, e.g., Barndorff-Nielsen [6] and Schoutens [34].

The following simple corollary to Proposition 2.6 is proved in Appendix A.2:

Corollary 2.9. *A CDF F is semi-heavy tailed satisfying* (1.1) *if and only if*

$$F(u) = 1 - \exp\left\{ -\int_{-\infty}^{u} c(x)\mathrm{d}x \right\} \quad \textit{for } u \in \mathbb{R}, \tag{2.6}$$

for some measurable function $c \geq 0$ *that satisfies*

$$\lim_{x\to\infty} c(x) = \alpha \quad \textit{and}$$

$$\lim_{u\to\infty} \int_{-\infty}^{u} \left(c(x) - \alpha\, \mathbf{1}_{[0,\infty)}(x) + \frac{\rho}{x}\, \mathbf{1}_{[1,\infty)}(x) \right) \mathrm{d}x = \ln\left(\frac{C}{\alpha}\right). \tag{2.7}$$

If any of these equivalent conditions hold so that both of them hold, then $F \in \mathcal{L}(\alpha)$.

Corollary 2.9 shows which distributions in Example 2.7 are semi-heavy:

Example 2.10. The distributions in Example 2.7 are semi-heavy only if $\beta = 0$ as they have $c(x) = \alpha + \beta \cos(\mathrm{e}^x - 1)$ and $c(x) = \alpha + \beta \cos(x)$ for $x \geq 0$ in (2.6).

By Corollary 2.9 semi-heavy-tailed CDF's belong to $\mathcal{L}(\alpha)$. However, by the following example a semi-heavy-tailed CDF belongs to $\mathcal{S}(\alpha)$ only if $\rho < -1$ in (1.1).

Example 2.11. For a semi-heavy-tailed CDF F with $\rho < -1$, Pakes [29], Corollary 2.1 ii and Lemma 2.3 (see also [30]), show that the limit (2.2) exists with value $2 \int_{\mathbb{R}} \mathrm{e}^{\alpha x}\mathrm{d}F(x) < \infty$, so that $F \in \mathcal{S}(\alpha)$. For a semi-heavy-tailed CDF F with $\rho \geq -1$ we have $\int_1^\infty \mathrm{e}^{\alpha x}\mathrm{d}F(x) = \infty$. Hence Pakes [29], p. 411 (see also [30]), shows that the ratio in (2.2) goes to infinity as $u \to \infty$, so that $F \in \mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$.

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3. Tauberian theorems for $\mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$

In this section we first introduce some standard notation for Lévy processes that will be used repeatedly from here on. Then we state and prove the Tauberian results mentioned in the Introduction that are of fundamental importance in Sections 6 and 7.

An *infinitely divisible* CDF F is characterized by a *characteristic triple* (ν, m, s^2) as

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x}\,\mathrm{d}F(x) = \exp\left\{\mathrm{i}\theta m + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta \mathbf{1}_{(-1,1)}(x)\,x\right)\mathrm{d}\nu(x) - \frac{\theta^2 s^2}{2}\right\}$$
$$\text{for } \theta \in \mathbb{R}. \tag{3.1}$$

Here ν is the *Lévy* (Borel) *measure* on $\mathbb{R}$ that satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\mathrm{d}\nu(x) < \infty$ while $m \in \mathbb{R}$ and $s^2 \geq 0$ are constants. This triple is unique.

A *Lévy process* is a continuous in probability and càdlàg stochastic process $\{\xi(t)\}_{t\geq 0}$ with $\xi(0) = 0$ that has stationary and independent increments. As the marginal distribution of $\xi(t)$ is infinitely divisible with characteristic triple $(\nu t, mt, s^2 t)$ where (ν, m, s^2) is the characteristic triple of $\xi(1)$, the latter triple is called the *characteristic triple* of ξ.

The Tauberian results Theorem 3.3, Corollary 3.5 and Theorem 3.6 establish that if ξ is a Lévy process with Lévy measure ν that satisfies the condition (3.3), possibly together with additional conditions, then we have the behaviour (3.4) the tails of the Lévy process. The claim (3.4) is a key condition in Sections 6 and 7.

Our first Tauberian result is derived from Braverman [12], Lemma 5, together with the following two results from the literature that are stated here for easy reference:

Theorem 3.1 (*Embrechts, Goldie and Veraverbeke [18], Sgibnev [36]*). *Given a constant $\alpha \geq 0$ and a Lévy process ξ with Lévy measure ν, we have*

$$\frac{\nu([1,\infty)\cap\cdot)}{\nu([1,\infty))} \in \mathcal{S}(\alpha) \Leftrightarrow \xi(t) \in \mathcal{S}(\alpha) \quad \textit{for some } t > 0 \Leftrightarrow \xi(t) \in \mathcal{S}(\alpha) \textit{ for } t > 0.$$

Moreover, if any of these conditions holds so that all of them hold, then we have

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\nu((u,\infty))} = t\,\mathbf{E}\{\mathrm{e}^{\alpha\xi(t)}\} \quad \textit{for } t > 0.$$

Theorem 3.2 (*Albin [1], Pakes [29,30]*). *Let $\alpha \geq 0$ be a constant and ξ a Lévy process with characteristic triple (ν, m, s^2). Write $\xi = \xi_1 + \xi_2$ where ξ_1 and ξ_2 are independent Lévy processes with characteristic triples $(\nu(\cdot\cap[1,\infty)), 0, 0)$ and $(\nu(\cdot\cap(-\infty,1)), m, s^2)$, respectively. We have*

$$\frac{\nu([1,\infty)\cap\cdot)}{\nu([1,\infty))} \in \mathcal{L}(\alpha) \Rightarrow \xi_1(t) \in \mathcal{L}(\alpha) \quad \textit{for } t > 0 \Rightarrow \xi(t) \in \mathcal{L}(\alpha) \textit{ for } t > 0.$$

Moreover, if $\xi_1(t) \in \mathcal{L}(\alpha)$ for $t > 0$, then we have

$$\mathbf{P}\{\xi(t) > u\} \sim \mathbf{P}\{\xi_1(t) > u\} \int_{\mathbb{R}} \mathrm{e}^{\alpha x}\,\mathrm{d}F_{\xi_2(t)}(x) \quad \textit{as } u \to \infty \textit{ for } t > 0. \tag{3.2}$$

Here is our first Tauberian result. This result is due to Braverman [13], Theorem 1, but our proof is very much shorter. See also Section 8 on priority:

Theorem 3.3 (*Braverman [13]*). *Let ξ be a Lévy process that satisfies*

$$F(\cdot) \equiv \frac{\nu([1 \vee \cdot, \infty))}{\nu([1, \infty))} \in \mathcal{L}(\alpha) \quad \text{for some } \alpha > 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{1 - F(u)}{1 - F \star F(u)} = 0. \tag{3.3}$$

Then it holds that

$$\xi(t) \in \mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha) \quad \text{for } t > 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{P}\{\xi(t) > u\}} = 0 \quad \text{for } 0 < s < t. \tag{3.4}$$

Proof. Let ξ have triple (ν, m, s^2) and write $\xi = \xi_1 + \xi_2$ where ξ_1 and ξ_2 are independent Lévy processes with triples $(\nu(\cdot \cap [1, \infty)), 0, 0)$ and $(\nu(\cdot \cap (-\infty, 1)), m, s^2)$, respectively. From Theorem 3.2 we have $\xi_1(t), \xi(t) \in \mathcal{L}(\alpha)$. As ξ_1 is a compound Poisson process with jump CDF F, and as the second requirement of (3.3) means that F is light-tailed in the sense of Braverman [12], Definition 1, it follows from Braverman [12], Lemma 5, that ξ_1 satisfies the second requirement of (3.4). As tail probabilities for ξ_1 are proportional to those of ξ by (3.2) we see that the second requirement of (3.4) holds also for ξ. Hence $\xi(t) \notin \mathcal{S}(\alpha)$ by Theorem 3.1, which finishes the proof of all claims of the theorem. □

In Corollary 3.5 and Theorem 3.6 below we specialize Theorem 3.3 to semi-heavy-tailed Lévy measures with $\rho > -1$ and $\rho = -1$, respectively (see Example 2.11). We also express the tail probability $\mathbf{P}\{\xi(t) > u\}$ as $u \to \infty$ in terms of the characteristic triple, which makes possibly more explicit results in our applications in Section 7.

For the statement of Corollary 3.5 and Theorem 3.6, consider the moment generating function (MGF) of a Lévy process ξ given by

$$\phi(t, \lambda) = \mathbf{E}\{e^{-\lambda\xi(t)}\} = \left(\mathbf{E}\{e^{-\lambda\xi(1)}\}\right)^t = \phi(1, \lambda)^t \equiv \phi(\lambda)^t \quad \text{for } t > 0 \text{ and } \lambda \in \mathbb{R}. \tag{3.5}$$

Writing (ν, m, s^2) for the characteristic triple of ξ, Sato [33], Theorem 25.17, shows that

$$\begin{aligned} \phi(t, \lambda) < \infty \quad &\text{for some } t > 0 \Leftrightarrow \phi(t, \lambda) < \infty \quad \text{for } t > 0 \\ &\Leftrightarrow \int_{\mathbb{R} \setminus (-1,1)} e^{-\lambda x} d\nu(x) < \infty. \end{aligned} \tag{3.6}$$

We will need the following functions μ and V given by [cf. (3.1)]

$$\begin{cases} \mu(\lambda) = -\dfrac{\phi'(\lambda)}{\phi(\lambda)} = \displaystyle\int_{\mathbb{R}} \left(x\, e^{-\lambda x} - x\, \mathbf{1}_{(-1,1)}(x)\right) d\nu(x) + m - \lambda s^2 \\ V(\lambda) = -\mu'(\lambda) = \displaystyle\int_{\mathbb{R}} x^2 e^{-\lambda x} d\nu(x) + s^2 \end{cases} \tag{3.7}$$

for $\lambda \in \mathbb{R}$ such that the definition makes sense. We will also need the inverse function $\mu^{\leftarrow}(u)$ of μ which will be well-defined in all cases we encounter for $u \in \mathbb{R}$ sufficiently large as we will have $\lim_{\lambda \downarrow -\alpha} \mu(\lambda) = \infty$ with $\mu'(\lambda) = -V(\lambda) < 0$ (see below).

We will need the following well-known Tauberian result for compound Poisson processes with semi-heavy Lévy measures. Our version of the result is stated slightly differently than in the literature to better suit our purposes, but it is easy to see that it is equivalent to the results in the literature.

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Theorem 3.4 (*Embrechts, Jensen, Maejima and Teugels [19], Homble and McCormick [23, 24], Jensen [25]*). *For a compound Poisson process ξ with Lévy measure ν that is absolutely continuous sufficiently far out to the right with*

$$\frac{\mathrm{d}\nu(u)}{\mathrm{d}u} \sim C\, u^{\rho}\mathrm{e}^{-\alpha u} \quad \text{as } u \to \infty \text{ for some constants } C, \alpha > 0 \text{ and } \rho > -1, \tag{3.8}$$

we have, with the notation (3.5) and (3.7),

$$\mathbf{P}\{\xi(t) > u\} \sim \frac{\mathrm{e}^{u\mu^{\leftarrow}(u/t)}\phi(\mu^{\leftarrow}(u/t))^t}{\alpha\sqrt{2\pi t\, V(\mu^{\leftarrow}(u/t))}} \quad \text{as } u \to \infty \text{ for } t > 0.$$

The following corollary to Theorems 3.3 and 3.4 addresses processes which have semi-heavy-tailed Lévy measures with $\rho > -1$:

Corollary 3.5. *Let ξ be a Lévy process with characteristic triple (ν, m, s^2) such that ν is absolutely continuous sufficiently far out to the right and satisfies (3.8). Write $\xi = \xi_1 + \xi_2$ where ξ_1 and ξ_2 are independent Lévy processes with triplets $(\nu(\cdot \cap [1,\infty)), 0, 0)$ and $(\nu(\cdot \cap (-\infty, 1)), m, s^2)$, respectively, and let ϕ_1, μ_1, $\mu_1^{\leftarrow}$ and V_1 denote the quantities ϕ, μ, $\mu^{\leftarrow}$ and V in (3.5) and (3.7) calculated for the process ξ_1 instead of ξ. Eq. (3.4) holds and*

$$\mathbf{P}\{\xi(t) > u\} \sim \mathbf{E}\{\mathrm{e}^{\alpha\xi_2(t)}\}\frac{\mathrm{e}^{u\mu_1^{\leftarrow}(u/t)}\phi_1(\mu_1^{\leftarrow}(u/t))^t}{\alpha\sqrt{2\pi t\, V_1(\mu_1^{\leftarrow}(u/t))}} \quad \text{as } u \to \infty \text{ for } t > 0. \tag{3.9}$$

Proof. As (3.3) holds by (3.8) and Example 2.11 we get (3.4) from Theorem 3.3. As ξ_1 is a compound Poisson process that satisfies (3.8) Theorem 3.4 shows that

$$\mathbf{P}\{\xi_1(t) > u\} \sim \frac{\mathrm{e}^{u\mu_1^{\leftarrow}(u/t)}\phi_1(\mu_1^{\leftarrow}(u/t))^t}{\alpha\sqrt{2\pi t\, V_1(\mu_1^{\leftarrow}(u/t))}} \quad \text{as } u \to \infty \text{ for } t > 0.$$

Hence (3.9) follows from (3.2) [as $\xi_1(t) \in \mathcal{L}(\alpha)$ by the proof of Theorem 3.3]. □

By further calculations one can phrase (3.9) in terms of ϕ, μ, $\mu^{\leftarrow}$ and V instead of ϕ_1, μ_1, $\mu_1^{\leftarrow}$ and V_1 in special cases: See the proof of (7.8) below for an example on this!

In Theorem 3.6 we address processes which have semi-heavy-tailed Lévy measures with $\rho = -1$. Now the Tauberian arguments have to be developed in detail from scratch as there are no suitable results in the literature to take off from. However, the idea of the proof to use Esscher transforms to find tail probabilities is well-known and is also used in the companion paper Albin and Sundén [2], where we give a bibliography.

Theorem 3.6. *If ξ is a Lévy process with Lévy measure ν that is absolutely continuous sufficiently far out to the right and satisfies (3.8) with $\rho = -1$, then (3.4) holds. If in addition $Ct > 1$ and*

$$\liminf_{x\downarrow 0} \frac{x}{\ln(1/x)}\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} = \infty, \tag{3.10}$$

then we have

$$\mathbf{P}\{\xi(t) > u\} \sim \frac{\sqrt{C}(Ct)^{Ct}\mathrm{e}^{-Ct}}{\alpha\,\Gamma(Ct+1)}\frac{\mathrm{e}^{u\mu^{\leftarrow}(u/t)}\phi(\mu^{\leftarrow}(u/t))^t}{\sigma(\mu^{\leftarrow}(u/t))} \quad \text{as } u \to \infty. \tag{3.11}$$

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Proof. We get (3.4) from Theorem 3.3 as (3.3) holds by Example 2.11.

To prove (3.11) consider the Esscher transform of $\xi(t)$, which with the notation (3.5) and (3.7) is a random variable $Z_{t,\lambda}$ with CDF defined by the change of measure

$$\mathrm{d}F_{Z_{t,\lambda}}(x) = \frac{\mathrm{e}^{-\lambda x}\mathrm{d}F_{\xi(t)}(x)}{\phi(\lambda)^t} \quad \text{for } t > 0 \text{ and } \lambda \in \mathbb{R} \text{ such that } \phi(\lambda) < \infty. \tag{3.12}$$

By elementary calculations we have $\mathbf{E}\{Z_{t,\lambda}\} = t\mu(\lambda)$ and $\mathbf{Var}\{Z_{t,\lambda}\} = tV(\lambda) \equiv t\,\sigma(\lambda)^2$. Further, (3.1) and (3.7) show that $(Z_{t,\lambda} - t\mu(\lambda))/\sigma(\lambda)$ has CHF $g_{t,\lambda}$ given by

$$\begin{aligned} g_{t,\lambda}(\theta) &= \frac{\phi(\lambda - \mathrm{i}\theta/\sigma(\lambda))^t}{\phi(\lambda)^t}\,\mathrm{e}^{-\mathrm{i}t\mu(\lambda)\theta/\sigma(\lambda)} \\ &= \exp\left\{ t\int_{\mathbb{R}} \left(\cos\left(\frac{\theta x}{\sigma(\lambda)}\right) - 1\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) \right. \\ &\quad \left. + \mathrm{i}t\int_{\mathbb{R}} \left(\sin\left(\frac{\theta x}{\sigma(\lambda)}\right) - \frac{\theta x}{\sigma(\lambda)}\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) - \frac{\theta^2 s^2 t}{2V(\lambda)}\right\}. \end{aligned} \tag{3.13}$$

By (3.7), (3.8) and elementary calculations we have

$$\mu(\lambda) \sim \frac{C}{\alpha + \lambda} \quad \text{and} \quad V(\lambda) \sim \frac{C}{(\alpha+\lambda)^2} \quad \text{as } \lambda \downarrow -\alpha. \tag{3.14}$$

Let Γ_{Ct} denote a gamma distributed random variable with PDF $f_{\Gamma_{Ct}}(x) = x^{Ct-1}\mathrm{e}^{-x}/\Gamma(Ct)$ for $x > 0$. By (3.8), (3.14) and the change of variable $x = \sigma(\lambda)\,y$ in (3.13) we have

$$\begin{aligned} g_{t,\lambda}(\theta) &\to \exp\left\{ Ct\int_0^\infty \frac{\cos(\theta y) - 1}{y}\,\mathrm{e}^{-\sqrt{C}y}\mathrm{d}y + \mathrm{i}Ct\int_0^\infty \frac{\sin(\theta y) - \theta y}{y}\,\mathrm{e}^{-\sqrt{C}y}\mathrm{d}y \right\} \\ &= \exp\left\{ -\frac{Ct\ln(1+\theta^2/C)}{2} + \mathrm{i}Ct\left(\arctan\left(\frac{\theta}{\sqrt{C}}\right) - \frac{\theta}{\sqrt{C}}\right)\right\} \\ &= \frac{1}{(1 - \mathrm{i}\theta/\sqrt{C})^{Ct}}\mathrm{e}^{-\mathrm{i}\sqrt{C}\,t\theta} \\ &= \mathbf{E}\left\{\exp\left[\mathrm{i}\theta(\Gamma_{Ct} - Ct)/\sqrt{C}\right]\right\} \equiv g_t(\theta) \quad \text{as } \lambda \downarrow -\alpha, \end{aligned} \tag{3.15}$$

cf. Erdélyi, Magnus, Oberhettinger and Tricomi [21], Equations 4.2.1, 4.7.59 and 4.7.82.

A key step in the proof is to prove that

$$\limsup_{\lambda \downarrow -\alpha} \int_{|\theta| > K} \left| g_{t,\lambda}(\theta) \right| \mathrm{d}\theta \to 0 \quad \text{as } K \to \infty, \tag{3.16}$$

as this shows that $g_{t,\lambda}$ is integrable for $\lambda > -\alpha$ small enough (since $g_{t,\lambda}$ is bounded by 1), so that $(Z_{t,\lambda} - t\mu(\lambda))/\sigma(\lambda)$ has a continuous PDF $f_{(Z_{t,\lambda} - t\mu(\lambda))/\sigma(\lambda)}$, which by (3.15) and (3.16) satisfies (using bounded convergence)

$$\begin{aligned} &\limsup_{\lambda\downarrow -\alpha}\ \sup_{x\in\mathbb{R}} \left| f_{(Z_{t,\lambda} - t\mu(\lambda))/\sigma(\lambda)}(x) - f_{(\Gamma_{Ct} - Ct)/\sqrt{C}}(x)\right| \\ &\le \limsup_{K\to\infty}\limsup_{\lambda\downarrow -\alpha}\left(\int_{|\theta|\le K}\left|g_{t,\lambda}(\theta) - g_t(\theta)\right|\mathrm{d}\theta + \int_{|\theta|>K}\left(|g_{t,\lambda}(\theta)| + |g_t(\theta)|\right)\mathrm{d}\theta\right) \\ &= 0. \end{aligned} \tag{3.17}$$

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From (3.12) and (3.17) together with (3.14), we get

$$\begin{aligned} f_{\xi(t)}(t\mu(\lambda)-x/\lambda) &= \frac{\mathrm{e}^{t\lambda\mu(\lambda)-x}\phi(\lambda)^t}{\sigma(\lambda)}\, f_{(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}\left(-\frac{x}{\lambda\,\sigma(\lambda)}\right) \\ &\sim \frac{\mathrm{e}^{t\lambda\mu(\lambda)-x}\phi(\lambda)^t}{\sigma(\lambda)}\, f_{(\Gamma_{Ct}-Ct)/\sqrt{C}}(0) \\ &= \frac{\sqrt{C}(Ct)^{Ct}\mathrm{e}^{-Ct}}{\Gamma(Ct+1)}\frac{\mathrm{e}^{t\lambda\mu(\lambda)-x}\phi(\lambda)^t}{\sigma(\lambda)} \quad \text{as } \lambda\downarrow-\alpha. \end{aligned} \tag{3.18}$$

Note that $\sup_{x\in\mathbb{R}} f_{(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}(x)$ is bounded for $\lambda > -\alpha$ small enough by (3.17) [as $Ct > 1$], so that we may integrate (3.18) using bounded convergence to obtain

$$\begin{aligned} \mathbf{P}\{\xi(t) > t\mu(\lambda)\} &= \int_0^\infty \frac{f_{\xi(t)}(t\mu(\lambda)-x/\lambda)}{-\lambda}\mathrm{d}x \\ &\sim \int_0^\infty \frac{\sqrt{C}(Ct)^{Ct}\mathrm{e}^{-Ct}}{(-\lambda)\,\Gamma(Ct+1)}\frac{\mathrm{e}^{t\lambda\mu(\lambda)-x}\phi(\lambda)^t}{\sigma(\lambda)}\mathrm{d}x \\ &\sim \frac{\sqrt{C}(Ct)^{Ct}\mathrm{e}^{-Ct}}{\alpha\,\Gamma(Ct+1)}\frac{\mathrm{e}^{t\lambda\mu(\lambda)}\phi(\lambda)^t}{\sigma(\lambda)} \quad \text{as } \lambda\downarrow-\alpha. \end{aligned}$$

From this in turn we get (3.11) by a change of variable in the limit.

In order to finish the proof of the theorem it remains to prove (3.16). To that end pick constants $\varepsilon\in(0,1)$ and $A\geq 1$ such that $(1-\varepsilon)^5 Ct \geq 1$ and $\varepsilon\, t\mathrm{d}\nu(x)/\mathrm{d}x \geq 16\ln(1/x)/x$ for $x\in(0,1/A)$, cf. (3.10). As $1-\cos(x)\geq \frac{1}{4}x^2$ for $|x|\leq 1$ we have [as $\mathrm{e}^{-\lambda x}\geq 1$ for $x\geq 0$]

$$\begin{aligned} \varepsilon\, t\int_{\mathbb{R}}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) &\geq \varepsilon\, t\int_0^{\sigma(\lambda)/|\theta|}\frac{x^2\theta^2}{4\,V(\lambda)}\mathrm{d}\nu(x) \\ &\geq \int_0^{\sigma(\lambda)/|\theta|}\frac{4\,\theta^2}{V(\lambda)}\,x\ln(1/x)\mathrm{d}x \\ &= 1+\ln\left(\frac{\theta^2}{V(\lambda)}\right) \quad \text{for } |\theta|\geq A\sigma(\lambda) \end{aligned}$$

and $\lambda > -\alpha$ small enough. As (3.8) shows that $\mathrm{d}\nu(x)/\mathrm{d}x \geq (1-\varepsilon)\,C\,\mathrm{e}^{-\alpha x}/x$ for $x\geq B$, for some constant $B > 0$ large enough to make $B+2\pi/A \leq B/(1-\varepsilon)$, we further have

$$\begin{aligned} &(1-\varepsilon)\,t\int_{\mathbb{R}}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) \\ &\geq (1-\varepsilon)^2 Ct\sum_{k=0}^{\infty}\int_{B/\sigma(\lambda)+2\pi k/|\theta|}^{B/\sigma(\lambda)+2\pi(k+1)/|\theta|}(1-\cos(\theta x))\frac{\mathrm{e}^{-(\alpha+\lambda)\sigma(\lambda)x}}{x}\mathrm{d}x \\ &\geq (1-\varepsilon)^2 Ct\sum_{k=0}^{\infty}\frac{2\pi}{|\theta|}\frac{\mathrm{e}^{-(\alpha+\lambda)\sigma(\lambda)(B/\sigma(\lambda)+2\pi(k+1)/|\theta|)}}{B/\sigma(\lambda)+2\pi(k+1)/|\theta|} \\ &\geq (1-\varepsilon)^2 Ct\int_{B/\sigma(\lambda)}^{\infty}\frac{\mathrm{e}^{-(\alpha+\lambda)\sigma(\lambda)(x+2\pi/(A\sigma(\lambda)))}}{x+2\pi/(A\sigma(\lambda))}\mathrm{d}x \\ &\geq (1-\varepsilon)^2 Ct\,\mathrm{e}^{-2\pi(\alpha+\lambda)/A}\frac{B}{B+2\pi/A}\int_{B/\sigma(\lambda)}^{\infty}\frac{\mathrm{e}^{-(\alpha+\lambda)\sigma(\lambda)x}}{x}\mathrm{d}x \end{aligned}$$

$$\geq (1-\varepsilon)^4 Ct \int_{B(\alpha+\lambda)}^{\infty} \frac{\mathrm{e}^{-x}}{x} \mathrm{d}x$$
$$\geq (1-\varepsilon)^5 Ct \ \ln\left(\frac{1}{B(\alpha+\lambda)}\right) \quad \text{for } |\theta| \geq A\sigma(\lambda)$$

and $\lambda > -\alpha$ small enough, where the last inequality is an elementary calculation.

By the estimates of the previous paragraph together with (3.13) and (3.14) we obtain

$$\begin{aligned}\int_{|\theta|\geq A\sigma(\lambda)} |g_{t,\lambda}(\theta)|\mathrm{d}\theta &= \int_{|\theta|\geq A\sigma(\lambda)} \exp\left\{-t\int_{\mathbb{R}}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\right\}\mathrm{d}\theta \\ &\leq \int_{|\theta|\geq A\sigma(\lambda)} \mathrm{e}^{(1-\varepsilon)^5 Ct\gamma}(B(\alpha+\lambda))^{(1-\varepsilon)^5 Ct}\frac{V(\lambda)}{\theta^2}\mathrm{d}\theta \\ &= \mathrm{e}^{(1-\varepsilon)^5 Ct\gamma}\frac{2(B(\alpha+\lambda))^{(1-\varepsilon)^5 Ct}\sigma(\lambda)}{A} \to 0 \quad \text{as } \lambda\downarrow -\alpha. \end{aligned} \tag{3.19}$$

Moreover, we have, using Erdélyi, Magnus, Oberhettinger and Tricomi [21], Eq. 4.7.59, together with the inequality $1-\cos(x) \leq x^2/2$ and (3.14),

$$\begin{aligned} & t\int_{\mathbb{R}}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) \\ &\geq (1-\varepsilon)\,Ct\int_B^{\infty}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\frac{\mathrm{e}^{-(\alpha+\lambda)x}}{x}\mathrm{d}x \\ &\geq (1-\varepsilon)\,Ct\int_0^{\infty}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\frac{\mathrm{e}^{-(\alpha+\lambda)x}}{x}\mathrm{d}x - Ct\int_0^B\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\frac{1}{x}\mathrm{d}x \\ &\geq \frac{(1-\varepsilon)\,Ct}{2}\ln\left(1+\frac{\theta^2}{(\alpha+\lambda)^2 V(\lambda)}\right) - Ct\int_0^B \frac{\theta^2 x}{2\,V(\lambda)}\mathrm{d}x \\ &\geq \frac{(1-\varepsilon)\,Ct}{2}\ln\left(\frac{\theta^2}{2C}\right) - Ct\frac{A^2B^2}{4} \end{aligned}$$

for $|\theta| \leq A\sigma(\lambda)$ and $\lambda > -\alpha$ small enough.

Hence we have (for $\lambda > -\alpha$ small enough)

$$\begin{aligned}\int_{K\leq|\theta|\leq A\sigma(\lambda)} |g_{t,\lambda}(\theta)|\mathrm{d}\theta &\leq 2\int_K^{\infty}\exp\left\{Ct\frac{A^2B^2}{4} - \frac{(1-\varepsilon)\,Ct}{2}\ln\left(\frac{\theta^2}{2C}\right)\right\}\mathrm{d}\theta \\ &= 2\exp\left\{Ct\frac{A^2B^2}{4}\right\}\frac{(2C)^{(1-\varepsilon)Ct/2}}{(1-\varepsilon)Ct-1}K^{1-(1-\varepsilon)Ct},\end{aligned}$$

which goes to 0 as $K\to\infty$ since $(1-\varepsilon)\,Ct > 1$. Recalling (3.19) this gives (3.16). □

4. Subexponential Lévy processes

The following result we will later extend from long-tailed processes to exponential ones.

Theorem 4.1 (*Berman [9], Marcus [28], Willekens [39]*). *For a Lévy process ξ we have*

$$\xi(h)\in\mathcal{L} \Leftrightarrow \sup_{t\in[0,h]}\xi(t)\in\mathcal{L}.$$

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Moreover, if any of these memberships holds so that both of them hold, then

$$\lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} = 1. \tag{4.1}$$

The following simple converse to Theorem 4.1 is new and will be used in Section 6 to prove that $H > 1$ in (1.3) for $\mathcal{C} = \mathcal{S}(\alpha)$.

Theorem 4.2. *For a Lévy process ξ that satisfies (4.1), but is not a subordinator, one of the following two conditions holds:*

1. $\xi(h) \in \mathcal{L}$;
2. $$\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}} = 0 \quad \textit{for all } t \in (0, h).$$

Proof. Let (4.1) hold and assume that the liminf in Condition 2 takes a value $\ell(t) > 0$ for some $t \in (0, h)$. To show that Condition 1 holds note that

$$\begin{aligned} 0 &= \lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \left(\mathbf{P}\left\{\sup_{s\in[0,h]} \xi(s) > u\right\} - \mathbf{P}\{\xi(h) > u\}\right) \\ &\geq \limsup_{u\to\infty} \frac{\mathbf{P}\{\xi(h) \leq u, \xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}} \\ &\geq \mathbf{P}\{\xi(h-t) \leq -\varepsilon\} \limsup_{u\to\infty} \frac{\mathbf{P}\{u < \xi(t) \leq u+\varepsilon\}}{\mathbf{P}\{\xi(h) > u\}} \\ &= \ell(t)\, \mathbf{P}\{\xi(h-t) \leq -\varepsilon\} \limsup_{u\to\infty} \frac{\mathbf{P}\{u < \xi(t) \leq u+\varepsilon\}}{\mathbf{P}\{\xi(t) > u\}} \quad \text{for } \varepsilon > 0. \end{aligned} \tag{4.2}$$

As ξ is not a subordinator we have $\mathbf{P}\{\xi(h-t) \leq -\varepsilon\} > 0$ for ε small enough, see, e.g., Sato [33], Theorem 24.7. Therefore (4.2) shows that

$$\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u+\varepsilon\}}{\mathbf{P}\{\xi(t) > u\}} = 1.$$

Hence $\xi(t) \in \mathcal{L}$ by Lemma 2.4, so that $\xi(h) \in \mathcal{L}$ by the existence of the limit $\ell(t)$. □

While subordination of ξ implies (4.1), as does Condition 1 of Theorem 4.2 by Theorem 4.1, Condition 2 does not imply (4.1) as is exemplified by Brownian motion.

5. $\mathcal{O}$-exponential Lévy processes

In this section we extend Theorems 4.1 and 4.2 from the class $\mathcal{L}$ to $\mathcal{OL}$. The extension of Theorem 4.1 will be used in Section 6 in the proof of (1.2) and (1.3) for $\mathcal{C} = \mathcal{L}(\alpha)$.

Theorem 5.1. *For a Lévy process ξ we have*

$$\xi(h) \in \mathcal{OL} \Leftrightarrow \sup_{t\in[0,h]} \xi(t) \in \mathcal{OL}.$$

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Moreover, if any of these memberships holds so that both of them hold, then

$$\limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t)>u\right\} < \infty. \tag{5.1}$$

Proof. The fact that $\xi(h) \in \mathcal{OL}$ implies $\sup_{t\in[0,h]} \xi(t) \in \mathcal{OL}$ and (5.1) follows as

$$\mathbf{P}\{\xi(h)>x\} \le \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t)>x\right\} \le \frac{\mathbf{P}\{\xi(h)>x-1\}}{\mathbf{P}\{\inf_{t\in[0,h]} \xi(t)>-1\}} \le \frac{C\,\mathbf{P}\{\xi(h)>x\}}{\mathbf{P}\{\inf_{t\in[0,h]} \xi(t)>-1\}}$$

for x large enough for some constant $C > 0$, where the middle inequality follows from Sato [33], Remark 45.9. Conversely, if $\sup_{t\in[0,h]} \xi(t) \in \mathcal{OL}$, then by the same inequality

$$\begin{aligned} &\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(h)>u+x\}}{\mathbf{P}\{\xi(h)>u\}} \\ &\quad \ge \liminf_{u\to\infty} \mathbf{P}\left\{\inf_{t\in[0,h]} \xi(t)>-1\right\} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t)>u+x+1\right\} \Big/ \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t)>u\right\} > 0 \end{aligned}$$

for $x > 0$, so that $\xi(h) \in \mathcal{OL}$ by Lemma 2.8. □

Theorem 5.2. *For a Lévy process ξ such that $-\xi$ is not a subordinator and* (5.1) *holds one of the following two conditions holds:*

1. $\xi(h) \in \mathcal{OL}$;
2. $$\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(t)>u\}}{\mathbf{P}\{\xi(h)>u\}} = 0 \quad \textit{for } t \in (0,h).$$

Proof. If (5.1) holds and the liminf in Condition 2 equals $\ell(t) > 0$ for a $t \in (0, h)$, then

$$\begin{aligned} \infty &> \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t)>u\right\} \\ &\ge \mathbf{P}\{\xi(h-t)\ge\varepsilon\} \limsup_{u\to\infty} \frac{\mathbf{P}\{\xi(t)>u-\varepsilon\}}{\mathbf{P}\{\xi(h)>u\}} \\ &\ge \ell(t)\mathbf{P}\{\xi(h-t)\ge\varepsilon\} \limsup_{u\to\infty} \frac{\mathbf{P}\{\xi(h)>u-\varepsilon\}}{\mathbf{P}\{\xi(h)>u\}} \quad \text{for } \varepsilon>0. \end{aligned}$$

As $-\xi$ is not a subordinator we get $\mathbf{P}\{\xi(h-t) \ge \varepsilon\} > 0$ for $\varepsilon > 0$ small enough as in the proof of Theorem 4.2. Hence we see that $\xi(h) \in \mathcal{OL}$ using Lemma 2.8. □

6. Exponential Lévy processes

For Lévy processes in $\mathcal{S}(\alpha)$ Braverman and Samorodnitsky [14], Theorem 3.1, proved that

$$\lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t)>u\right\} = H \quad \text{exists with value } H \in [1,\infty). \tag{6.1}$$

Although Braverman [11], Theorem 2.1, expresses H in terms of the characteristic triple, he also notes that the expression typically cannot be evaluated except for subordinators.

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The next theorem extends Theorem 4.1 from $\mathcal{L}$ to $\mathcal{L}(\alpha)$ as well as (6.1) from $\mathcal{S}(\alpha)$ to $\mathcal{L}(\alpha)$ assuming (6.2) given below. It seems that specific processes in $\mathcal{L}(\alpha)$ always satisfy (6.2) (see Sections 3 and 7), making our result very useful in practice, while we are unsure of the real theoretical significance of (6.2). Our proof is quite short and transparent while in the literature already proofs of (6.1) for $\mathcal{S}(\alpha)$ are long and difficult.

Theorem 6.1. *For a constant $\alpha \geq 0$ and a Lévy process ξ such that*

$$L(t) = \lim_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}} \quad \textit{exists for } t \in (0,h) \tag{6.2}$$

we have

$$\xi(h) \in \mathcal{L}(\alpha) \Leftrightarrow \sup_{t\in[0,h]} \xi(t) \in \mathcal{L}(\alpha) \quad \textit{for } \alpha \geq 0.$$

Moreover, if any of these memberships holds so that both of them hold, then (6.1) holds. In that case we have $H = 1$ if $L(t) = 0$ for $t \in (0,h)$.

Proof. Assume that $\xi(h) \in \mathcal{L}(\alpha)$. Note that for each $t \in (0,h)$ with $L(t) > 0$ we have $\xi(t) \in \mathcal{L}(\alpha)$ by (6.2). This in turn by inspection of (2.1) means that

$$\lim_{u\to\infty} \mathbf{P}\{\xi(t) - u > x \mid \xi(t) > u\} = \mathrm{e}^{-\alpha x} \quad \text{for } x \geq 0. \tag{6.3}$$

Letting η be an $\exp(\alpha)$ distributed random variable that is independent of ξ (6.3) gives

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} \\
&\geq \limsup_{a\downarrow 0} \liminf_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\max_{k=0,\dots,\lfloor h/a\rfloor} \xi(h-ka) > u\right\} \\
&= \limsup_{a\downarrow 0} \liminf_{u\to\infty} \sum_{k=0}^{\lfloor h/a\rfloor} \frac{\mathbf{P}\{\xi(h-ka) > u\}}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\bigcap_{\ell=0}^{k-1} \{\xi(h-\ell a) \leq u\} \,\middle|\, \xi(h-ka) > u\right\} \\
&= \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor h/a\rfloor} L(h-ka)\, \mathbf{P}\left\{\bigcap_{\ell=0}^{k-1} \{\xi((k-\ell)a) + \eta \leq 0\}\right\}
\end{aligned} \tag{6.4}$$

(where $\bigcap_{\ell=0}^{-1}$ is the empty intersection, that is, the whole sample space). For a matching upper bound, note that the strong Markov property gives

$$\begin{aligned}
&\mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u + x\right\} \\
&\leq \mathbf{P}\left\{\max_{k=0,\dots,\lfloor h/a\rfloor} \xi(h-ka) > u\right\} + \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u + x\right\} \mathbf{P}\left\{\inf_{t\in[0,a]} \xi(t) < -x\right\} \\
&\quad \text{for } x > 0.
\end{aligned}$$

From this together with (6.4) and the fact that $\xi(h) \in \mathcal{L}(\alpha)$ we get

$$\limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\}$$

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$$= \limsup_{x\downarrow 0} \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u+x\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u+x\right\}$$

$$\le \limsup_{x\downarrow 0} \liminf_{a\downarrow 0} \limsup_{u\to\infty} \frac{e^{\alpha x}}{\mathbf{P}\{\xi(h) > u\}}$$

$$\times \mathbf{P}\left\{\max_{k=0,\ldots,\lfloor h/a\rfloor} \xi(h-ka) > u\right\} \Big/ \mathbf{P}\left\{\inf_{t\in[0,a]} \xi(t) \ge -x\right\}$$

$$\le \liminf_{a\downarrow 0} \sum_{k=0}^{\lfloor h/a\rfloor} L(h-ka)\,\mathbf{P}\left\{\bigcap_{\ell=0}^{k-1}\{\xi((k-\ell)a)+\eta \le 0\}\right\}. \tag{6.5}$$

From (6.4) together with (6.5) we conclude that (6.1) holds with

$$H = \lim_{a\downarrow 0} \sum_{k=0}^{\lfloor h/a\rfloor} L(h-ka)\,\mathbf{P}\left\{\bigcap_{\ell=0}^{k-1}\{\xi((k-\ell)a)+\eta \le 0\}\right\}. \tag{6.6}$$

Here $H \ge 1$ by (6.1) with $H = 1$ if $L(t) = 0$ for $t \in (0,h)$ by (6.6), while $H < \infty$ by Theorem 5.1.

Conversely, assume that $\sup_{t\in[0,h]} \xi(t) \in \mathcal{L}(\alpha)$. To finish the proof we have to show that $\xi(h) \in \mathcal{L}(\alpha)$. Assume that $\alpha > 0$ as we are done otherwise by Theorem 4.1. Observe that it is enough to show that given any $x \ge 0$ we have

$$\lim_{n\to\infty} \frac{\mathbf{P}\{\xi(h) > u_n + x\}}{\mathbf{P}\{\xi(h) > u_n\}} = e^{-\alpha x} \tag{6.7}$$

for any sequence $u_n \to \infty$ as $n \to \infty$ such that the limit (6.7) really exists. This is so because the ratio in (6.7) is bounded so that every subsequence of that ratio has a further subsequence that converges to the limit $e^{-\alpha x}$. It follows that (2.1) holds for $x \ge 0$, which in turn gives (2.1) in general by an elementary argument.

Consider the distributions supported on $[0,\infty)$ with CDF given by

$$F_n(x) = \mathbf{P}\{\xi(h) \le u_n + x \mid \xi(h) > u_n\} = 1 - \frac{\mathbf{P}\{\xi(h) > u_n + x\}}{\mathbf{P}\{\xi(h) > u_n\}} \quad \text{for } x \ge 0.$$

For suitable constants $N \in \mathbb{N}$ and $C, \varepsilon > 0$, Theorem 5.1 gives [use (5.1) in the first step]

$$\limsup_{x\to\infty} \sup_{n\ge N} (1 - F_n(x))$$

$$\le \limsup_{x\to\infty} C \sup_{n\ge N} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_n + x\right\} \Big/ \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_n\right\}$$

$$\le \limsup_{x\to\infty} C \sup_{n\ge N} \prod_{k=1}^{\lfloor x\rfloor} \frac{\mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_n + k\right\}}{\mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_n + k - 1\right\}}$$

$$\le \limsup_{x\to\infty} C \sup_{n\ge N} \prod_{k=1}^{\lfloor x\rfloor} (1+\varepsilon)\, e^{-\alpha} = 0.$$

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Hence the sequence $\{F_n\}_{n=1}^{\infty}$ is tight in the sense of weak convergence. Therefore Prohorov's theorem shows that there exists a weakly convergent subsequence $F_{n_k} \stackrel{\mathrm{d}}{\to} F$.

Letting η be a random variable with CDF F that is independent of ξ (6.4) gives

$$\liminf_{k\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u_{n_k}\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_{n_k}\right\} \geq \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor h/a \rfloor} L(h-ka)\,\mathbf{P}\left\{\bigcap_{\ell=0}^{k-1} \{\xi((k-\ell)a) + \eta \leq 0\}\right\}.$$

Using that $\sup_{t\in[0,h]} \xi(t) \in \mathcal{L}(\alpha)$ we further get the following version of (6.5):

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u_{n_k}\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_{n_k}\right\} \\
&= \limsup_{x\downarrow 0} \limsup_{k\to\infty} \frac{\mathrm{e}^{\alpha x}}{\mathbf{P}\{\xi(h) > u_{n_k}\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_{n_k} + x\right\} \\
&\leq \liminf_{a\downarrow 0} \sum_{k=0}^{\lfloor h/a \rfloor} L(h-ka)\,\mathbf{P}\left\{\bigcap_{\ell=0}^{k-1} \{\xi((k-\ell)a) + \eta \leq 0\}\right\}.
\end{aligned}$$

With the notation (6.6) we thus obtain the following version of (6.1):

$$\mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_{n_k}\right\} \sim H\,\mathbf{P}\{\xi(h) > u_{n_k}\} \quad \text{as } k \to \infty$$

with $H \in [1,\infty)$ as before. This gives the required (6.7) since $\sup_{t\in[0,h]} \xi(t) \in \mathcal{L}(\alpha)$. □

We immediately get the following powerful corollary to Theorem 6.1. See also Braverman [13], Theorem 2, and Section 8 below on priority issues:

Corollary 6.2. *For a Lévy process ξ satisfying (3.4) we have (6.1) with $H = 1$.*

We now complete the result (6.1) of Braverman [11] and Braverman and Samorodnitsky [14] for $\mathcal{S}(\alpha)$ to a result in the fashion of Theorem 4.1. We also show that $H > 1$ unless ξ is a subordinator which does not follow from the mentioned literature.

Corollary 6.3. *For a constant $\alpha \geq 0$ and a Lévy process ξ we have*

$$\frac{\nu([1,\infty)\cap\cdot)}{\nu([1,\infty))} \in \mathcal{S}(\alpha) \Leftrightarrow \xi(h) \in \mathcal{S}(\alpha) \Leftrightarrow \sup_{t\in[0,h]} \xi(t) \in \mathcal{S}(\alpha) \quad \text{and (6.2)}.$$

Moreover, if any of these memberships holds so that all of them hold, then (6.1) holds. In that case we have $H > 1$ unless $\alpha = 0$ or ξ is a subordinator.

Proof. The left equivalency in the corollary follows from Theorem 3.1, as does the fact that $\xi(h) \in \mathcal{S}(\alpha)$ implies (6.2). Hence Theorem 6.1 shows that $\xi(h) \in \mathcal{S}(\alpha)$ also implies $\sup_{t\in[0,h]} \xi(t) \in \mathcal{S}(\alpha)$ as (6.1) holds. Conversely, the fact that $\sup_{t\in[0,h]} \xi(t) \in \mathcal{S}(\alpha)$ and (6.2) imply $\xi(h) \in \mathcal{S}(\alpha)$ follows from Theorem 6.1 alone, again as (6.1) holds.

As any of the equivalent statements in the corollary implies that (6.2) holds with $\xi(h) \in \mathcal{L}(\alpha)$, Theorem 6.1 shows that they also imply (6.1).

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If $\alpha > 0$, then Condition 1 of Theorem 4.2 fails. As $\xi(h) \in \mathcal{S}(\alpha)$ implies that the limit in (6.2) is strictly positive, by Theorem 3.1, also Condition 2 of Theorem 4.2 fails. Since ξ is not a subordinator Theorem 4.2 thus shows that

$$\limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} > 1,$$

which combines with (6.1) to show that $H > 1$. □

7. Applications

We now consider applications of our results to GZ, CGMY and GH processes.

7.1. GZ processes

The GZ process was introduced by Grigelionis [22], thereby generalizing the z processes of Prentice [32]. The GZ process is a Lévy process with characteristic triple

$$\left(\frac{d\nu(x)}{dx}, m, s^2\right) = \left(\frac{2\delta}{|x|(1-e^{-2\pi|x|/\zeta})}\left(e^{2\pi\beta_1 x}\mathbf{1}_{(-\infty,0)}(x) + e^{-2\pi\beta_2 x}\mathbf{1}_{(0,\infty)}(x)\right), m, 0\right), \tag{7.1}$$

where $\beta_1, \beta_2, \delta, \zeta > 0$ and $m \in \mathbb{R}$ are parameters.

Theorem 7.1. *For a GZ Lévy process (6.1) holds with $H = 1$. If in addition $h > 1/(2\delta)$, then we have*

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{e^{(A-2\delta\gamma)h}}{2\pi\beta_2\Gamma(2\delta h)} u^{2\delta h-1} e^{-2\pi\beta_2 u} \quad \text{as } u \to \infty, \tag{7.2}$$

where γ is Euler's constant and

$$A = 2\pi\beta_2 m + \int_{\mathbb{R}} \left(e^{2\pi\beta_2 x}\mathbf{1}_{(-\infty,1)}(x) - 1 - 2\pi\beta_2\mathbf{1}_{(-1,1)}(x)\,x\right) d\nu(x) + \int_1^\infty \left(e^{2\pi\beta_2 x}\frac{d\nu(x)}{dx} - \frac{2\delta}{x}\right) dx.$$

Proof. By inspection of (7.1) we see that (3.8) holds with $C = 2\delta$, $\rho = -1$ and $\alpha = 2\pi\beta_2$. Hence (3.4) holds by Theorem 3.6, so that Corollary 6.2 gives (6.1) with $H = 1$.

If $h > 1/(2\delta)$, then the hypothesis of the second part of Theorem 3.6 holds, as (3.10) holds by inspection of (7.1). Hence (3.11) applies. By (3.14) we have $\mu^{\leftarrow}(u) + 2\pi\beta_2 \sim 2\delta/u$ as $u \to \infty$, so that by dominated convergence [note that $\mu^{\leftarrow}(u/h) + 2\pi\beta_2 \geq 0$ and $(\mu^{\leftarrow}(u/h) + 2\pi\beta_2)\,u/(2h\delta) \geq 1/2$ for u large enough] and a change of variable

$$\begin{aligned}\int_1^\infty e^{-\mu^{\leftarrow}(u/h)\,x} d\nu(x) &= \int_1^\infty e^{-(\mu^{\leftarrow}(u/h)+2\pi\beta_2)\,x}\left(e^{2\pi\beta_2 x}\frac{d\nu(x)}{dx} - \frac{2\delta}{x}\right) dx + 2\delta\int_1^\infty e^{-(\mu^{\leftarrow}(u/h)+2\pi\beta_2)\,x}\frac{dx}{x} \\ &= \int_1^\infty \left(e^{2\pi\beta_2 x}\frac{d\nu(x)}{dx} - \frac{2\delta}{x}\right) dx + 2\delta\int_{2h\delta/u}^\infty \frac{e^{-x}}{x} dx + o(1) \quad \text{as } u \to \infty.\end{aligned} \tag{7.3}$$

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Using (3.1) together with (3.11), (3.14) and (7.3), elementary calculations give

$$\mathbf{P}\{\xi(h)>u\} \sim \frac{(2\delta h)^{2\delta h}\mathrm{e}^{Ah}}{2\pi\beta_2\Gamma(2\delta h)}\frac{\mathrm{e}^{-2\pi\beta_2 u}}{u}\exp\left\{2\delta h\int_{2\delta h/u}^{\infty}\frac{\mathrm{e}^{-x}}{x}\mathrm{d}x\right\}$$
$$\sim \frac{(2\delta h)^{2\delta h}\mathrm{e}^{Ah}}{2\pi\beta_2\Gamma(2\delta h)}\frac{\mathrm{e}^{-2\pi\beta_2 u}}{u}\left(\frac{\mathrm{e}^{-\gamma}u}{2\delta h}\right)^{2\delta h} \quad \text{as } u\to\infty,$$

which establishes (7.2): Here the last asymptotic relation follows from, e.g., Erdélyi, Magnus, Oberhettinger and Tricomi [20], Equations 6.9.25 and 6.7.13. □

Example 7.2. The Meixner process of Schoutens and Teugels [35] is a GZ process with $\beta_1, \beta_2 > 0$ and $\beta_1+\beta_2 = 1$. Thus it satisfies (6.1) with $H = 1$. Further, (7.2) holds when $h > 1/(2\delta)$.

Remark 7.3. According to Barndorff-Nielsen, Kent and Sørensen [7], Theorem 5.2, if for a constant $\alpha > 0$ and a PDF f the function

$$F_k(x) = \int_0^x y^k \mathrm{e}^{\alpha y} f(y)\mathrm{d}y, \quad x > 0,$$

has an ultimately monotone derivative with a Laplace transform that satisfies

$$\int_0^\infty x^k \mathrm{e}^{(\alpha+s)x} f(x)\mathrm{d}x \sim C\,\Gamma(k-\rho)(-s)^{-(k+\rho+1)} \quad \text{as } s\uparrow 0, \tag{7.4}$$

for some constants $C > 0$, $k\in\mathbb{N}$ and $\rho > -k-1$, then f satisfies (1.1). However, in general one cannot tell whether F_k has an ultimately monotone derivative just by inspection of the Laplace transform. And should such additional information on f be available the Tauberian result should typically not be needed anyway.

For example, Grigelionis [22], Corollary 1, deduces that

$$f_{\xi(t)}(u) \sim \left(\frac{2\pi\,\Gamma(\beta_1+\beta_2)}{\zeta\,\Gamma(\beta_1)\,\Gamma(\beta_2)}\right)^{2\delta t}\frac{u^{2\delta t-1}}{\Gamma(2\delta t)}\exp\left\{-\frac{2\pi\beta_2(u-mt)}{\zeta}\right\} \quad \text{as } u\to\infty$$

for GZ processes from information like (7.4) only, with the property that F_k has a monotone derivative waived as "standard calculations": We find this argument incomplete!

7.2. CGMY processes

The CGMY Lévy process of Carr, Geman, Madan and Yor [15] has characteristic triple

$$\left(\frac{\mathrm{d}\nu(x)}{\mathrm{d}x}, m, s^2\right)$$
$$= \left(C_-(-x)^{-1-Y_-}\mathrm{e}^{Gx}\mathbf{1}_{(-\infty,0)}(x) + C_+x^{-1-Y_+}\mathrm{e}^{-Mx}\mathbf{1}_{(0,\infty)}(x), m, 0\right), \tag{7.5}$$

where $C_-, C_+, G, M > 0$, $Y_-, Y_+ < 2$ and $m\in\mathbb{R}$ are parameters.

Theorem 7.4. *For a CGMY process* (6.1) *holds. Further, we have* $H > 1$ *and*

$$\mathbf{P}\{\xi(h)>u\} \sim \frac{C_+h}{M}\exp\left\{hMm + h\int_{\mathbb{R}}\left(\mathrm{e}^{Mx}-1-M\mathbf{1}_{(-1,1)}(x)\,x\right)\mathrm{d}\nu(x)\right\}\frac{\mathrm{e}^{-Mu}}{u^{1+Y_+}} \tag{7.6}$$

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as $u \to \infty$ *for* $Y_+ > 0$, *while* $H = 1$ *and*

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{M^{C_+h-1} e^{Bh}}{\Gamma(C_+h)} u^{C_+h-1} e^{-Mu} \tag{7.7}$$

as $u \to \infty$ *for* $Y_+ = 0$, *where* γ *is Euler's constant and*

$$B = Mm + \int_{\mathbb{R}} \left(e^{Mx} \mathbf{1}_{(-\infty,0)}(x) - 1 - M\mathbf{1}_{(-1,1)}(x)\, x \right) d\nu(x),$$

and $H = 1$ *and*

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{(C_+h\Gamma(1-Y_+))^{-1/(2(1-Y_+))}}{M\sqrt{2\pi(1-Y_+)}\, u^{(1-Y_+/2)/(1-Y_+)}} \times \exp\left\{ -Mu + \frac{(1+1/\Gamma(1-Y_+))\, u^{-Y_+/(1-Y_+)}}{(C_+h\Gamma(1-Y_+))^{-1/(1-Y_+)}} + Bh \right\} \tag{7.8}$$

as $u \to \infty$ *for* $Y_+ < 0$.

Proof. For $Y_+ > 0$, (7.5) and Example 2.11 show that $\nu([1,\infty) \cap \cdot)/\nu([1,\infty)) \in \mathcal{S}(M)$, so that Corollary 6.3 gives $\xi(h) \in \mathcal{S}(M)$ and (6.1) with $H > 1$ (as ξ is not a subordinator). Further, we get (7.6) by insertion of (7.5) in the second part of Theorem 3.1.

For $Y_+ = 0$, (7.5) shows that (3.8) holds with $\rho = -1$, so that Theorem 3.6 gives (3.4). Hence (6.1) holds with $H = 1$ by Corollary 6.2. To prove (7.7), write $\xi = \xi_1 + \xi_2$ where ξ_1 and ξ_2 are independent Lévy processes with triplets $(\nu((0,\infty) \cap \cdot), C_+(1 - e^{-M})/M, 0)$ and $(\nu((-\infty,0) \cap \cdot), m - C_+(1 - e^{-M})/M, 0)$, respectively. Then ξ_1 is a gamma subordinator, see, e.g., Schoutens [34], Section 5.3.3, and satisfies

$$\mathbf{P}\{\xi_1(h) > u\} \sim \frac{M^{C_+h-1}}{\Gamma(C_+h)} u^{C_+h-1} e^{-Mu} \quad \text{as } u \to \infty, \tag{7.9}$$

so that $\xi_1(h) \in \mathcal{L}(M)$. As $\mathbf{E}\{e^{\beta\xi_2(h)}\} < \infty$ for $\beta \geq 0$ [recall (3.6)] it follows from Pakes [29], Lemma 2.1 (see also [30]), that [cf. (3.2)]

$$\mathbf{P}\{\xi(h) > u\} \sim \mathbf{P}\{\xi_1(h) > u\}\mathbf{E}\{e^{M\xi_2(h)}\} \quad \text{as } u \to \infty. \tag{7.10}$$

Inserting (7.9) and calculating the MGF in (7.10) using (3.1) and (7.5) we get (7.7).

For $Y_+ < 0$, (7.5) shows that (3.8) holds with $\rho = -1 - Y_+ > -1$, $\alpha = M$ and $C = C_+$, so that Corollary 3.5 gives (3.4). Hence (6.1) holds with $H = 1$ by Corollary 6.2, with the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ as $u \to \infty$ in (6.1) given by (3.9), by Corollary 3.5. To evaluate (3.9) we note that (3.7) and (7.5) give

$$\mu_1(\lambda) = C_+ \int_1^\infty x^{-Y_+} e^{-(M+\lambda)x} dx = C_+\Gamma(1-Y_+)(M+\lambda)^{Y_+-1} - \frac{C_+}{1-Y_+} + o(1)$$

$$V_1(\lambda) = C_+ \int_1^\infty x^{1-Y_+} e^{-(M+\lambda)x} dx \sim C_+\Gamma(2-Y_+)(M+\lambda)^{Y_+-2}$$

as $\lambda \downarrow -M$. It follows that

$$\mu_1^{\leftarrow}(u) = -M + \left(\frac{u - C_+/(1-Y_+)}{C_+\Gamma(1-Y_+)} \right)^{1/(Y_+-1)} + o\left(u^{1/(Y_+-1)-1}\right) \quad \text{as } u \to \infty,$$

so that

$$\begin{aligned} u\mu_1^{\leftarrow}(u/h) &\sim -Mu + (C_+ h\,\Gamma(1-Y_+))^{1/(1-Y_+)} u^{Y_+/(Y_+-1)} \\ V_1(\mu_1^{\leftarrow}(u/h)) &\sim C_+\Gamma(2-Y_+)(C_+ h\,\Gamma(1-Y_+))^{(Y_+-2)/(1-Y_+)} u^{(Y_+-2)/(Y_+-1)} \\ &= (1-Y_+)\,h^{-1}(C_+ h\,\Gamma(1-Y_+))^{-1/(1-Y_+)} u^{(Y_+-2)/(Y_+-1)} \end{aligned}$$

as $u \to \infty$. Moreover, we have

$$\begin{aligned} \ln\left(\mathbf{E}\{e^{M\xi_2(h)}\}\phi_1(\lambda)^h\right) &\sim h\int_0^\infty x^{-1-Y_+} e^{-(M+\lambda)x} dx + Bh \\ &= C_+ h\,\Gamma(-Y_+)(M+\lambda)^{Y_+} + Bh \end{aligned}$$

as $\lambda \downarrow -M$, so that

$$\begin{aligned} &\ln\left(\mathbf{E}\{e^{M\xi_2(h)}\}\phi_1(\mu_1^{\leftarrow}(u/h))^h\right) \sim C_+ h (C_+ h\,\Gamma(1-Y_+))^{Y_+/(1-Y_+)} u^{Y_+/(Y_+-1)} + Bh \\ &\quad \sim (C_+ h\,\Gamma(1-Y_+))^{1/(1-Y_+)}\Gamma(1-Y_+)^{-1} u^{Y_+/(Y_+-1)} + Bh \end{aligned}$$

as $u \to \infty$. Inserting the above findings into (3.9) we readily obtain (7.8). □

Braverman [11] and Braverman and Samorodnitsky [14] apply to CGMY for $Y_+ > 0$.

CGMY processes with $Y_-, Y_+ < 0$ are compound Poisson and light tailed in the sense of Braverman [12], Definition 1, by Example 2.11. Hence Braverman's Theorem 1 applies if the process has non-negative drift while his Theorem 3 applies if $0 < M < 1$ and the drift is negative, in both cases to yield (6.1) with $H = 1$.

Example 7.5. As the variance gamma processes of Madan and Seneta [27] are CGMY processes with $C_- = C_+$ and $Y_- = Y_+ = 0$ they satisfy (6.1) with $H = 1$ and the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ as $u \to \infty$ in (6.1) given by (7.7).

Example 7.6. The Kou [26] jump-diffusion Lévy process has characteristic triple

$$\left(\frac{d\nu(x)}{dx}, m, s^2\right) = \left((1-p)\lambda e^{\lambda_- x}\mathbf{1}_{(-\infty,0)}(x) + p\lambda e^{-\lambda_+ x}\mathbf{1}_{(0,\infty)}(x), m,\ s^2\right), \tag{7.11}$$

where $p \in (0, 1)$, $m \in \mathbb{R}$ and $\lambda, \lambda_-, \lambda_+, s^2 > 0$ are parameters.

Besides the Gaussian component s^2, (7.11) is (7.5) with $Y_- = Y_+ = -1$. Hence the proof of Theorem 7.4 carries over to show that (6.1) holds with $H = 1$ and the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ as $u \to \infty$ in (6.1) given by (3.9). The evaluation of (3.9) in the proof of Theorem 7.4 can be modified in a straightforward manner to include the Gaussian component, but we omit the details.

7.3. GH processes

The GH Lévy process introduced by Barndorff-Nielsen [3,4] has characteristic triple

$$\begin{aligned} &\left(\frac{d\nu(x)}{dx}, m, s^2\right) \\ &= \left(\frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{\exp\{-|x|\sqrt{2y+\varrho^2}\}}{\pi^2 y\,\left(J_{|\zeta|}(\delta\sqrt{2y})^2 + Y_{|\zeta|}(\delta\sqrt{2y})^2\right)} dy + \zeta^+ e^{-\varrho|x|}\right), m, 0\right), \end{aligned} \tag{7.12}$$

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where $\beta, \zeta, m \in \mathbb{R}$, $\delta > 0$ and $\varrho > |\beta|$ are parameters. Here J_ζ denotes the Bessel function and Y_ζ the Bessel function of the second kind, respectively.

Theorem 7.7. *For a GH process (6.1) holds where $H = 1$ if $\zeta \geq 0$ while $H > 1$ if $\zeta < 0$.*

Proof. Using the facts from Watson [38], Equations 3.1.8, 3.51.1, 3.52.3 and 3.54.1-2, about the asymptotic behaviour of $J_\zeta(y)$ and $Y_\zeta(y)$ as $y \downarrow 0$ we readily obtain

$$\int_0^\infty \frac{\exp\{-|x|\sqrt{2y+\varrho^2}\}}{\pi^2 y \left(J_\zeta(\delta\sqrt{2y})^2 + Y_\zeta(\delta\sqrt{2y})^2\right)} \mathrm{d}y \sim \int_0^\infty \frac{\exp\{-x(\varrho + y/\varrho)\}}{\pi^2 y Y_\zeta(\delta\sqrt{2y})^2} \mathrm{d}y$$

$$\sim \begin{cases} \displaystyle\int_0^\infty \frac{\delta^{2\zeta}[\sin(\pi\zeta)\Gamma(1-\zeta)]^2 y^{\zeta-1}\exp\{-x(\varrho + y/\varrho)\}}{\pi^2 2^\zeta} \mathrm{d}y & \text{for } \zeta \in [0,\infty)\setminus\mathbb{N} \\ \displaystyle\int_0^\infty \frac{\delta^{2\zeta} y^{\zeta-1}\exp\{-x(\varrho + y/\varrho)\}}{2^\zeta \Gamma(\zeta)^2} \mathrm{d}y & \text{for } \zeta \in \mathbb{N}\setminus\{0\} \\ \displaystyle\int_0^\infty \frac{\exp\{-x(\varrho + y/\varrho)\}}{y\ln(y/\varrho)^2} \mathrm{d}y & \text{for } \zeta = 0 \end{cases}$$

as $x \to \infty$, so that by insertion in (7.12)

$$\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} \sim \begin{cases} \dfrac{2^\zeta[\sin(\pi\zeta)\Gamma(1+\zeta)]^2\exp\{-(\varrho-\beta)x\}}{\pi^2\delta^{2\zeta}\varrho^\zeta x^{1-\zeta}} & \text{for } \zeta \in (-\infty,0]\setminus\mathbb{N} \\ \dfrac{2^\zeta \exp\{-(\varrho-\beta)x\}}{\delta^{2\zeta}\varrho^\zeta\Gamma(-\zeta)^2 x^{1-\zeta}} & \text{for } \zeta \in (-\mathbb{N})\setminus\{0\} \\ \dfrac{\exp\{-(\varrho-\beta)x\}}{\ln(2)x} & \text{for } \zeta = 0 \\ \dfrac{\zeta \mathrm{e}^{-(\varrho-\beta)x}}{x} & \text{for } \zeta > 0. \end{cases} \tag{7.13}$$

For $\zeta \geq 0$, (7.13) shows that (3.8) holds with $\rho = -1$, so that Theorem 3.6 gives (3.4). Hence Corollary 6.2 shows that (6.1) holds with $H = 1$. For $\zeta < 0$, (7.13) and Example 2.11 show that $\nu([1,\infty)\cap\cdot)/\nu([1,\infty)) \in \mathcal{S}(\varrho-\beta)$, so that Corollary 6.3 gives (6.1) with $H > 1$ (as ξ is not a subordinator). □

Braverman [11] and Braverman and Samorodnitsky [14] apply to GH for $\zeta < 0$.

Example 7.8. The normal inverse Gaussian process introduced by Barndorff-Nielsen [4,5] is a GH processes with $\zeta = -\frac{1}{2}$. Thus it satisfies (6.1) with $H > 1$.

Example 7.9. The hyperbolic process introduced by Barndorff-Nielsen [3] is a GH process with $\zeta = 1$. Thus is satisfies (6.1) with $H = 1$.

Remark 7.10. When $\zeta < 0$ the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ as $u \to \infty$ in (6.1) for GH processes is given by the second part of Theorem 3.1 together with integration of (7.13). When $\zeta \geq 0$ the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ can be calculated from (3.11) in the fashion of Theorem 7.1 for h large enough. We have omitted these calculations to avoid additional technicalities.

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8. Two priority issues

Theorem 3.3 is due to Braverman [13], Theorem 1.

Braverman showed us his result during Fall 2004, so we were aware of his finding when submitting our paper (although, frankly, we had forgotten about it for the final version of our article, so we had to be reminded about it).

Braverman [13], Theorem 2, states that the conclusion of Corollary 6.2 holds under the hypothesis of Theorem 3.3. Thus our Corollary 6.2 is more general than Braverman [13], Theorem 2. However, from a practical point of view, one may argue the importance of the added generality.

Our version of Corollary 6.2 is implicit in Chapter 5 of the thesis of Bengtsson [8] (now named Sundén) that already appeared during Fall 2004, which is also acknowledged by Braverman [13], Remark 1.

It should be noted that there is a more or less complete difference between the approach and methods of proof of Theorem 3.3 and Corollary 6.2 and those of the corresponding results of Braveman.

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Appendix. Technical details of Section 2

Here we prove Proposition 2.6 and Corollary 2.9. We remark that these results are not very far from some standard results on regular variation that can be found in e.g., Bingham, Goldie and Teugels [10], as is indeed indicated by their proofs: The results and proofs are just supplied as a service to the reader not very expert in regular variation.

A.1. Proof of Proposition 2.6

We have $F \in \mathcal{L}(\alpha)$ if and only if $1 - F(\ln(\cdot)) \in \mathcal{R}(-\alpha)$ and $F \in \mathcal{OL}$ if and only if $1-F(\ln(\cdot)) \in \mathcal{OR}$, see, e.g., Shimura and Watanabe [37], p. 451. By the representation theorems for $\mathcal{R}(-\alpha)$ and $\mathcal{OR}$ (see e.g., Bingham, Goldie and Teugels [10], Theorems 1.3.1 and 2.2.7) a function $1 - F(\ln(\cdot))$ belongs to $\mathcal{R}(-\alpha)$ $[\mathcal{OR}]$ if and only if

$$1 - F(\ln(u)) = \hat{c}(u) \exp\left\{-\int_0^u \frac{\hat{a}(x)}{x} \mathrm{d}x\right\} \quad \text{for } u \in \mathbb{R} \text{ large enough}, \tag{A.1}$$

for some measurable functions $\hat{a}$ and $\hat{c}$ such that $\lim_{x\to\infty} \hat{a}(x) = \alpha$ and $\lim_{x\to\infty} \hat{c}(x) > 0$ exists $[\limsup_{x\to\infty} |\hat{a}(x)| < \infty$ and $0 < \liminf_{x\to\infty} \hat{c}(x) \le \limsup_{x\to\infty} \hat{c}(x) < \infty]$. Since F is

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absolutely continuous with $\lim_{u\to-\infty} F(u) = 0$ and $\lim_{u\to\infty} F(u) = 1$ we can rewrite (A.1) as

$$\begin{aligned} 1 - F(u) &= \exp\left\{-\int_0^{e^u} \frac{\hat{a}(x)+\hat{b}(x)}{x}\mathrm{d}x\right\} \\ &= \exp\left\{-\int_{-\infty}^{u}\left(\hat{a}(e^x)+\hat{b}(e^x)\right)\mathrm{d}x\right\} \quad \text{for } u\in\mathbb{R}, \end{aligned}$$

where $a(x) \equiv \hat{a}(e^x)$ and $b(x) \equiv \hat{b}(e^x)$ satisfies (2.4) and (2.5), respectively, depending on whether $F \in \mathcal{L}(\alpha)$ or $F \in \mathcal{OL}$. Finally, as F is non-decreasing we have $a + b \geq 0$. □

A.2. *Proof of Corollary 2.9*

Integrating (1.1) we readily obtain

$$1 - F(u) = \int_u^\infty f(x)\mathrm{d}x \sim \frac{C}{\alpha}\, u^\rho e^{-\alpha u} \sim \frac{f(u)}{\alpha} \quad \text{as } u\to\infty. \tag{A.2}$$

In particular we see that $F \in \mathcal{L}(\alpha)$, so that (2.3) holds with $a + b \geq 0$ as in (2.4), where

$$\lim_{u\to\infty}\int_{-\infty}^{u}\left(a(x)+b(x)-\alpha\,\mathbf{1}_{[0,\infty)}(x)+\frac{\rho}{x}\,\mathbf{1}_{[1,\infty)}(x)\right)\mathrm{d}x = \ln\left(\frac{C}{\alpha}\right)$$

by (A.2). Writing $c = a + b$, we thus have (2.6) with $c \geq 0$ as well as the second part of (2.7). Differentiating both sides of (2.6) and using (A.2) we get

$$f(u) = c(u)\exp\left\{-\int_{-\infty}^{u} c(x)\mathrm{d}x\right\} = c(u)\ (1 - F(u)) \sim c(u)\frac{f(u)}{\alpha} \quad \text{as } u\to\infty, \tag{A.3}$$

which gives the first part of (2.7).

Conversely, if (2.6) and (2.7) hold, then it is immediate that (A.2) holds so that $F \in \mathcal{L}(\alpha)$, while (1.1) follows from (A.3). □

References

[1] J.M.P. Albin, A note on the closure of convolution power mixtures of exponential distributions, J. Austral. Math. Soc. (2007) http://www.math.chalmers.se/palbin/exponential.pdf (in press).
[2] J.M.P. Albin, M. Sundén, On the asymptotic behaviour of Lévy processes. Part II: Superexponential processes, Preprint, 2007. http://www.math.chalmers.se/palbin/levyII.pdf.
[3] O.E. Barndorff-Nielsen, Exponentially decreasing distributions for the logarithm of particle size, Proc. Roy. Soc. London Ser. A 353 (1977) 401–419.
[4] O.E. Barndorff-Nielsen, Hyperbolic distributions and distributions on hyperbolae, Scand. J. Statist. 5 (1978) 151–157.
[5] O.E. Barndorff-Nielsen, Normal inverse Gaussian distributions and stochastic volatility models, Scand. J. Statist. 24 (1997) 1–13.
[6] O.E. Barndorff-Nielsen, Processes of normal inverse Gaussian type, Finance Stoch. 2 (1998) 41–68.
[7] O. Barndorff-Nielsen, J. Kent, M. Sørensen, Normal variance-mean mixtures and z distributions, Internat. Statist. Rev. 50 (1982) 145–159.
[8] M. Bengtsson, On the asymptotic behaviour of Lévy processes, Licentiate Thesis, Department of Mathematics, Chalmers University of Technology, 2004. http://www.math.chalmers.se/palbin/licmattias.pdf. Bengtsson is now named Sundén.
[9] S.M. Berman, The supremum of a process with stationary independent and symmetric increments, Stochastic Process. Appl. 23 (1986) 281–290.
[10] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, Cambridge University Press, Cambridge, 1987.

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[11] M. Braverman, Suprema and sojourn times of Lévy processes with exponential tails, Stochastic Process. Appl. 68 (1997) 265–283.

[12] M. Braverman, Suprema of compound Poisson processes with light tails, Stochastic Process. Appl. 90 (2000) 145–156.

[13] M. Braverman, On a class of Lévy processes, Statist. Probab. Lett. 75 (2005) 179–189.

[14] M. Braverman, G. Samorodnitsky, Functionals of infinitely divisible stochastic processes with exponential tails, Stochastic Process. Appl. 56 (1995) 207–231.

[15] P. Carr, H. Geman, D.B. Madan, M. Yor, Stochastic volatility for Lévy processes, Math. Finance 13 (2003) 345–382.

[16] P. Embrechts, M. Goldie, On closure and factorization properties of subexponential and related distributions, J. Austral. Math. Soc. (Ser. A) 29 (1980) 243–256.

[17] P. Embrechts, M. Goldie, On convolution tails, Stochastic Process. Appl. 13 (1982) 263–278.

[18] P. Embrechts, M. Goldie, N. Veraverbeke, Subexponentiality and infinite divisibility, Zeitschrift Wahrsch. 49 (1979) 335–347.

[19] P. Embrechts, J.L. Jensen, M. Maejima, J.L. Teugels, Approximations for compound Poisson and Pólya processes, Adv. Appl. Probab. 17 (1985) 623–637.

[20] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, vol. I, McGraw-Hill, New York, 1953.

[21] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Tables of Integral Transforms, vol. I, McGraw-Hill, New York, 1954.

[22] B. Grigelionis, Generalized z-distributions and related stochastic processes, Lithuanian Math. J. 41 (2001) 239–251.

[23] P. Homble, W.P. McCormick, Weak limit results for the extremes of a class of shot noise processes, Technical Report, Department of Statistics, University of Georgia, 1993. http://www.stat.uga.edu/tech_reports/papers/shot.noise.pdf.

[24] P. Homble, W.P. McCormick, Weak limit results for the extremes of a class of shot noise processes, J. Appl. Probab. 32 (1995) 707–726.

[25] J.L. Jensen, Uniform saddlepoint approximations, Adv. Appl. Probab. 20 (1988) 622–634.

[26] S. Kou, A jump-diffusion model for option pricing, Management Sci. 48 (2002) 1086–1101.

[27] D.B. Madan, E. Seneta, Chebyshev polynomial approximations and characteristic function estimation, J. Roy. Statist. Soc. Ser. B 49 (1987) 163–169.

[28] M.B. Marcus, ξ-Radial Processes and Random Fourier Series, American Mathematical Society, Providence, 1987.

[29] A.G. Pakes, Convolution equivalence and infinite divisibility, J. Appl. Probab. 41 (2004) 407–424.

[30] A.G. Pakes, Convolution equivalence and infinite divisibility: Corrections and corollaries, J. Appl. Probab. 44 (2007) 295–305.

[31] E.J.G. Pitman, Subexponential distribution functions, J. Austral. Math. Soc. (Ser. A) 29 (1980) 337–347.

[32] R.L. Prentice, Discrimination among some parametric models, Biometrika 62 (1975) 607–614.

[33] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge, 1999.

[34] W. Schoutens, Lévy processes in Finance, Wiley, New York, 2003.

[35] W. Schoutens, J.L. Teugels, Lévy processes, polynomials and martingales, Comm. Statist. Stochastic Models 14 (1998) 335–349.

[36] M.S. Sgibnev, Asymptotics of infinitely divisible distribution in $\mathbb{R}$, Sibirsk. Mat. Zh. 31 (1990) 135–140 (in Russian). English translation in Siberian Math. J., 31, 115–119.

[37] T. Shimura, T. Watanabe, Infinite divisibility and generalized subexponentiality, Bernoulli 11 (2005) 445–469.

[38] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1944.

[39] E. Willekens, On the supremum of an infinitely divisible process, Stochastic Process. Appl. 26 (1987) 173–175.

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On the Asymptotic Behaviour of Lévy processes.

Part II: Superexponential Lévy processes

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Abstract

We study tail probabilities of suprema of Lévy processes with superexponential marginal distributions over compact intervals. Several of the processes for which the asymptotics are studied here for the first time have recently become important in financial modeling. Hence our results may be of importance in the assessment of financial risk.

Short title: Superexponential Lévy Processes

1 Introduction

In this work, which may be considered as a second part of Albin and Sundén [3], we study extremes of superexponential Lévy processes.

By a Lévy process we mean a stochastically continuous process $\xi = \{\xi(t)\}_{t\geq 0}$ starting at $\xi(0) = 0$, that has stationary and independent increments. Writing $\kappa(x) = x/(1 \vee |x|)$ for $x \in \mathbb{R}$, the finite dimensional distributions of a Lévy process are fully determined by its so called characteristic triple (ν, m, s^2) through the relation

$$\mathbf{E}\big\{\mathrm{e}^{i\theta\xi(t)}\big\} = \exp\bigg\{it\theta m + t\int_{\mathbb{R}}\big(\mathrm{e}^{i\theta x} - 1 - i\theta\kappa(x)\big)\,d\nu(x) - \frac{t\theta^2 s^2}{2}\bigg\} \quad \text{for } \theta \in \mathbb{R} \text{ and } t \geq 0.$$

Here $m \in \mathbb{R}$ and $s^2 \geq 0$ are constants while ν is the so called Lévy measure on $\mathbb{R}$ that satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\,d\nu(x) < \infty$.

We call a Lévy process ξ superexponential if $\mathbf{E}\{\mathrm{e}^{\alpha\xi(1)}\} < \infty$ for $\alpha \geq 0$. It follows from Sato [23], Theorem 25.17, that ξ is superexponential if and only if it has a well-defined Laplace transform

$$\phi_t(\lambda) = \mathbf{E}\{\mathrm{e}^{-\lambda\xi(t)}\} = \phi_1(\lambda)^t < \infty \quad \text{for } \lambda \leq 0 \text{ and } t > 0. \tag{1.1}$$

Under technical conditions on the superexponential Lévy process ξ, we establish the existence of a constant $H \geq 1$ such that

$$\lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}}\mathbf{P}\bigg\{\sup_{t\in[0,h]} \xi(t) > u\bigg\} = H \quad \text{for } h > 0. \tag{1.2}$$

For the verification of these technical conditions, we provide a series of lemmas, propositions and theorems, which in turn are established by means of some new Tauberian techniques that we develop for this purpose.

We also find the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ for large u in (1.2), in closed form, so that the asymptotic behaviour of $\mathbf{P}\{\sup_{t\in[0,h]} \xi(t) > u\}$ is fully understood.

We now state our main result on extremes of superexponential Lévy processes. See Albin and Sundén [3] on bibliographic information for results on this type. In addition, two relevant references not mentioned there are Albin [1] and Braverman [9, 11].

Theorem 1.1. *Let ξ be a separable superexponential Lévy process with infinite upper end-point [see (2.1) below]. Assume that there exist functions $w > 0$ and $q > 0$, with w continuous, such that*

$$\frac{\xi(aq(u))}{w(u)} \stackrel{d}{\to} \zeta(a) \quad \textit{as } u \to \infty \textit{ for } a > 0 \tag{1.3}$$

and

$$L(t,x) = \lim_{u\to\infty} \frac{\mathbf{P}\{\xi(h-q(u)t) > u + xw(u)\}}{\mathbf{P}\{\xi(h) > u\}} \quad \textit{exists for } t > 0 \textit{ and } x \in \mathbb{R} \tag{1.4}$$

with $L(0,x) = \mathrm{e}^{-x}$. *Further, assume that* $\zeta(a)$ *is continuously distributed for* $a > 0$, *or that* $L(t,\cdot)$ *is a continuous function for* $t > 0$. *If*

$$\lim_{T\to\infty} \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{ \sup_{t\in[0,h-Tq(u)]} \xi(t) > u \right\} = 0, \tag{1.5}$$

then the limit (1.2) exists with value $H \in [1,\infty)$.

The constant H in (1.2) is a rather complicated functional of the quantities ζ and L, see the proof of Theorem 1.1 in Section 4 for more information. Clearly, it seems, in general H cannot be calculated in closed form. However, as we will see below, in some cases we encounter H really can be calculated. Also, in other cases, qualitative information such as whether $H > 1$ or not can be established.

The structure of the paper is as follows: In Section 2 we give an array of results that are helpful to verify two of the key conditions (1.3) and (1.4) of Theorem 1.1. In the same fashion, Section 3 is devoted to develop a tool to verify the condition (1.5). In Section 4 we give the proof of Theorem 1.1, while in the concluding Section 5 we consider an array of examples of application of our results to important specific superexponential Lévy processes. Except for the first example (Brownian motion), the conclusions of these examples are new.

Section 2 is very technical and makes up for more than half the bulk of the paper on its own. We belive that for most readers it is best to skip the details of Section 2 in a first reading and instead move on to study Sections 3 and 4. Then, when need arises in the examples of Section 5 to verify the conditions of Theorem 1.1, it might be more appropriate to go back to study selected details of Section 2.

2 Tauberian results for superexponential processes

The condition (1.4) of Theorem 1.1 is closely related to the notion of Type I domain of attraction of extremes:

Definition 2.1. *A random variable X belongs to the* Type I domain of attraction *of extremes, with* auxiliary function $w(u) > 0$, *if*

$$\lim_{u\to\infty} \frac{\mathbf{P}\{X > u + xw(u)\}}{\mathbf{P}\{X > u\}} = \mathrm{e}^{-x} \quad \textit{for } x \in \mathbb{R}.$$

The auxiliary function in Definition 2.1 satisfies $\lim_{u\to\infty} w(u)/u = 0$ and can be chosen to be continuous (see e.g, Bingham, Goldie and Teugels [8], Lemma 3.10.1 and Corollary 3.10.9). Further, $\tilde{w}$ is another auxiliary for X function if and only if $\lim_{u\to\infty} \tilde{w}(u)/w(u) = 1$.

Feigin and Yashchin [15], Theorems 2 and 3, give a scheme to deduce the asymptotics of the right tail of a probability distribution function from the the left tail of its Laplace transform. The usefulness of this to establish Type I attraction was noted in a particular case by Davis and Resnick, [12], Section 3, see also Rootzén [20, 21]. Balkema, Klüppelberg and Resnick [4, 5, 6] and Balkema, Klüppelberg and Stadtmüller [7] characterized convergence of the Esscher transforms (exponential families), which are the key ingredient of proofs in this area. But they impose conditions on densities that we are not comfortable with. And it is not that convergence which is our goal, but to find the actual tail behaviour and to show Type I attraction. In fact, we have to deal with random variables, the distribution of which depend on how far out we are in the tail (an "external parameter"). This makes the existing literature non-applicable anyway.

Our main proposed route to verify the condition (1.4) as well as to find the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ for large u in Theorem 1.1 will be the following Theorem 2.2. To prepare for that result, recall that the right end-point $\sup\{x \in \mathbb{R} : \mathbf{P}\{\xi(t) > x\} > 0\}$ of a Lévy process ξ is infinite for some $t > 0$ if and only if

$$\sup\big\{x \in \mathbb{R} : \mathbf{P}\{\xi(t) > x\} > 0\big\} = \infty \quad \text{for each } t > 0 \tag{2.1}$$

(see e.g., Sato [23], Theorem 24.10). Moreover, for a Lévy process ξ with Laplace transform (1.1), we introduce the following notation:

$$\begin{cases} \mu(\lambda) = -\dfrac{\phi_1'(\lambda)}{\phi_1(\lambda)} = \displaystyle\int_{\mathbb{R}} \big(x\mathrm{e}^{-\lambda x} - \kappa(x)\big)\, d\nu(x) + m - \lambda s^2 & \text{for } \lambda \le 0, \\ \sigma(\lambda)^2 = -\mu'(\lambda) = \displaystyle\int_{\mathbb{R}} x^2 \mathrm{e}^{-\lambda x}\, d\nu(x) + s^2 & \text{for } \lambda \le 0, \\ \mu^{\leftarrow}(u) = \inf\{\lambda \in \mathbb{R} : \mu(\lambda) \le u\} & \text{for } u > 0 \text{ large enough.} \end{cases} \tag{2.2}$$

The so called Esscher transform of ξ_t is defined to be a random variable $X_{t,\lambda}$ having probability distribution

$$dF_{X_{t,\lambda}}(x) = \frac{\mathrm{e}^{\lambda x} dF_{\xi(t)}(x)}{\phi_t(\lambda)}, \tag{2.3}$$

where $F_{\xi(t)}$ denotes the cummulative probability distribution function of ξ_t. It is easy to see that $\mu(\lambda)$ and $\sigma(\lambda)^2$ are the mean and variance of $X_{t,\lambda}$, respectively.

The following Theorem 2.2 will be crucial for us to verify the condition (1.4) of Theorem 1.1. It is a development of a scheme of Feigin and Yashchin [15], and Davis and Resnick [12], with additional input from Albin [2], to establish Type I attraction for infinitely divisible probability distributions. The key to verify the rather involved conditions of Theorem 2.2 in turn is Proposition 2.8 below, which gives sufficient conditions in terms of the characteristic triple for the conditions of Theorem 2.2 to hold.

Theorem 2.2. *Let ξ be a superexponential Lévy process with characteristic triple (ν, m, s^2) and infinite upper end-point (2.1). With the notation (2.2), assume that*

$$\lim_{\lambda \to -\infty} \lambda^2 \sigma(\lambda)^2 = \infty, \tag{2.4}$$

$$\lim_{\lambda \to -\infty} \int_{|x| > \varepsilon \sigma(\lambda)} \frac{x^2}{\sigma(\lambda)^2} \mathrm{e}^{-\lambda x} d\nu(x) = 0 \quad \textit{for } \varepsilon > 0 \tag{2.5}$$

and

$$\lim_{K \to \infty} \limsup_{\lambda \to -\infty} \int_{|\theta| > K} \exp\left\{-t\left[\int_{\mathbb{R}} \left(1 - \cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right) \mathrm{e}^{-\lambda x} d\nu(x) + \frac{\theta^2 s^2}{2\sigma(\lambda)^2}\right]\right\} d\theta = 0 \tag{2.6}$$

for t in a neighborhood of $h > 0$. Further assume that the following limit exists

$$\lim_{\lambda \to -\infty} \frac{\lambda \mu(\lambda)}{\lambda \mu(\lambda) + \ln(\phi_1(\lambda))} = L. \tag{2.7}$$

With the notation

$$w(u) = -\frac{1}{\mu^{\leftarrow}(u/h)} \quad \textit{and} \quad q(u) = \frac{1}{\ln(\phi_1(\mu^{\leftarrow}(u/h)))}, \tag{2.8}$$

we have $\lim_{u \to \infty} q(u)/w(u) = 0$,

$$\lim_{u \to \infty} \frac{\mathbf{P}\{\xi(h - q(u)t) > u + xw(u)\}}{\mathbf{P}\{\xi(h) > u\}} = \mathrm{e}^{-t-x} \quad \textit{for } x \in \mathbb{R} \textit{ and } t > 0, \tag{2.9}$$

as well as

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{\mathrm{e}^{u\mu^{\leftarrow}(u/h)} \phi_1(\mu^{\leftarrow}(u/h))^h}{\sqrt{2\pi h}\, \sigma(\mu^{\leftarrow}(u/h))(-\mu^{\leftarrow}(u/h))} \quad \textit{as } u \to \infty. \tag{2.10}$$

Proof. Our first aim is to establish asymptotic normality of a normalized Esscher transform $(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)$ to be defined below. To that end, note that by (2.1), we have

$$\int_{-1}^{0}(-x)d\nu(x)=\infty \quad \text{or} \quad \nu((0,\infty))>0 \quad \text{or} \quad s>0 \tag{2.11}$$

(see e.g. Sato [23], Theorem 24.7). For the function μ it is therefore easy to see that

$$\begin{aligned}\mu(\lambda)&=\int_{-\infty}^{0}\big(\mathrm{e}^{-\lambda x}x-\kappa(x)\big)\,d\nu(x)+\int_{0}^{\infty}\big(\mathrm{e}^{-\lambda x}x-\kappa(x)\big)\,d\nu(x)+(-\lambda)s^2+m\\&\to\infty \quad \text{as } \lambda\to-\infty\end{aligned} \tag{2.12}$$

[note that all terms on the right of the equality in (2.12) are non-negative]. Further, observe that

$$Q(\lambda)\equiv\frac{1}{\ln(\phi_1(\lambda))}=\bigg(\int_{\mathbb{R}}\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,d\nu(x)+m\lambda+\frac{\lambda^2s^2}{2}\bigg)^{-1}$$

satisfies

$$Q(\lambda)>0 \qquad \text{for } \lambda \text{ sufficiently small, with} \qquad \lim_{\lambda\to-\infty}Q(\lambda)=0: \tag{2.13}$$

This follows readily when $\nu((0,\infty))>0$ or $s^2=0$ [recall (2.11)] using that

$$\int_{-1}^{0}\big(\mathrm{e}^{-\lambda x}-1+\lambda x\big)\,d\nu(x)=o(\lambda^2) \quad \text{as } \lambda\to-\infty. \tag{2.14}$$

This in turn is so because

$$\begin{aligned}\int_{-1}^{0}\big(\mathrm{e}^{-\lambda x}-1+\lambda x\big)\,d\nu(x)&=\bigg[\frac{\mathrm{e}^{-\lambda x}-1+\lambda x}{x^2}\int_{-1}^{x}y^2d\nu(y)\bigg]_{-1}^{0}\\&\quad-\lambda^2\int_{0}^{-\lambda}\frac{x\,\mathrm{e}^{-x}+x+2\,\mathrm{e}^{-x}-2}{x^3}\bigg(\int_{-1}^{x/\lambda}y^2d\nu(y)\bigg)dx\\&\to\lambda^2\int_{-1}^{0}y^2d\nu(y)\bigg(\frac{1}{2}-\int_{0}^{\infty}\frac{x\,\mathrm{e}^{-x}+x+2\,\mathrm{e}^{-x}-2}{x^3}dx\bigg)\\&=0 \quad \text{as } \lambda\to-\infty\end{aligned} \tag{2.15}$$

(as the right-most inner integral of the calculation equals 1/2). Further, if instead $\nu((0,\infty))=s^2=0$, then (2.13) holds since (2.11) ensures that

$$\lim_{\lambda\to-\infty}\frac{1}{|\lambda|}\int_{-\infty}^{0}\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,d\nu(x)\geq\lim_{\lambda\to-\infty}\frac{1}{2}\int_{-\infty}^{2/\lambda}-\kappa(x)d\nu(x)=\infty.$$

As a final preparation we observe that

$$\lim_{\lambda\to-\infty}\int_{\mathbb{R}}\bigg(\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\bigg)\,\mathrm{e}^{-\lambda x}\,d\nu(x)=0 \quad \text{for } \theta\in\mathbb{R}. \tag{2.16}$$

This is so because (2.5) gives

$$\limsup_{\lambda\downarrow-\alpha}\int_{|x|>\varepsilon\sigma(\lambda)}\left|\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\right|\mathrm{e}^{-\lambda x}\,d\nu(x)\le\limsup_{\lambda\downarrow-\alpha}\int_{|x|>\varepsilon\sigma(\lambda)}\frac{2|\theta|\,x^2}{\varepsilon\,\sigma(\lambda)^2}\mathrm{e}^{-\lambda x}\,d\nu(x)$$
$$=0\quad\text{for }\varepsilon>0,$$

while, by Taylor expansion, given any $\delta>0$, and for $\varepsilon=\varepsilon(\theta)>0$ sufficiently small,

$$\limsup_{\lambda\downarrow-\alpha}\int_{|x|\le\varepsilon\sigma(\lambda)}\left|\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\right|\mathrm{e}^{-\lambda x}\,d\nu(x)\le\limsup_{\lambda\downarrow-\alpha}\int_{|x|\le\varepsilon\sigma(\lambda)}\frac{\delta\theta^2x^2}{\sigma(\lambda)^2}\mathrm{e}^{-\lambda x}\,d\nu(x)$$
$$\le\delta\theta^2.$$

Let $Z_{t,\lambda}$ be the Esscher transform of $\xi(h-Q(\lambda)t)$ given by

$$dF_{Z_{t,\lambda}}(x)=\frac{\mathrm{e}^{-\lambda x}dF_{\xi(h-Q(\lambda)t)}(x)}{\phi_{h-Q(\lambda)t}(\lambda)}\quad\text{for }x\in\mathbb{R}\text{ and }t>0,$$

for $\lambda\le0$ sufficiently small [recall (2.3)]. Notice that, writing

$$\big(d\nu_{t,\lambda}(x),m_{t,\lambda},s^2_{t,\lambda}\big)=(h-Q(\lambda)t)\Big(\mathrm{e}^{-\lambda x}\,d\nu(x),\;m-\int_{\mathbb{R}}\kappa(x)\,(1-\mathrm{e}^{-\lambda x})\,d\nu(x)-\lambda s^2,\;s^2\Big),$$

the random variable $Z_{t,\lambda}$ has characteristic function

$$\mathbf{E}\{\mathrm{e}^{i\theta Z_{t,\lambda}}\}=\exp\Big\{i\theta m_{t,\lambda}+\int_{\mathbb{R}}\big(\mathrm{e}^{i\theta x}-1-i\theta\kappa(x)\big)\,d\nu_{t,\lambda}(x)-\frac{\theta^2s^2_{t,\lambda}}{2}\Big\}$$

for $\theta\in\mathbb{R}$ and $t>0$, for $\lambda\le0$ sufficiently small. Hence the random variable $Z_{t,\lambda}$ is infinitely divisible with characteristic triple $(\nu_{t,\lambda},m_{t,\lambda},s^2_{t,\lambda})$. Observing that

$$\mathbf{E}\{Z_{t,\lambda}\}=(h-Q(\lambda)t)\mu(\lambda)\equiv\mu_{t,\lambda}$$

it follows that

$$\mathbf{E}\{\mathrm{e}^{i\theta Z_{t,\lambda}}\}=\exp\Big\{i\theta\mu_{t,\lambda}+\int_{\mathbb{R}}(\mathrm{e}^{i\theta x}-1-i\theta x)d\nu_{t,\lambda}(x)-\frac{\theta^2s^2_{t,\lambda}}{2}\Big\}$$

(see e.g., Sato [23], p. 39). Hence the characteristic function $g_{t,\lambda}$ of $(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)$ is given by

$$g_{t,\lambda}(\theta)=\Big(\exp\Big\{-\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)$$
$$-\,i\int_{\mathbb{R}}\Big(\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)-\frac{\theta^2s^2}{2\sigma(\lambda)^2}\Big\}\Big)^{h-Q(\lambda)t}.$$

Here (2.5) and (2.16) together with (2.13) and a Taylor expansion readily give $\lim_{\lambda\to-\infty}$ $g_{t,\lambda}(\theta)=\mathrm{e}^{-h\theta^2/2}$ for $\theta\in\mathbb{R}$ and $t>0$. Since $1-\cos(x)\ge x^2/4$ for $|x|\le1$ we further have

$$\int_{\mathbb{R}}|g_{t,\lambda}(\theta)|\,d\theta=\int_{\mathbb{R}}\exp\Big\{-(h-Q(\lambda)t)\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\mathrm{e}^{-\lambda x}\,d\nu(x)\Big\}d\theta$$
$$\le\int_{|\theta|>K}\exp\Big\{-(h-Q(\lambda)t)\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\mathrm{e}^{-\lambda x}\,d\nu(x)\Big\}d\theta$$
$$+\int_{|\theta|\le K}\exp\Big\{-(h-Q(\lambda)t)\,\frac{\theta^2}{4\sigma(\lambda)^2}\int_{|x|\le\sigma(\lambda)/K}x^2\mathrm{e}^{-\lambda x}\,d\nu(x)\Big\}d\theta.$$

Here the first term on the right-hand side can be made arbitrarily small as $\lambda \to -\infty$ and $K \to \infty$ (on that order) using (2.6). For the second term on the right-hand side, (2.5) and (2.13) show that there exists a constant $\delta = \delta(K) \in (0,1)$ such that

$$\int_{|\theta|\le K} \exp\left\{ -(h-Q(\lambda)t)\,\frac{\theta^2}{4\sigma(\lambda)^2} \int_{|x|\le\sigma(\lambda)/K} x^2 \mathrm{e}^{-\lambda x}\, d\nu(x) \right\} d\theta$$
$$\le \int_{|\theta|\le K} \exp\left\{ -h(1-\delta)\,\frac{\theta^2}{4} \right\} d\theta \quad \text{for } \lambda \text{ small enough..}$$

The integrability of $|g_{t,\lambda}|$ established in the previous paragraph together with the Riemann-Lebesgue lemma show that $(Z_{t,\lambda} - \mu_{t,\lambda})/\sigma(\lambda)$ has a well-defined continuous probability density function $f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}$ for λ small enough. Furthermore, using (2.6) again, we readily see that

$$\begin{aligned}
&\limsup_{\lambda\downarrow-\infty} \sup_{x\in\mathbb{R}} \left| f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(x) - \frac{1}{\sqrt{2\pi h}}\,\mathrm{e}^{-x^2/(2h)} \right| \\
&\le \limsup_{K\to\infty} \limsup_{\lambda\downarrow-\infty} \left(\int_{|\theta|\le K} \left| g_{t,\lambda}(\theta) - \mathrm{e}^{-h\theta^2/2} \right| d\theta + \int_{|\theta|>K} \left(|g_{t,\lambda}(\theta)| + \mathrm{e}^{-h\theta^2/2} \right) d\theta \right) \qquad (2.17) \\
&= 0.
\end{aligned}$$

Observing that

$$f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(x) = \frac{\mathrm{e}^{-\lambda(\sigma(\lambda)x+\mu_{t,\lambda})} f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} + \sigma(\lambda)x)\,\sigma(\lambda)}{\phi_{h-Q(\lambda)t}(\lambda)}$$

for $x \in \mathbb{R}$ and $t \ge 0$. Hence (2.12) together with (2.4) and (2.17) show that

$$\begin{aligned}
f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} + x/\lambda) &= \mathrm{e}^x \frac{f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(x/(\lambda\sigma(\lambda)))\,\mathrm{e}^{\lambda\mu_{t,\lambda}} \phi_{h-Q(\lambda)t}(\lambda)}{\sigma(\lambda)} \\
&\sim \mathrm{e}^x \frac{\mathrm{e}^{\lambda\mu_{t,\lambda}} \phi_1(\lambda)^{h-Q(\lambda)t}}{\sqrt{2\pi h}\,\sigma(\lambda)} \qquad (2.18) \\
&\sim \mathrm{e}^{x-t} \frac{\mathrm{e}^{h\lambda\mu(\lambda)} \phi_1(\lambda)^h}{\sqrt{2\pi h}\,\sigma(\lambda)} \quad \text{as } \lambda \to -\infty.
\end{aligned}$$

We are now prepared to establish (2.9): By the the asymptotic behaviour (2.18) of $f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} + x/\lambda)$ together with application of (2.4) and (2.17), we get

$$\begin{aligned}
&\lim_{\lambda\to-\infty} \frac{-\lambda\,\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda} - y/\lambda\}}{f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} - x/\lambda)} \\
&= \mathrm{e}^x \lim_{\lambda\to-\infty} \int_y^\infty \frac{f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} - z/\lambda)}{f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda})}\, dz \qquad (2.19) \\
&= \mathrm{e}^x \lim_{\lambda\to-\infty} \int_y^\infty \mathrm{e}^{-z} \frac{f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(-z/(\lambda\sigma(\lambda)))}{f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(0)}\, dz \\
&= \mathrm{e}^{x-y} \quad \text{for } x, y \in \mathbb{R}.
\end{aligned}$$

Observing that

$$\frac{-\lambda f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} + (Lt-y)/\lambda)}{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda} + Lt/\lambda\}} \quad y \geq 0, \tag{2.20}$$

is a probability density function, (2.19) and the theorem of Scheffé [24] show that

$$\begin{aligned}
&\lim_{\lambda\to-\infty} \frac{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda}+(Lt-x)/\lambda\}}{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda}+Lt/\lambda\}} \\
&\quad = \lim_{\lambda\to-\infty} \int_x^\infty \frac{f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} + (Lt-y)/\lambda)}{-Q(\lambda)\mu(\lambda)\,\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda}+Lt/\lambda\}}\,dy \\
&\quad = \int_x^\infty \mathrm{e}^{-y} dy = \mathrm{e}^{-x} \quad \text{for } x \geq 0.
\end{aligned} \tag{2.21}$$

Using this in turn, together with (2.18) and (2.19), we readily obtain

$$\lim_{\lambda\to-\infty} \frac{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda} + (Lt-x)/\lambda\}}{\mathbf{P}\{\xi(h) > \mu_{t,\lambda} + Lt/\lambda\}} = \mathrm{e}^{-x-t} \quad \text{for } x,t \geq 0.$$

As (2.7) shows that, given any $\varepsilon > 0$,

$$\mu_{t,\lambda} + \frac{Lt+\varepsilon}{\lambda} \leq h\mu(\lambda) \leq \mu_{t,\lambda} + \frac{Lt-\varepsilon}{\lambda} \quad \text{for } \lambda \text{ small enough,}$$

we may now conclude that

$$\begin{aligned}
&\limsup_{\lambda\to-\infty} \frac{\mathbf{P}\{\xi(h-Q(\lambda)t) > h\mu(\lambda) - x\lambda\}}{\mathbf{P}\{\xi(h) > h\mu(\lambda)\}} \\
&\leq \limsup_{\lambda\to-\infty} \frac{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda} + (Lt+ |!\varepsilon-x)/\lambda\}}{\mathbf{P}\{\xi(h) > \mu_{t,\lambda} + (Lt-\varepsilon)/\lambda\}} \\
&= \limsup_{\lambda\to-\infty} \frac{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda} + (Lt+\varepsilon-x)/\lambda\}}{\mathbf{P}\{\xi(h) > \mu_{t,\lambda} + Lt/\lambda\}} \frac{\mathbf{P}\{\xi(h) > \mu_{t,\lambda} + Lt/\lambda\}}{\mathbf{P}\{\xi(h) > \mu_{t,\lambda} + (Lt-\varepsilon)/\lambda\}} \\
&= \mathrm{e}^{2\varepsilon-x-t} \\
&\to \mathrm{e}^{-x-t} \quad \text{as } \varepsilon \downarrow 0.
\end{aligned}$$

Treating the corresponing lim inf in an entirely similar fashion, it follows that

$$\lim_{\lambda\to-\infty} \frac{\mathbf{P}\{\xi(h-Q(\lambda)t) > h\mu(\lambda) - x\lambda\}}{\mathbf{P}\{\xi(h) > h\mu(\lambda)\}} = \mathrm{e}^{-x-t} \quad \text{for } x,t \geq 0. \tag{2.22}$$

As μ is continuous and eventually strictly decreasing [by (2.4)], with $\mu(\lambda) \to \infty$ if and only if $\lambda \to -\infty$, we may substitute $\lambda = \mu^{\leftarrow}(u)$ in (2.22), to obtain

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\xi(h-q(hu)t) > hu + xw(hu)\}}{\mathbf{P}\{\xi(h) > hu\}} = \mathrm{e}^{-x-t} \quad \text{for } x,t \geq 0. \tag{2.23}$$

From (2.23) it is a simple matter to establish (2.9) in full generality with $x \in \mathbb{R}$ rather for $x \geq 0$ only. Further, the asymptotics (2.10) follow from inspection of (2.18) and (2.19). Finally, by inspection of (2.8), the limit $\lim_{u\to\infty} q(u)/w(u) = 0$ holds if $\lim_{\lambda\to-\infty} \lambda/\ln(\phi_1(\lambda)) = 0$. However, this latter limit holds by the arguments we use to establish (2.13). This finishes the proof of all claims of the theorem. □

Remark 2.3. For $\hat{h} \in (0, h)$ it is possible, with extra work, to prove a version of Theorem 2.2 where (2.9) holds uniformly (in an obvious sense) for $t \in [0, (h - \hat{h})/q(u)]$. As we do not need this extension, we do not elaborate on it.

To check all the technical conditions of Theorem 2.2 we provide Proposition 2.8 below, the proof of which involves the following concepts of regular variation at 0:

Definition 2.4. *A monotone function $f > 0$ is* regularly varying *as $x \uparrow 0$ with index $\alpha \in \mathbb{R}$, denoted $f \in \mathcal{R}_{0^-}(\alpha)$, if*

$$\lim_{x \uparrow 0} \frac{f(yx)}{f(x)} = y^{\alpha} \quad \textit{for } y > 0.$$

Definition 2.5. *A monotone function $f > 0$ is O-*regularly varying *as $x \uparrow 0$, with Matuszewska indices $-\infty < \alpha \le \beta < \infty$, denoted $f \in \mathcal{OR}_{0^-}(\alpha, \beta)$, if for some constant $x_0 < 0$ and for each $\varepsilon > 0$, there exists a constant $C \ge 1$, such that*

$$\frac{y^{\beta+\varepsilon}}{C} \le \frac{f(yx)}{f(x)} \le C y^{\alpha-\varepsilon} \quad \textit{for } x \in [x_0, 0) \textit{ and } y \in (0, 1],$$

where α and β are the largest and smallest numbers, respectively, such that these two inequalities hold.

By Potter's theorem (see e.g. Bingham, Goldie and Teugels [8], Theorem 1.5.6), we have $\mathcal{R}_{0^-}(\alpha) \subseteq \mathcal{OR}_{0^-}(\alpha, \alpha)$ for $\alpha \in \mathbb{R}$.

The next lemma which is used in the proof of Proposition 2.8 is a version at 0 of the Stieltjes version of Karamata's theorem for one-sided indices at ∞ (see e.g., Bingham, Goldie and Teugels [8], Section 2.6.2):

Lemma 2.6. *For $U \in \mathcal{OR}_{0^-}(\alpha, \beta)$ nondecreasing with $-2 < \alpha \le \beta < 0$, we have*

$$0 < \liminf_{x \uparrow 0} \frac{1}{x^2 U(x)} \int_x^0 y^2 \, dU(y) \le \limsup_{x \uparrow 0} \frac{1}{x^2 U(x)} \int_x^0 y^2 \, dU(y) < \infty. \qquad (2.24)$$

Proof. We have $\lim_{x \uparrow 0} x^2 U(x) = 0$ because

$$\limsup_{x \uparrow 0} x^2 \frac{U(x)}{U(x_0)} \le \limsup_{x \uparrow 0} \frac{C x^{2+\alpha-\varepsilon}}{x_0^{\beta+\varepsilon}} = 0 \quad \text{for } \varepsilon > 0 \text{ small enough.}$$

From this in turn we get the upper bound noticing that

$$\int_x^0 \frac{y^2 \, dU(y)}{x^2 U(x)} = 2 \int_x^0 \frac{(-y) U(y)}{x^2 U(x)} \, dy - 1 = 2 \int_0^1 \frac{z U(zx)}{U(x)} \, dz - 1 \le 2 \int_0^1 C z^{\alpha+1-\varepsilon} dz - 1$$

where the right-hand side is finite for $\varepsilon > 0$ small enough. Further, as we have

$$\limsup_{z\downarrow 0} z\int_z^1 \frac{U(yx)}{U(x)}\,dy \le \limsup_{z\downarrow 0} z\int_z^1 Cy^{\alpha-\varepsilon}dy = \limsup_{z\downarrow 0}\frac{C(z-z^{\alpha+2-\varepsilon})}{\alpha+1-\varepsilon} = 0$$

for $x \in [x_0, 0)$ and $\varepsilon > 0$ small enough, Fatou's Lemma gives

$$\liminf_{x\uparrow 0}\int_x^0 \frac{y^2\,dU(y)}{x^2U(x)} = 2\liminf_{x\uparrow 0}\int_0^1 \frac{zU(zx)}{U(x)}dz - 1 \ge 2\int_0^1\left(\int_z^1 \liminf_{x\uparrow 0}\frac{U(yx)}{U(x)}dy\right)dz - 1. \tag{2.25}$$

Since $U(yx)/U(x) \ge 1$ is a nondecreasing function of $y \in (0,1)$, the lim inf on the left in (2.24) can be 0 only if $\liminf_{x\uparrow 0} U(yx)/U(x) = 1$ for $y \in (0,1)$, as otherwise the right-hand side of (2.25) is strictly greater than $2\int_0^1(\int_z^1 dy)dz - 1 = 0$. And so the lim inf on the left in (2.24) must be strictly greater than 0, because

$$\liminf_{x\uparrow 0}\frac{U(yx)}{U(x)} \ge \frac{y^{\beta+\varepsilon}}{C} > 1 \quad \text{for } \varepsilon, y > 0 \text{ small enough.}$$

□

Our second lemma, which is also used in the proof of Proposition 2.8, is a version at 0 of the de Haan-Stadtmüller theorem (see e.g., Bingham, Goldie and Teugels [8], Theorem 2.10.2):

Lemma 2.7. *For $U \in \mathcal{OR}_{0^-}(\alpha,\beta)$ nonincreasing with $0 < \alpha \le \beta < \infty$ and $\int_{-\infty}^1 e^{-\lambda x}\,d(-U)(x) < \infty$ for λ small enough, we have*

$$0 < \liminf_{\lambda\to-\infty}\int_{-\infty}^0 \frac{e^{-\lambda x}\,d(-U)(x)}{U(1/\lambda)} \le \limsup_{\lambda\to-\infty}\int_{-\infty}^0 \frac{e^{-\lambda x}\,d(-U)(x)}{U(1/\lambda)} < \infty.$$

Proof. As $U(-x/\lambda)/U(1/\lambda) \le 1$ with

$$\frac{U(-x/\lambda)}{U(1/\lambda)} \le C(-x)^{\alpha-\varepsilon} < 1 \quad \text{for } x \in [-1,0) \text{ and } \lambda \text{ small enough,}$$

we have

$$\liminf_{\lambda\to-\infty}\int_{-\infty}^0 \frac{\mathrm{e}^{-\lambda x}d(-U)(x)}{U(1/\lambda)} \ge \liminf_{\lambda\to-\infty}\int_{1/\lambda}^0 \frac{\mathrm{e}^{-\lambda x}d(-U)(x)}{U(1/\lambda)} = \frac{1}{\mathrm{e}} - \limsup_{\lambda\to-\infty}\int_{-1}^0 \mathrm{e}^x\frac{U(-x/\lambda)}{U(1/\lambda)}dx$$

with the right-hand side is strictly positive [cf. the concluding argument for (2.24) in the proof of Lemma 2.6]. Turning upwards the fact that $\lim_{\lambda\to-\infty}\mathrm{e}^{\lambda/2}/U(1/\lambda) = 0$ gives

$$\begin{aligned}\int_{-\infty}^0 \frac{\mathrm{e}^{-\lambda x}\,d(-U)(x)}{U(1/\lambda)} &= \frac{\mathrm{e}^{\lambda}U(-1)}{U(1/\lambda)} - \int_{-1}^0 \mathrm{e}^{-\lambda x}\frac{U(x)}{U(1/\lambda)}\,dx + \frac{\mathrm{e}^{\lambda/2}}{U(1/\lambda)}\int_{-\infty}^1 \mathrm{e}^{-\lambda x/2}\,d(-U)(x)\\ &\le \frac{\mathrm{e}^{\lambda}U(-1)}{U(1/\lambda)} + \frac{\mathrm{e}^{\lambda/2}}{U(1/\lambda)}\int_{-\infty}^1 \mathrm{e}^{-\lambda x/2}\,d(-U)(x)\\ &\to 0 \quad \text{as } \lambda\to-\infty.\end{aligned}$$

□

As have been mentioned already, the following proposition is a key result to verify the conditions of Theorem 2.2:

Proposition 2.8. *For a superexponential Lévy process ξ with characteristic triple (ν, m, s^2) and infinite upper end-point (2.1), we have the following implications:*

1. *If $s^2 > 0$, then (2.4) and (2.6) hold.*
2. *If $s^2 > 0$ and $\nu((0,\infty)) = 0$, then (2.4)-(2.7) hold.*
3. *If $\nu((0,\infty)) > 0$, then (2.4) and (2.7) hold.*
4. *Equations (2.4), (2.5) and (2.7) hold if $\nu((0,\infty)) > 0$ and there exists a non-decreasing function g such that*

$$\lim_{x\to\infty} \frac{g(x)}{\ln(x)} = \infty \quad \textit{and} \quad \int_1^\infty \exp\{g(x)x\} d\nu(x) < \infty. \tag{2.26}$$

5. *Equations (2.4)-(2.7) hold if*

$$x_0 \equiv \sup\big\{x : \nu((x,\infty)) > 0\big\} \in (0,\infty)$$

and ν is absolutely continuous with a version of $d\nu(x)/dx$ that is bounded, strictly positive for $x \in (x_1, x_2)$ for some $0 < x_1 < x_2 \le x_0$ and satisfies

$$\frac{d\nu(x)}{dx} \sim Cx^{-1-\rho} \quad \textit{as } x \downarrow 0^+ \quad \textit{for some constants } C > 0 \textit{ and } \rho \in (0,2). \tag{2.27}$$

6. *Equations (2.4)-(2.7) hold if ν is absolutely continuous with $\sup\big\{x : \nu((x,\infty)) > 0\big\} = \infty$, if ν satisfies (2.26) and (2.27), and if ν has a version of $d\nu(x)/dx$ that is ultimately decreasing.*
7. *Equation (2.4) holds if*

$$\nu((-\infty,\cdot)) \in \mathcal{OR}_{0^-}(\alpha,\beta) \quad \textit{for some constants } -2 < \alpha \le \beta < 0. \tag{2.28}$$

8. *If $\nu((0,\infty)) = 0$ and (2.4) holds, then (2.5) holds.*
9. *If $\nu((0,\infty)) = 0$ and (2.28) holds, then (2.4)-(2.6) hold.*
10. *If ξ is selfdecomposable, then (2.4) and (2.5) hold.*

11. If $\nu((0,\infty))=0$ and $d\nu(x)=k(x)dx/|x|^2$ for $x<0$ where $k>0$ is non-decreasing, then (2.4)-(2.6) hold.

12. Equations (2.4)-(2.7) hold if $\nu((0,\infty))=0$ and

$$\nu((-\infty,\cdot))\in\mathcal{R}_{0^-}(\alpha) \quad \textit{for some constant} \quad -2<\alpha<-1. \tag{2.29}$$

Proof. Statement 1 of the proposition is quite immediate.

To prove Statement 2, notice that

$$\limsup_{\lambda\to-\infty}\int_{|x|>\varepsilon\sigma(\lambda)}\frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}\,d\nu(x)\le\limsup_{\lambda\to-\infty}\frac{1}{s^2}\int_{-\infty}^{0}x^2\mathrm{e}^{-\lambda x}\,d\nu(x)=0 \tag{2.30}$$

when $s^2>0$ and $\nu((0,\infty))=0$, so that (2.5) holds. In view of Statement 1 it thus remains to prove (2.7). To that end it is sufficient to show that the limit

$$\lim_{\lambda\to-\infty}\frac{\ln(\phi_1(\lambda))}{\lambda\mu(\lambda)}=\lim_{\lambda\to-\infty}\frac{\int_{\mathbb{R}}\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,d\nu(x)+m\lambda+\lambda^2s^2/2}{\int_{\mathbb{R}}\big(\lambda x\mathrm{e}^{-\lambda x}-\lambda\kappa(x)\big)\,d\nu(x)-m\lambda-\lambda^2s^2} equiv \tilde{L} \tag{2.31}$$

exists and is not equal to -1. As it is obvious that

$$\int_{-\infty}^{-1}\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,d\nu(x)=O(\lambda)\quad\text{and}\quad\int_{-\infty}^{-1}\big(\lambda x\mathrm{e}^{-\lambda x}-\lambda\kappa(x)\big)\,d\nu(x)=O(\lambda)$$

as $\lambda\to-\infty$, (2.31) with $\tilde{L}=-\frac{1}{2}$ will in turn follow provided that we prove that

$$\int_{-1}^{0}\big(\mathrm{e}^{-\lambda x}-1+\lambda x\big)\,d\nu(x)=o(\lambda^2)\quad\text{and}\quad\int_{-1}^{0}\big(\lambda x\mathrm{e}^{-\lambda x}-\lambda x\big)\,d\nu(x)=o(\lambda^2)$$

as $\lambda\to-\infty$. The first of these asymptotic relations is established in (2.15). The second asymptotic relation follows in a simlar fashion noticing that, by integration by parts,

$$\begin{aligned}\int_{-1}^{0}\big(\lambda x\mathrm{e}^{-\lambda x}-\lambda x\big)\,d\nu(x)&=-\lambda^2\int_{0}^{-\lambda}\frac{x\mathrm{e}^{-x}+\mathrm{e}^{-x}-1}{x^2}\bigg(\int_{-1}^{x/\lambda}y^2d\nu(y)\bigg)dx\\&\sim\lambda^2\int_{-1}^{0}y^2d\nu(y)\bigg(-1-\int_0^\infty\frac{x\mathrm{e}^{-x}+\mathrm{e}^{-x}-1}{x^2}dx\bigg),\end{aligned}$$

where the second inner integral on the righ-hand side equals -1, so that the whole expression under consideration is $o(\lambda^2)$ as $\lambda\to-\infty$, as required.

To prove Statement 3, notice that (2.14) readily gives (2.4). Further, by inspection of the proof of Statement 2, (2.7) holds with $L=1$ if

$$\lim_{\lambda\to-\infty}\int_0^\infty\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,d\nu(x)\Big/\bigg(\int_0^\infty\big(\lambda\kappa(x)-\lambda x\mathrm{e}^{-\lambda x}\big)\,d\nu(x)\bigg)=0$$

and

$$\lim_{\lambda\to-\infty}\frac{1}{\lambda^2}\int_0^\infty\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,d\nu(x)=\lim_{\lambda\to-\infty}\frac{1}{\lambda^2}\int_0^\infty\big(\lambda\kappa(x)-\lambda x\,\mathrm{e}^{-\lambda x}\big)\,d\nu(x)=\infty.$$

However, both these requirements are quite obvious consequences of the fact that

$$\int_0^1\big(\lambda\kappa(x)-\lambda x\mathrm{e}^{-\lambda x}\big)\,d\nu(x)\geq\int_0^1\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,d\nu(x)\geq 0.$$

To prove Statement 4, notice that by (2.26), the function $G(x)=g(\sqrt{x})$ is non-decreasing with

$$\lim_{x\to\infty}\frac{G(x)}{\ln(x)}=\infty\quad\text{and}\quad\int_1^\infty\exp\big\{G(x^2)x\big\}\,d\nu(x)<\infty. \tag{2.32}$$

As we must have $\nu((\underline{x},\infty))>0$ for some $\underline{x}>0$, (2.32) gives that

$$\begin{aligned}\liminf_{\lambda\to-\infty}\frac{G(\varepsilon^2\sigma(\lambda)^2)}{-\lambda}&\geq\liminf_{\lambda\to-\infty}\frac{1}{-\lambda}\,G\bigg(\varepsilon^2\int_{\underline{x}}^\infty x^2\mathrm{e}^{-\lambda x}d\nu(x)\bigg)\\&\geq\liminf_{\lambda\to-\infty}\frac{1}{-\lambda}\,G\Big(\varepsilon^2\underline{x}^2\nu((\underline{x},\infty))\mathrm{e}^{-\lambda\underline{x}}\Big)\\&=\infty\quad\text{for }\varepsilon>0.\end{aligned}$$

From this in turn we readily obtain, making use of (2.32) again [see also (2.30)],

$$\begin{aligned}&\limsup_{\lambda\to-\infty}\int_{|x|>\varepsilon\sigma(\lambda)}\frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}d\nu(x)\\&\leq\limsup_{\lambda\to-\infty}\frac{1}{\sigma(\lambda)^2}\int_{-\infty}^0x^2\mathrm{e}^{-\lambda x}d\nu(x)+\Big(\sup_{x<0}x^2\mathrm{e}^x\Big)\bigg(\limsup_{\lambda\to-\infty}\frac{1}{\lambda^2\sigma(\lambda)^2}\int_{|x|>\varepsilon\sigma(\lambda)}\mathrm{e}^{-2\lambda x}d\nu(x)\bigg)\\&\leq 0+\Big(\sup_{x<0}x^2\mathrm{e}^x\Big)\bigg(\int_1^\infty\exp\big\{G(x^2)x\big\}d\nu(x)\bigg)\Big(\limsup_{\lambda\to-\infty}\sup_{x>\varepsilon\sigma(\lambda)}\exp\big\{-2\lambda x-G(x^2)x\big\}\Big)\\&=0\quad\text{for }\varepsilon>0.\end{aligned}$$

Hence (2.5) holds. The statement now follows from Statement 3.

To prove Statement 5, notice that Statement 4 shows that (2.4), (2.5) and (2.7) hold. Using the elementary inequality $1-\cos(x)\geq x^2/4$ for $|x|\leq 1$ we further get

$$\begin{aligned}&\limsup_{\lambda\to-\infty}\int_{K<|\theta|\leq\sigma(\lambda)/x_0}\exp\bigg\{-t\int_{\mathbb{R}}\bigg(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\bigg)\,\mathrm{e}^{-\lambda x}d\nu(x)\bigg\}d\theta\\&\qquad\leq\limsup_{\lambda\to-\infty}\int_{|\theta|>K}\exp\bigg\{-\int_0^{x_0}\frac{t\theta^2x^2}{4\sigma(\lambda)^2}\mathrm{e}^{-\lambda x}d\nu(x)\bigg\}d\theta\\&\qquad=\int_{|\theta|>K}\exp\Big\{-\frac{t\theta^2}{4}\Big\}d\theta\\&\qquad\to 0\quad\text{as }K\to\infty.\end{aligned} \tag{2.33}$$

Further, using (2.27) to find a $\delta\in(0,1\wedge x_0)$ such that $d\nu(x)/dx\geq\frac{1}{2}Cx^{-1-\rho}$ for $x\in(0,\delta)$, we get in a similar fashion

$$\begin{aligned}
&\limsup_{\lambda\to-\infty}\int_{\sigma(\lambda)/x_0<|\theta|<\sigma(\lambda)\sqrt{-\lambda}}\exp\Big\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)\Big\}d\theta\\
&\quad\le\limsup_{\lambda\to-\infty}\int_{\sigma(\lambda)/x_0<|\theta|<\sigma(\lambda)\sqrt{-\lambda}}\exp\Big\{-\int_{(\delta\sigma(\lambda)/|\theta|)/2}^{\delta\sigma(\lambda)/|\theta|}\frac{Ct\theta^2x^{1-\rho}}{8\sigma(\lambda)^2}\mathrm{e}^{-\lambda x}dx\Big\}d\theta\\
&\quad\le\limsup_{\lambda\to-\infty}\int_{|\theta|>\sigma(\lambda)/x_0}\exp\Big\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\mathrm{e}^{\delta\sqrt{-\lambda}/2}\Big\}d\theta\\
&\quad=0,
\end{aligned}\tag{2.34}$$

where we made use of the simple fact that

$$\limsup_{\lambda\to-\infty}\sigma(\lambda)^2\mathrm{e}^{\lambda x_0}=0\tag{2.35}$$

to get the last equality. Finally, we have, based in part on a slight modification of (2.34), and noticing the quick oscillations of the cosinus function,

$$\begin{aligned}
&\limsup_{\lambda\to-\infty}\int_{|\theta|>\sigma(\lambda)\sqrt{-\lambda}}\exp\Big\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)\Big\}d\theta\\
&\le\limsup_{\lambda\to-\infty}\int_{|\theta|>\sigma(\lambda)\sqrt{-\lambda}}\exp\Big\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}-\frac{t}{2}\int_{x_1}^{x_2}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)\Big\}d\theta\\
&\le\limsup_{\lambda\to-\infty}\int_{|\theta|>\sigma(\lambda)\sqrt{-\lambda}}\exp\Big\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}-t\,\frac{x_2-x_1}{4}\inf_{x\in(x_1,x_2)}\frac{d\nu(x)}{dx}\,\mathrm{e}^{-\lambda x_1}\Big\}d\theta\\
&=0,
\end{aligned}\tag{2.36}$$

using (2.35) at the end again. Putting (2.33), (2.34) and (2.36) together we arrive at (2.6).

To prove Statement 6, notice that Statement 4 shows that (2.4), (2.5) and (2.7) hold. As

$$2\ln(y)-\lambda y-yg(y)-\big(2\ln(x)-\lambda x-xg(x)\big)\le(y-x)(2-\lambda-g(x))\quad\text{for }1\le x\le y$$

by the first part of (2.26), we can further find a function $x_0(\lambda)$ with $\lim_{\lambda\to-\infty}x_0(\lambda)=\infty$ and

$$\lim_{\lambda\to-\infty}\exp\{\lambda\varepsilon\}x_0(\lambda)=0\quad\text{for }\varepsilon>0,\tag{2.37}$$

such that $2\ln(x)-\lambda x-xg(x)$ is non-increasing for $x\ge x_0(\lambda)$, giving, by the second part of (2.26),

$$\int_{x_0(\lambda)}^\infty x^2\mathrm{e}^{-\lambda x}d\nu(x)\le\mathrm{e}^{2\ln(x_0(\lambda))-\lambda x_0(\lambda)-x_0(\lambda)g(x_0(\lambda))}\int_1^\infty\mathrm{e}^{xg(x)}d\nu(x)\to0\tag{2.38}$$

as $\lambda \to -\infty$. Now by (2.38), the argument for (2.33) in the proof of Statement 5 carries over to show that

$$\lim_{K\to\infty} \limsup_{\lambda\to-\infty} \int_{K<|\theta|\le\sigma(\lambda)/x_0(\lambda)} \exp\Big\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\mathrm{e}^{-\lambda x}d\nu(x)\Big\}d\theta = 0. \quad (2.39)$$

Notice that (2.38) also gives

$$\sigma(\lambda)^2 \sim \int_0^{x_0(\lambda)} x^2\mathrm{e}^{-\lambda x}\,d\nu(x) \le x_0(\lambda)^2\mathrm{e}^{-\lambda x_0(\lambda)}\int_0^\infty (1\wedge x^2)\,d\nu(x) \quad \text{as } \lambda\to-\infty,$$

from which we readily conclude that (2.37) implies

$$\lim_{\lambda\to-\infty}\frac{\exp\{-\lambda\varepsilon\}}{x_0(\lambda)^\rho\ln(\sigma(\lambda))} = \infty \quad \text{for } \varepsilon>0. \quad (2.40)$$

By (2.40) in turn, there exists a function $f(\lambda)>0$ with $\lim_{\lambda\to-\infty} f(\lambda)=0$ such that

$$\lim_{\lambda\to-\infty}\frac{\exp\{-\lambda f(\lambda)\}}{x_0(\lambda)^\rho\ln(\sigma(\lambda))} = \infty. \quad (2.41)$$

Selecting $\delta\in(0,1)$ such that $d\nu(x)/dx \ge \frac{1}{2}Cx^{-1-\rho}$ for $x\in(0,\delta)$, the analogue of (2.34) in the proof of Statement 5 now becomes

$$\begin{aligned}
&\limsup_{\lambda\to-\infty}\int_{\sigma(\lambda)/x_0(\lambda)<|\theta|<\delta\sigma(\lambda)/(2f(\lambda))}\exp\Big\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\mathrm{e}^{-\lambda x}d\nu(x)\Big\}d\theta\\
&\le \limsup_{\lambda\to-\infty}\int_{\sigma(\lambda)/x_0(\lambda)<|\theta|<\delta\sigma(\lambda)/(2f(\lambda))}\exp\Big\{-\int_{(\delta\sigma(\lambda)/|\theta|)/2}^{\delta\sigma(\lambda)/|\theta|}\frac{Ct\theta^2x^{1-\rho}}{8\sigma(\lambda)^2}\mathrm{e}^{-\lambda x}dx\Big\}d\theta\\
&\le \limsup_{\lambda\to-\infty}\int_{|\theta|>\sigma(\lambda)/x_0(\lambda)}\exp\Big\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\mathrm{e}^{-\lambda f(\lambda)}\Big\}d\theta\\
&= \limsup_{\lambda\to-\infty}\frac{2}{\rho}\Big(\frac{Ct(1-2^{\rho-2})}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\mathrm{e}^{-\lambda f(\lambda)}\Big)^{-1/\rho}\Gamma\Big(\frac{1}{\rho},\Big(\frac{\sigma(\lambda)}{x_0(\lambda)}\Big)^\rho\frac{Ct(1-2^{\rho-2})}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\mathrm{e}^{-\lambda f(\lambda)}\Big)\\
&= \limsup_{\lambda\to-\infty}\frac{2}{\rho}\Big(\frac{\sigma(\lambda)}{x_0(\lambda)}\Big)^{1-\rho}\Big(\frac{Ct(1-2^{\rho-2})}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\mathrm{e}^{-\lambda f(\lambda)}\Big)^{-1}\\
&\qquad\qquad\times\exp\Big\{-\Big(\frac{\sigma(\lambda)}{x_0(\lambda)}\Big)^\rho\frac{Ct(1-2^{\rho-2})}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\mathrm{e}^{-\lambda f(\lambda)}\Big\}\\
&= 0,
\end{aligned} \quad (2.42)$$

by well-known asymptotics for the incomplete Gamma function $\Gamma(1/\rho,\cdot)$, and provided that

$$\lim_{\lambda\to-\infty}\frac{\sigma(\lambda)x_0(\lambda)^{\rho-1}}{\exp\{-\lambda f(\lambda)\}}\exp\Big\{-\frac{t\exp\{-\lambda f(\lambda)\}}{x_0(\lambda)^\rho}\Big\} = 0 \quad \text{for } t>0,$$

the latter fact which in turn holds when (2.41) does. Finally, the analogue of (2.36) becomes

$$
\begin{aligned}
&\limsup_{\lambda\to-\infty}\int_{|\theta|>\delta\sigma(\lambda)/(2f(\lambda))}\exp\bigg\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}\,d\nu(x)\bigg\}d\theta\\
&\le\limsup_{\lambda\to-\infty}\int_{|\theta|>\delta\sigma(\lambda)/(2f(\lambda))}\\
&\qquad\qquad\exp\bigg\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}-\frac{t}{2}\int_1^{x_0(\lambda)}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}\,d\nu(x)\bigg\}d\theta\\
&\le\limsup_{\lambda\to-\infty}\int_{|\theta|>\delta\sigma(\lambda)/(2f(\lambda))}\exp\bigg\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}-\frac{t}{4}\int_1^{x_0(\lambda)}\mathrm{e}^{-\lambda x-1}\,d\nu(x)\bigg\}d\theta\\
&\le\limsup_{\lambda\to-\infty}\int_{|\theta|>\delta\sigma(\lambda)/(2f(\lambda))}\exp\bigg\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}-\frac{t\,\mathrm{e}^{-1}}{4}\,\frac{\sigma(\lambda)^2}{x_0(\lambda)^2}\bigg\}d\theta\\
&=\limsup_{\lambda\to-\infty}\frac{2}{\rho}\Big(\frac{Ct(1-2^{\rho-2})}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\Big)^{-1/\rho}\Gamma\Big(\frac{1}{\rho},\Big(\frac{\delta\sigma(\lambda)}{2f(\lambda)}\Big)^\rho\frac{Ct(1-2^{\rho-2})}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\exp\bigg\{-\frac{t\,\mathrm{e}^{-1}}{4}\,\frac{\sigma(\lambda)^2}{x_0(\lambda)^2}\bigg\}\\
&=\limsup_{\lambda\to-\infty}\frac{2}{\rho}\Big(\frac{\delta\sigma(\lambda)}{2f(\lambda)}\Big)^{1-\rho}\Big(\frac{Ct(1-2^{\rho-2})}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\Big)^{-1}\\
&\qquad\qquad\qquad\qquad\times\exp\bigg\{-\Big(\frac{\delta\sigma(\lambda)}{2f(\lambda)}\Big)^\rho\frac{Ct(1-2^{\rho-2})}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}-\frac{t\,\mathrm{e}^{-1}}{4}\,\frac{\sigma(\lambda)^2}{x_0(\lambda)^2}\bigg\}\\
&=0,
\end{aligned}
\tag{2.43}
$$

by the already cited properties of the incomplete Gamma function, and provided that

$$
\lim_{\lambda\to-\infty}\sigma(\lambda)f(\lambda)^{\rho-1}\exp\bigg\{-t\,\frac{\sigma(\lambda)^2}{x_0(\lambda)^2}\bigg\}=0\quad\text{for } t>0,
$$

the latter fact which in turn holds provided that

$$
\lim_{\lambda\to-\infty}\frac{\sigma(\lambda)^2}{x_0(\lambda)^2\ln(\sigma(\lambda))}=\infty,
\tag{2.44}
$$

because (2.41) readily gives that $f(\lambda)^{-1}=o(\sigma(\lambda))$ as $\lambda\to-\infty$. However, it is also readily seen that (2.40) implies (2.44), using that $\sigma(\lambda)/x_0(\lambda)\to\infty$ by (2.37), and that $\sigma(\lambda)^{2-\rho}\mathrm{e}^{\lambda\varepsilon}\to\infty$. Putting (2.39)-(2.43) together we now finally arrive at (2.6), which in turn completes the proof of Statement 6.

To prove Statement 7, notice that Lemma 2.6 gives

$$
0<\frac{1}{C_1}\le\liminf_{x\uparrow0}\int_x^0\frac{y^2d\nu(y)}{x^2\nu((-\infty,x))}\le\limsup_{x\uparrow0}\int_x^0\frac{y^2d\nu(y)}{x^2\nu((-\infty,x))}\le C_1<\infty
\tag{2.45}
$$

for some constant $C_1\ge1$. As this also shows that $\int_\cdot^0 y^2d\nu(y)$ belongs to $\mathcal{OR}_{0^-}(\alpha+2,\beta+2)$, Lemma 2.7 now in turn gives

$$0 < \frac{1}{C_2} \leq \liminf_{\lambda\to-\infty}\Big(\int_{1/\lambda}^0 y^2 d\nu(y)\Big)^{-1}\int_{-\infty}^0 \mathrm{e}^{-\lambda x} d\Big(-\int_x^0 y^2 d\nu(y)\Big)$$
$$\leq \limsup_{\lambda\to-\infty}\Big(\int_{1/\lambda}^0 y^2 d\nu(y)\Big)^{-1}\int_{-\infty}^0 \mathrm{e}^{-\lambda x} d\Big(-\int_x^0 y^2 d\nu(y)\Big) \leq C_2 < \infty$$

for some constant $C_2 \geq 1$. And so we get (2.4) in the following manner [recall (2.11)]:

$$\begin{aligned}\liminf_{\lambda\to-\infty}\lambda^2\sigma(\lambda)^2 &\geq \liminf_{\lambda\to-\infty}\lambda^2\int_{-\infty}^0 \mathrm{e}^{-\lambda x} d\Big(-\int_x^0 y^2\, d\nu(y)\Big)\\ &\geq \frac{1}{C_2}\liminf_{\lambda\to-\infty}\lambda^2\int_{1/\lambda}^0 y^2\, d\nu(y)\\ &\geq \frac{1}{C_1C_2}\liminf_{\lambda\to-\infty}\nu((-\infty,1/\lambda))\\ &= \infty\end{aligned}\tag{2.46}$$

To prove Statement 8, using that $-\varepsilon\sigma(\lambda) < 1/\lambda$ for λ small enough, we get (2.5) in the following manner:

$$\begin{aligned}&\limsup_{\lambda\to-\infty}\int_{-\infty}^{-\varepsilon\sigma(\lambda)}\frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}\, d\nu(x)\\ &\quad\leq \Big(\sup_{x<0}x^2\mathrm{e}^{x/2}\Big)\limsup_{\lambda\to-\infty}\,\mathrm{e}^{\varepsilon\lambda\sigma(\lambda)/2}\,\nu((-\infty,-\varepsilon\sigma(\lambda)))\Big/\Big(\frac{1}{\mathrm{e}}\int_{1/\lambda}^0 x^2\, d\nu(x)\Big)\\ &\quad= 0 \quad \text{for } \varepsilon>0.\end{aligned}$$

To prove Statement 9, in view of Statements 7 and 8, it is enough to prove that (2.6) holds. Note that, since $\nu((0,\infty)) = 0$, the arguments that were use to establish (2.46) carry over to show that

$$\frac{1}{C_1C_2} \leq \liminf_{\lambda\to-\infty}\frac{\nu((-\infty,1/\lambda))}{\lambda^2\sigma(\lambda)^2} \leq \limsup_{\lambda\to-\infty}\frac{\nu((-\infty,1/\lambda))}{\lambda^2\sigma(\lambda)^2} \leq C_1C_2. \tag{2.47}$$

Further, using the inequality $1-\cos(x) \geq x^2/4$ for $|x| \leq 1$ we have by (2.45) and (2.47)

$$\begin{aligned}\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\mathrm{e}^{-\lambda x}\, d\nu(x) &\geq \frac{\theta^2}{4\,\mathrm{e}\,\sigma(\lambda)^2}\int_{\max\{-\sigma(\lambda)/|\theta|,1/\lambda\}}^0 x^2\, d\nu(x)\\ &\geq \frac{1}{8C_1\mathrm{e}}\min\Big\{\nu((-\infty,-\sigma(\lambda)/|\theta|)), \frac{\nu((-\infty,1/\lambda))\theta^2}{\lambda^2\sigma(\lambda)^2}\Big\}\\ &\geq \frac{1}{8C_1\mathrm{e}}\min\Big\{\frac{|\theta|^{-\beta-\varepsilon}\nu((-\infty,-\sigma(\lambda)))}{C}, \frac{\theta^2}{2C_1C_2}\Big\}\end{aligned}$$

for $|\theta| > 1$ and λ small enough. As the fact that $\lim_{\lambda\to-\infty}\sigma(\lambda) = 0$ gives $\lim_{\lambda\to-\infty}\nu((-\infty,-\sigma(\lambda))) = \infty$ [recall (2.11)], it follows that (2.6) holds.

To prove Statement 10, by Statements 1 and 3 we may assume that $\nu((0,\infty)) = 0$ and $s^2 = 0$. It is enough to prove (2.4), as Statement 8 then gives (2.5). Recall that selfdecomposability means that $d\nu(x) = k(x)/|x|$ where $k > 0$ is non-decreasing (see e.g., Sato, [23], Corollary 15.11). From (2.11) we get in addition that $\lim_{x\uparrow 0} k(x) = \infty$. And so we get (2.4) as follows:

$$\begin{aligned}
\liminf_{\lambda\to-\infty} \lambda^2\sigma(\lambda)^2 &\geq \liminf_{\lambda\to-\infty} \int_{1/\lambda}^{0} \lambda^2 x^2 \mathrm{e}^{-\lambda x}\, d\nu(x) \\
&\geq \frac{1}{\mathrm{e}} \liminf_{\lambda\to-\infty} \int_{1/\lambda}^{0} \lambda^2(-x)k(x)\, dx \\
&\geq \frac{1}{\mathrm{e}} \liminf_{\lambda\to-\infty} \lambda^2(-0^-) \int_{1/\lambda}^{0^-} k(y)\, dy + \frac{1}{\mathrm{e}} \liminf_{\lambda\to-\infty} \lambda^2 \int_{1/\lambda}^{0} \Big(\int_{1/\lambda}^{x} k(y)\, dy\Big)\, dx \\
&\geq \frac{1}{2\,\mathrm{e}} \liminf_{\lambda\to-\infty} k(1/\lambda) \\
&= \infty.
\end{aligned}$$

To prove Statement 11, by Statement 1 we may assume that $s^2 = 0$. Further, ξ is selfdecomposable (see the proof of Statement 10), so that Statement 10 gives (2.5). Noticing that

$$\frac{d}{dx} - \int_x^0 \frac{y^2}{x}\mathrm{e}^{-\lambda y}\, d\nu(y) = \int_x^0 \frac{k(y)}{x^2}\mathrm{e}^{-\lambda y}\, dy + \frac{k(x)}{x}\mathrm{e}^{-\lambda x} = \int_x^0 \frac{-y}{x^2}\frac{d}{dy}(\mathrm{e}^{-\lambda y}k(y))\, dy \geq 0 \tag{2.48}$$

it is now an easy matter to finish off the proof: Using that $1-\cos(x) \geq x^2/4$ for $|x| \leq 1$ we get (2.6), as (2.48) together with (2.5) give that

$$\int_{\mathbb{R}} \Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\, \mathrm{e}^{-\lambda x}\, d\nu(x) \geq \int_{-\sigma(\lambda)/|\theta|}^{0} \frac{\theta^2 x^2 \mathrm{e}^{-\lambda x}}{4\sigma(\lambda)^2}\, d\nu(x) \geq \int_{-\sigma(\lambda)}^{0} \frac{x^2|\theta|\,\mathrm{e}^{-\lambda x}}{4\sigma(\lambda)^2}\, d\nu(x) \geq \frac{|\theta|}{8}$$

for λ small enough and $|\theta| \geq 1$.

To prove Statement 12, in view of Statement 2 we may assume that $s^2 = 0$. By (2.54) below we have

$$\int_{-\infty}^{0} \big(\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x)\big)\, d\nu(x) \sim -\Gamma(1+\alpha)\, \nu((-\infty, 1/\lambda)) \quad \text{as } \lambda \to -\infty. \tag{2.49}$$

Moreover, by (2.53) below together with Feller's Tauberian theorem (see e.g., Bingham, Goldie and Teugels [8], Theorem 1.7.1′), we have

$$\begin{aligned}
&\int_{-\infty}^{0} \big(\lambda x\mathrm{e}^{-\lambda x} - \lambda\kappa(x)\big)\, {}^{|!}d\nu(x) \\
&= \int_{-1}^{0} \lambda x\, (\mathrm{e}^{-\lambda x} - 1)\, d\nu(x) + \int_{-\infty}^{-1} \big(\lambda x\mathrm{e}^{-\lambda x} + \lambda\big)\, d\nu(x) \\
&= \lambda(\mathrm{e}^{\lambda} - 1)\nu((-\infty, -1)) + \int_{-1}^{0} \big((\lambda^2 x - \lambda)\mathrm{e}^{-\lambda x} + \lambda\, \nu((-\infty, x)\big)\, dx + O(\lambda)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda^2 \int_{-1}^{0} (\lambda x - 2)\mathrm{e}^{-\lambda x} d\Big(\int_x^0 \Big(\int_{-1}^{y} \nu((-\infty, z))\, dz\Big) dy\Big) + O(\lambda) \\
&\sim \lambda^3 \,\frac{(1/\lambda)^3 \Gamma(4+\alpha)\,\nu((-\infty,-1/\lambda))}{-(\alpha+1)\,(2+\alpha)} - 2\lambda^2 \,\frac{(1/\lambda)^2 \Gamma(3+\alpha)\,\nu((-\infty,-1/\lambda))}{-(\alpha+1)\,(2+\alpha)} \qquad (2.50) \\
&= -\Gamma(2+\alpha)\,\nu((-\infty,-1/\lambda)) \quad \text{as } \lambda \to -\infty,
\end{aligned}
$$

where $\alpha < -1$ ensures that $\lim_{\lambda\to-\infty} \nu((-\infty,1/\lambda))/(-\lambda) = \infty$. Putting (2.49) and (2.50) together we see that (2.31) holds with $\tilde{L} = 1+\alpha$. □

The next proposition gives sufficient conditions for condition (1.4) of Theorem 1.1 to hold in terms of the characteristic triplet:

Proposition 2.9. *Let ξ be a superexponential Lévy process with characteristic triple (ν, m, s^2) and infinite upper end-point (2.1). With the notation (2.8) we have the following implications (with obvious notation):*

1. *If $\nu((0,\infty)) > 0$, then $\dfrac{\xi(aq(u))}{w(u)} \stackrel{d}{\to} 0$ as $u \to \infty$ for $a > 0$;*
2. *If $\nu((0,\infty)) = 0$ and $s^2 > 0$, then $\dfrac{\xi(aq(u))}{w(u)} \stackrel{d}{\to} N(0,2a)$ as $u \to \infty$ for $a > 0$.*
3. *If $\nu((0,\infty)) = 0$ and $s^2 = 0$ and (2.29) holds, then*
$$\frac{\xi(aq(u))}{w(u)} \stackrel{d}{\to} S_{-\alpha}\Big((a\cos(-\frac{\pi\alpha}{2}))^{-1/\alpha}, -1, 0\Big) \quad \text{as } u \to \infty \text{ for } a > 0.$$

Proof. We have weak convergence $\xi(aq(u))/w(u) \stackrel{\mathrm{d}}{\to} X$ if and only if we have convergence of the Laplace transform

$$
\begin{aligned}
\lim_{u\to\infty} \mathbf{E}\{\mathrm{e}^{-t\xi(aq(u))/w(u)}\} &= \lim_{u\to\infty} \phi_1(t/w(u))^{aq(u)} \\
&= \lim_{u\to\infty} \exp\{aq(u)\ln(\phi_1(t/w(u)))\} \\
&= \lim_{\lambda\to-\infty} \exp\Big\{\frac{a\ln(\phi_1(-t\lambda))}{-\lambda\mu(\lambda) - \ln(\phi_1(\lambda))}\Big\} \qquad (2.51) \\
&= \lim_{\lambda\to-\infty} \exp\Big\{a\,\frac{\int_{\mathbb{R}} \big(\mathrm{e}^{t\lambda x} - 1 - t\lambda\kappa(x)\big)\, d\nu(x) + mt\lambda + (t\lambda s)^2/2}{\int_{\mathbb{R}} \big(1 - \mathrm{e}^{-\lambda x}(1+\lambda x)\big)\, d\nu(x) + (\lambda s)^2/2}\Big\} \\
&= \mathbf{E}\{\mathrm{e}^{-tX}\} \quad \text{for } t \in (-1,0)
\end{aligned}
$$

(see e.g., Hoffmann-Jørgensen [17], pp. 377-378).

To prove Statement 1, notice that by arguing as for the proof of (2.7) in Statement 3 of Proposition 2.8, the limit in (2.51) is 1 when $\nu((0,\infty)) > 0$, which implies weak convergence to a degenerate random variable $X = 0$.

To prove Statement 2, notice that by arguing as for the proof of (2.7) in Statement 2 of Proposition 2.8, the limit in (2.51) is e^{at^2} when $\nu((0,\infty)) = 0$ and $s^2 > 0$, which implies weak convergence to a normal $\mathrm{N}(0, 2a)$ distributed random variable X.

To prove Statement 3, assume that $\nu((0,\infty)) = 0$ and $s^2 = 0$. Notice that by Karamata's theorem (see e.g., Bingham, Goldie and Teugels [8], Section 1.5.6),

$$-\int_x^0 y\nu((-\infty,y))\,dy \sim \frac{x^2\nu((-\infty,x))}{2+\alpha} \in \mathcal{R}_{0^-}(2+\alpha) \quad \text{as } x\uparrow 0.$$

Hence Feller's Tauberian theorem (see e.g., Bingham, Goldie and Teugels [8], Theorem 1.7.1′) gives

$$\begin{aligned}\int_{-\infty}^0 \big(1-\mathrm{e}^{-\lambda x}(1+\lambda x)\big)\,d\nu(x) &= \int_{-\infty}^0 \lambda^2\mathrm{e}^{-\lambda x}\,d\Big(-\int_x^0 y\nu((-\infty,y))\,dy\Big) \\ &\sim \Gamma(2+\alpha)\,\nu((-\infty,1/\lambda)) \quad \text{as } \lambda\to-\infty.\end{aligned} \tag{2.52}$$

Moreover, using Karamata's theorem again we get

$$\int_x^0\Big(\int_{-1}^y \nu((-\infty,z))\,dz\Big)dy \sim \frac{x^2\,\nu((-\infty,x))}{-(\alpha+1)\,(2+\alpha)} \in \mathcal{R}_{0^-}(2+\alpha) \quad \text{as } x\uparrow 0, \tag{2.53}$$

so that by Feller's Tauberian theorem

$$\begin{aligned}\int_{-\infty}^0 \big(\mathrm{e}^{t\lambda x}-1-t\lambda\kappa(x)\big)\,d\nu(x) &= o(1) + \int_{-1}^0 \big(t\lambda - \mathrm{e}^{t\lambda x}t\lambda\big)\nu((-\infty,x))\,dx \\ &\sim (t\lambda)^2\int_{-1}^0 \mathrm{e}^{t\lambda x}\,d\Big(\int_x^0\Big(\int_{-1}^y \nu((-\infty,z))\,dz\Big)dy\Big) \\ &\sim \frac{\Gamma(2+\alpha)\,\nu((-\infty,-1/(t\lambda)))}{-(\alpha+1)} \\ &\sim -(-t)^{-\alpha}\,\Gamma(1+\alpha)\,\nu((-\infty,1/\lambda)) \quad \text{as } \lambda\to-\infty\end{aligned} \tag{2.54}$$

for $t\in[-1,0)$. Since $\alpha<-1$ ensures that $\lim_{\lambda\to-\infty}\nu((-\infty,1/\lambda))/(-\lambda)=\infty$ it follows that the limit in (2.51) is $\mathrm{e}^{-a\,(-t)^{-\alpha}}$, which is the Laplace transform of the $-\alpha$-stable distribution in Statement 3 (see e.g., Samorodnitsky and Taqqu [22], Proposition 1.2.12).

□

By (2.1) we have $\alpha\le -1$ in (2.28) when $\nu((0,\infty)) = 0$ and $s^2 = 0$. But $\alpha=-1$ was not covered in Proposition 2.9 and turns out to behave differently than $\alpha<-1$:

Proposition 2.10. *Let ξ be a superexponential Lévy process with characteristic triple $(\nu, m, 0)$ and infinite upper end-point (2.1). Assume that $\nu((0,\infty)) = 0$ and that $\nu((-\infty,\cdot))\in\mathcal{R}_{0^-}(-1)$. Denoting*

$$w(u) = -\frac{1}{\mu^{\leftarrow}(u/h)} \quad \textit{and} \quad q(u) = \frac{1}{\ln(\phi_1(\mu^{\leftarrow}(u/h)))},$$

we have

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\xi(h-q(u)t) > u + xw(u)\}}{\mathbf{P}\{\xi(h) > u\}} = \begin{cases} \mathrm{e}^{-x} & \textit{for} \quad x \in \mathbb{R} \quad \textit{and} \quad t = 0 \\ 0 & \textit{for} \quad x \in \mathbb{R} \quad \textit{and} \quad t > 0 \end{cases} \tag{2.55}$$

and

$$\frac{\xi(aq(u))}{w(u)} \xrightarrow{d} a \quad \textit{as } u \to \infty \quad \textit{for } a > 0. \tag{2.56}$$

Proof. We still have (2.52) with $\alpha = -1$. However, by so called de Haan theory (see e.g., Bingham, Goldie and Teugels [8], Proposition 1.5.9a), (2.53) changes to

$$\int_{\cdot}^{0} \left(\int_{-1}^{y} \nu((-\infty, z)) dz \right) dy \in \mathcal{R}_{0^-}(1) \tag{2.57}$$

with

$$\lim_{x\uparrow 0} \frac{1}{x^2 \nu((-\infty, x))} \int_x^0 \left(\int_{-1}^{y} \nu((-\infty, z))\, dz \right) dy = \infty. \tag{2.58}$$

And so by Feller's Tauberian theorem the corresponding modification of (2.54) becomes

$$\begin{aligned} \int_{-\infty}^{0} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x)\right) d\nu(x) &= o(1) + \lambda^2 \int_{-1}^{0} \mathrm{e}^{-\lambda x}\, d\left(\int_x^0 \left(\int_{-1}^{y} \nu((-\infty, z))\, dz \right) dy \right) \\ &\sim \Gamma(2)\, \lambda^2 \int_{1/\lambda}^{0} \left(\int_{-1}^{y} \nu((-\infty, z))\, dz \right) \quad \text{as } \lambda \to -\infty, \end{aligned} \tag{2.59}$$

where the right-hand side is regularly varying by (2.57). Since (2.11) shows that

$$\lim_{\lambda\to-\infty} \frac{1}{(-\lambda)} \int_{-\infty}^{0} (\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x)) d\nu(x) = \infty,$$

we now readily obtain (2.56) in the following manner: For $t \in (-1, 0)$ we have

$$\begin{aligned} \lim_{u\to\infty} \mathbf{E}\{\mathrm{e}^{-t\xi(aq(u))/w(u)}\} &= \lim_{u\to\infty} \exp\left\{ \frac{a \ln(\phi_1(-t\mu^{\leftarrow}(u/h)))}{\ln(\phi_1(\mu^{\leftarrow}(u/h)))} \right\} \\ &= \lim_{\lambda\to-\infty} \exp\left\{ a\, \frac{\int_{-\infty}^{0} \left(\mathrm{e}^{t\lambda x} - 1 - t\lambda\kappa(x)\right) d\nu(x) + mt\lambda}{\int_{-\infty}^{0} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x)\right) d\nu(x) - m\lambda} \right\} \\ &= \lim_{\lambda\to-\infty} \exp\left\{ a\, \frac{(-t) \int_{-\infty}^{0} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x)\right) d\nu(x)}{\int_{-\infty}^{0} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x)\right) d\nu(x)} \right\} \\ &= \mathrm{e}^{-at}. \end{aligned}$$

Changing the definition of Q to $Q(\lambda) = 1/\ln(\phi_1(\lambda))$ in the proof of Theorem 2.2, that proof still goes through in essence. The only important change is that since

$$\lim_{\lambda\to-\infty} \frac{-\lambda\mu(\lambda) - \ln(\phi_1(\lambda))}{\ln(\phi_1(\lambda))} = 0$$

by (2.52) and (2.58)-(2.59) [recall that $\nu((-\infty,1/\lambda))/(-\lambda)\to\infty$], (2.18) changes to

$$f_{\xi(h-Q(\lambda)t)}(\mu_t(\lambda)+x/\lambda)\sim \mathrm{e}^{x-t\ln(\phi_1(\lambda))/(-\lambda\mu(\lambda)-\ln(\phi_1(\lambda)))}\,\frac{\mathrm{e}^{h\lambda\mu(\lambda)}\phi_1(\lambda)^h}{\sqrt{2\pi h}\,\sigma(\lambda)}\quad \text{as } \lambda\to-\infty.$$

This does not affect the validity of (2.19)-(2.21), while (2.22) and (2.23) change to

$$\lim_{u\to\infty}\frac{\mathbf{P}\{\xi(h-q(hu)t)>hu-xw(hu)\}}{\mathbf{P}\{\xi(h)>hu\}}=\begin{cases}\mathrm{e}^{-x} & \text{for } x\in\mathbb{R} \text{ and } t=0,\\ 0 & \text{for } x\in\mathbb{R} \text{ and } t>0.\end{cases}$$

From this in turn it follows that (2.55) holds as claimed. □

3 A general upper bound and consequences

We will study the probability $\mathbf{P}\{\sup_{t\in[0,h]}\xi(t)>u\}$ for a separable Lévy process ξ. As that probability coincide for all separable Lévy processes with the same finite dimensional distributions, it is enough to consider one specific such process: In proofs we can thus henceforth assume that ξ is càdlàg (right continuous with left limits).

The following simple general upper bound for the above mentioned probability will be an important tool for us:

Proposition 3.1. *For a separable Lévy process ξ we have*

$$\sup_{u\in\mathbb{R}}\frac{1}{\mathbf{P}\{\xi(h)>u-\varepsilon\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\leq\frac{1}{\inf_{t\in[0,h]}\mathbf{P}\{\xi(t)\geq-\varepsilon\}}\quad \textit{for } \varepsilon\geq 0.$$

Proof. Writing $T=\inf\{t>0:\xi(t)>u\}$ and $g(t)=\mathbf{P}\{\xi(t)\geq-\varepsilon\}$, we have

$$\begin{aligned}\mathbf{P}\{\xi(h)>u-\varepsilon\}&\geq\mathbf{E}\big\{\mathbf{P}\{T<h,\xi(h)-\xi(T)\geq-\varepsilon\,\big|\,T\}\big\}\\&=\mathbf{E}\big\{I_{T<h}\,\mathbf{P}\{\xi(h)-\xi(T)\geq-\varepsilon\,\big|\,T\}\big\}\\&=\mathbf{E}\big\{I_{T<h}\,g(h-T)\big\}\\&\geq\mathbf{P}\Big\{\sup_{t\in[0,h)}\xi(t)>u\Big\}\inf_{t\in[0,h]}g(t)\\&=\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\inf_{t\in[0,h]}g(t).\end{aligned}$$

□

A simple version of the following corollary to Proposition 3.1 for symmetric processes appears already in Doob [13], p. 106:

Corollary 3.2. *For a separable Lévy process ξ such that*

$$\liminf_{t\downarrow 0} \mathbf{P}\{\xi(t)>0\}>0 \quad \mathit{or} \quad \inf_{t\in[0,h]} \mathbf{P}\{\xi(t)\geq 0\}>0, \tag{3.1}$$

we have

$$\sup_{u\in\mathbb{R}} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t)>u\Big\}<\infty.$$

Proof. By Proposition 3.1 it is enough to show that the left condition in (3.1) implies that to the right. So assume that the left condition holds and that the right one does not. Then there exists a sequence $\{t_n\}_{n=1}^{\infty} \subseteq [0,h]$ such that

$$\mathbf{P}\{\xi(t_n)\geq 0\} \to \inf_{t\in[0,h]} \mathbf{P}\{\xi(t)\geq 0\}=0 \quad \text{as } n\to\infty.$$

Picking a convergent subsequence $\{t_n'\}_{n=1}^{\infty} \subseteq \{t_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} t_n' = t_0$, we get

$$\mathbf{P}\{\xi(t_0)>0\} \leq \liminf_{n\to\infty} \mathbf{P}\{\xi(t_n')>0\} \leq \liminf_{n\to\infty} \mathbf{P}\{\xi(t_n')\geq 0\}=0 \tag{3.2}$$

by continuity in probability of ξ. Hence the left condition in (3.1) gives $t_0>0$. And so ξ is supported on $(-\infty,0]$ by (3.2), which contradicts the left condition in (3.1). ☐

The next example addresses the difference between Proposition 3.1 and Corollary 3.2:

Example 3.3. Let $\{N(t)\}_{t\geq 0}$ be a unit rate Poisson process and $\{\eta_k\}_{k=1}^{\infty}$ independent Bernoulli distributed random variables satisfying $\mathbf{P}\{\eta_k=1\}=\mathbf{P}\{\eta_k=-1\}=\frac{1}{2}$. Rather spectacularly, Braverman [9], Section 4, shows that for the Lévy process $\xi(t)=\sum_{k=1}^{N(t)} \eta_k - t$, it holds that

$$1=\liminf_{u\to\infty} \frac{\mathbf{P}\big\{\sup_{t\in[0,h]} \xi(t)>u\big\}}{\mathbf{P}\{\xi(h)>u\}} < \limsup_{u\to\infty} \frac{\mathbf{P}\big\{\sup_{t\in[0,h]} \xi(t)>u\big\}}{\mathbf{P}\{\xi(h)>u\}}=\infty.$$

Hence neither Corollary 3.2 nor (3.1) holds for this process.

For an example of a Lévy process that does not satisfy (3.1) it is enough to consider $\xi(t)=N(t)-t$.

The following proposition can be very useful to verify that the condition (1.5) of Theorem 1.1 holds:

Proposition 3.4. *Let ξ be a separable Lévy process such that*

$$\inf_{t\in[0,h]} \mathrm{P}\{\xi(t)\geq 0\}>0$$

and

$$\lim_{u\to\infty}\frac{\mathbf{P}\{\xi(h-Tq(u))>u\}}{\mathbf{P}\{\xi(h)>u\}}=\mathrm{e}^{-T}.$$

It holds that

$$\lim_{T\to\infty}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\bigg\{\sup_{t\in[0,h-Tq(u)]}\xi(t)>u\bigg\}=0.$$

Proof. *By Proposition 3.1 together with the hypothesis of the proposition, we have that*

$$\begin{aligned}
&\lim_{T\to\infty}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\bigg\{\sup_{t\in[0,h-Tq(u)]}\xi(t)>u\bigg\}\\
&\quad\le\lim_{T\to\infty}\limsup_{u\to\infty}\frac{\mathbf{P}\{\xi(h-Tq(u))>u\}}{\mathbf{P}\{\xi(h)>u\}}\frac{1}{-}\inf_{t\in[0,h-Tq(u)]}\mathbf{P}\{\xi(t)\ge 0\}\\
&\quad\le\lim_{T\to\infty}\limsup_{u\to\infty}\frac{\mathbf{P}\{\xi(h-Tq(u))>u\}}{\mathbf{P}\{\xi(h)>u\}}\frac{1}{\inf_{t\in[0,h]}\mathbf{P}\{\xi(t\ge 0\}}\\
&\quad\le\lim_{T\to\infty}\mathrm{e}^{-T}\frac{1}{\inf_{t\in[0,h]}\mathbf{P}\{\xi(t)\ge 0\}}\\
&\quad=0.
\end{aligned}$$

□

4 Proof of Theorem 1.1

Proof of Theorem 1.1. Provided that $L(t,0)>0$ repeated use of (1.4) gives

$$\mathbf{P}\bigg\{\frac{\xi(h-q(u)t)-u}{w(u)}>x\,\bigg|\,\xi(h-q(u)t)>u\bigg\}=\frac{\mathbf{P}\{\xi(h-q(u)t)>u+xw(u)\}}{\mathbf{P}\{\xi(h-q(u)t)>u\}}\to\frac{L(t,x)}{L(t,0)}\tag{4.1}$$

as $u\to\infty$ for $x>0$. Let $\{\zeta_i(a)\}_{i=1}^{\infty}$ be independent random variables distributed as $\zeta(a)$. Further, let η_t be a possibly infinite valued random variable that is independent of $\{\zeta_i(a)\}_{i=1}^{\infty}$, that has the possibly improper cummulative probability distribution function $1-L(t,x)/L(t,0)$ when $L(t,0)>0$ and that is when $L(t,0)=0$. By (1.3) and (4.1) together with the assumed continuity properties of $\zeta(a)$ and $L(t,\cdot)$ we get

$$\begin{aligned}
&\liminf_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\bigg\{\sup_{t\in[0,h]}\xi(t)>u\bigg\}\\
&\quad\ge\lim_{T\to\infty}\limsup_{a\downarrow 0}\liminf_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\bigg\{\max_{k=0,\ldots,\lfloor T/a\rfloor}\xi(h-kaq(u))>u\bigg\}\\
&\quad=\lim_{T\to\infty}\limsup_{a\downarrow 0}\sum_{k=0}^{\lfloor T/a\rfloor}L(ka,0)\liminf_{u\to\infty}\mathbf{P}\bigg\{\bigcap_{\ell=0}^{k-1}\{\xi(h-\ell aq(u))\qquad\qquad(4.2)\\
&\qquad\qquad-\xi(h-kaq(u))+\xi(h-kaq(u))-u\le 0\}\,\bigg|\,\xi(h-kaq(u))>u\bigg\}\\
&\quad\ge\lim_{T\to\infty}\limsup_{a\downarrow 0}\sum_{k=0}^{\lfloor T/a\rfloor}L(ka,0)\,\mathbf{P}\bigg\{\bigcap_{\ell=0}^{k-1}\bigg\{\sum_{i=0}^{k-\ell}\zeta_i(a)+\eta_{ka}<0\bigg\}\bigg\}.
\end{aligned}$$

Here the first inequality is due to discretization, while the equality follows from the inclusion-exclusion formula and the fact that

$$\begin{aligned}&\mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor}\xi(h-kaq(u))>u\Big\}\\&\quad=\sum_{k=0}^{\lfloor T/a\rfloor}\mathbf{P}\Big\{\bigcap_{\ell=0}^{k-1}\{\xi(h-\ell aq(u))\le u\}\,\Big|\,\xi(h-kaq(u))>u\Big\}\mathbf{P}\{\xi(h-kaq(u))>u\}.\end{aligned}$$

The last inequality follows from (1.3) and the reasoning on the first few lines of this proof by means of dividing by $w(u)$ in the featured intersected events.

For an upper bound we make some preparations: The strong Markov property gives

$$\begin{aligned}&\mathbf{P}\Big\{\sup_{t\in[h-Tq(u),h]}\xi(t)>u+xw(u)\Big\}\\&\le\mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor}\xi(h-kaq(u))>u\Big\}\\&\quad+\mathbf{P}\Big\{\sup_{t\in[h-Tq(u),h]}\xi(t)>u+xw(u)\Big\}\mathbf{P}\Big\{\inf_{t\in[0,aq(u)]}\xi(t)\le -xw(u)\Big\}\quad\text{for } x>0.\end{aligned}\tag{4.3}$$

Further, by the continuity of w and the fact that $w(u)=o(u)$ [since $\xi(h)$ is Type I attracted], the functions u and $u+xw(u)$ range over the same values as $u\to\infty$ for any fixed $x>0$. Hence we have

$$\limsup_{u\to\infty}g(u)=\limsup_{u\to\infty}g(u+xw(u))\quad\text{for } x\in\mathbb{R}\text{ for any function } g.\tag{4.4}$$

From (1.3) together with basic theory of Lévy processes (see e.g., Sato [23], Theorem 8.7, together with Fristedt [16], p. 251), we have that $\{\xi(tq(u))/w(u)\}_{t\ge0}\stackrel{\mathrm{d}}{\to}\{\zeta(t)\}_{t\ge0}$ in the space $D[0,1]$ of càdlàg functions equipped with the Skorohod J_1 topology, where $\{\zeta(t)\}_{t\ge0}$ is a Lévy process. This gives that

$$\liminf_{a\downarrow0}\liminf_{u\to\infty}\mathbf{P}\Big\{\inf_{t\in[0,aq(u)]}\xi(t)>-xw(u)\Big\}\ge\liminf_{a\downarrow0}\mathbf{P}\Big\{\inf_{t\in[0,2a]}\zeta(t)>-\frac{x}{2}\Big\}=1\tag{4.5}$$

for $x>0$. Using (4.3)-(4.5) together with (1.4) and (1.3), we get in the fashion of (4.2)

$$\begin{aligned}&\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\\&\quad=\lim_{T\to\infty}\limsup_{x\downarrow0}\limsup_{u\to\infty}\frac{\mathrm{e}^x}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u+xw(u)\Big\}\\&\quad\le\lim_{T\to\infty}\limsup_{x\downarrow0}\liminf_{a\downarrow0}\limsup_{u\to\infty}\Big(\mathbf{P}\Big\{\inf_{t\in[0,aq(u)]}\xi(t)>-xw(u)\Big\}\Big)^{-1}\\&\qquad\qquad\times\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor}\xi(h-kaq(u))>u\Big\}\end{aligned}$$

$$+\lim_{T\to\infty}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\bigg\{\sup_{t\in[0,h-Tq(u)]}\xi(t)>u\bigg\}$$

$$\leq\lim_{T\to\infty}\liminf_{a\downarrow 0}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\bigg\{\max_{k=0,\ldots,\lfloor T/a\rfloor}\xi(h-kaq(u))>u\bigg\} \tag{4.6}$$

$$\leq\lim_{T\to\infty}\liminf_{a\downarrow 0}\sum_{k=0}^{\lfloor T/a\rfloor}L(ka,0)\,\mathbf{P}\bigg\{\bigcap_{l=k-1}^{0}\bigg\{\sum_{i=0}^{k-l}\zeta_i(a)+\eta_{ka}\leq 0\bigg\}\bigg\}.$$

Here the first equality this due to the Type I attraction, the first inequality is due to (4.3), the second inequality is due to (4.5) and (1.5), while the last inequality follows from the same arguments as were used to establsih (4.2).

By (1.5) together with (4.2) and 4.6), the following three limits exist and coincide

$$H=\lim_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\bigg\{\sup_{t\in[0,h]}\xi(t)>u\bigg\}$$

$$=\lim_{T\to\infty}\limsup_{a\downarrow 0}\sum_{k=0}^{\lfloor T/a\rfloor}L(ka,0)\,\mathbf{P}\bigg\{\bigcap_{\ell=k-1}^{0}\bigg\{\sum_{i=0}^{k-\ell}\zeta_i(a)+\eta_{ka}\leq 0\bigg\}\bigg\} \tag{4.7}$$

$$=\lim_{T\to\infty}\liminf_{a\downarrow 0}\sum_{k=0}^{\lfloor T/a\rfloor}L(ka,0)\,\mathbf{P}\bigg\{\bigcap_{\ell=k-1}^{0}\bigg\{\sum_{i=0}^{k-\ell}\zeta_i(a)+\eta_{ka}\leq 0\bigg\}\bigg\}.$$

As it is clear that $H\geq 1$ it only remains to show that $H<\infty$. However, this follows from applying (1.5) and (4.5) to the following version of (4.6), with $a>0$ small enough and $T>0$ large enough,

$$\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\bigg\{\sup_{t\in[0,h]}\xi(t)>u\bigg\}$$

$$\leq \mathrm{e}^{x}\bigg(\mathbf{P}\bigg\{\inf_{t\in[0,2a]}\zeta(t)>-\frac{x}{2}\bigg\}\bigg)^{-1}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\bigg\{\max_{k=0,\ldots,\lfloor T/a\rfloor}\xi(h-kaq(u))>u\bigg\}$$

$$+\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\bigg\{\sup_{t\in[0,h-Tq(u)]}\xi(t)>u\bigg\}$$

$$\leq\limsup_{x\downarrow 0}\bigg(\mathbf{P}\bigg\{\inf_{t\in[0,2a]}\zeta(t)>-\frac{x}{2}\bigg\}\bigg)^{-1}\sum_{k=0}^{\lfloor T/a\rfloor}L(ka,0)$$

$$+\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\bigg\{\sup_{t\in[0,h-Tq(u)]}\xi(t)>u\bigg\}\quad\text{for } x>0.$$

This concludes the proof of the full statement of Theorem 1.1. □

5 Examples on extremes of superexponential Lévy processes

Brownian motion is the canonical example of a superexponential Lévy process:

Example 5.1. Brownian motion with drift is a superexponential Lévy process ξ that has characteristic triple $(0, m, s^2)$ for some constants $m \in \mathbb{R}$ and $s^2 > 0$.

By Proposition 2.8 2, Brownian motion ξ satisfies (2.4)-(2.7), so that Theorem 2.2 shows that (1.4) holds. Further, Proposition 2.9 2 shows that (1.3) holds with $\zeta(a)$ $\mathrm{N}(0, 2a)$ distributed.

For $m \geq 0$ (1.5) follows readily from Corollary 3.2 together with (2.9). For $m < 0$ (3.1) does not hold. But a simple trick does the job: Let ξ_0 be a Lévy process with characteristic triple $(0, 0, s^2)$. Then Corollary 3.2 and (2.9) give (1.5) in the following way, using that $q(u) = o(w(u))$ by Theorem 2.2:

$$\begin{aligned}
\mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi(t) > u\Big\} &\leq \mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi_0(t) > u - m(h - Tq(u))\Big\} \\
&\leq \mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi_0(t) > u - mh - w(u)\Big\} \\
&\leq 2\,\mathbf{P}\{\xi_0(h - Tq(u)) > u - mh - w(u)\} \qquad (5.1)\\
&\leq 2\,\mathbf{P}\{\xi_0(h - Tq(u)) > u - m(h - Tq(u)) - w(u)\} \\
&= 2\,\mathbf{P}\{\xi(h - Tq(u)) > u - w(u)\} \\
&\sim 2\,\mathrm{e}^{1-T}\,\mathbf{P}\{\xi(h) > u)\} \quad \text{as } u \to \infty.
\end{aligned}$$

Notice that by Proposition 2.8 2 and Proposition 2.9 2, (1.3) and (1.4) hold with $\zeta(a)$ $\mathrm{N}(0, 2a)$ distributed and the functions w and q given by (2.8) for any Lévy process with characteristic triple (ν, m, s^2) such that $\nu((0, \infty)) = 0$ and $s^2 > 0$. But we cannot hope to show (1.5) as simply as for Brownian motion in this more general case. However, when (1.5) holds, then we must have $H = 2$ in (1.2) by well-known properties of Brownian motion.

Example 5.2. The Merton jump-diffusion [19] is a Lévy process ξ given by

$$\xi(t) = \gamma t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i \quad \text{for } t > 0,$$

where W is standard Brownian motion, $\sigma > 0$ a constant, N a Poisson process with intensity $\lambda > 0$ and the Y_i:s independent $N(0, \delta^2)$ distributed random variables.

Since $s^2 > 0$, (2.4) and (2.6) hold by Proposition 2.8 1. Further the Lévy measure is given by

$$\frac{d\nu}{dx} = \frac{\lambda}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}.$$

Choosing $g(x) = \sqrt{x}$ in Proposition 2.8 4 we see that (2.4), (2.5) and (2.7) hold. Hence Theorem 2.2 applies. Because $s^2 > 0$ we have $\inf_{t\in[0,h]} \mathbf{P}\{\xi(t) \geq 0\} > 0$ so that Proposition 3.4 applies. This means that Theorem (1.1) applies with $\zeta(a) = 0$. An inspection of (4.7) finally gives that $H = 1$.

Example 5.3. Pick a constant $\alpha \in (1,2)$. A totally skewed to the left α-stable Lévy process ξ has charactersitic triple $(\nu, m, 0)$, where

$$\frac{d\nu(x)}{dx} = \frac{\alpha\,\gamma^\alpha}{(-\Gamma(1-\alpha))\,(-\cos(\frac{\pi\alpha}{2}))(-x)^{\alpha+1}} \quad \text{for } x<0, \ \text{ for some constant } \gamma>0.$$

By Albin [1], Theorem 1, (1.2) holds with $H > 1$ for $m = \int_{\mathbb{R}}(\kappa(x) - x)d\nu(x)$. In this example we use Theorem 1.1 to extend Albin's result to a general m without using difficult results from the literature about α-stable distributions, contrary to what did Albin.

By Proposition 2.8 12, ξ satisfies (2.4)-(2.7) so that Theorem 2.2 shows that (1.4) holds. Further, Proposition 2.9 3 shows that (1.3) holds with $\zeta(a)$ having a $S_\alpha((-a\cos(\frac{\pi\alpha}{2}))^{1/\alpha}, -1, 0)$ distribution. For $\mathfrak{m} \geq 0$ (1.5) follows from Corollary 3.2 together with (2.9) using that

$$\mathbf{P}\{\xi(t)>0\} \geq \mathbf{P}\{(\xi(t)-\mathfrak{m}t)>0\} = \mathbf{P}\{t^{1/\alpha}(\xi(1)-\mathfrak{m}1)>0\} = \mathbf{P}\{(\xi(1)-\mathfrak{m}1)>0\} > 0$$

by self-similarity. For $\mathfrak{m} < 0$ we can use the trick (5.1) in exactly the same way as in Example 5.1 observing that again $q(u) = o(w(u))$.

By Proposition 2.8 3 and Proposition 2.9 3, (1.3) and (1.4) hold with $\zeta(a)$ having a $S_\alpha((-a\cos(\frac{\pi\alpha}{2}))^{1/\alpha}, -1, 0)$ distribution and the functions w and q given by (2.8) for Lévy processes such that $\nu((0,\infty)) = 0$ and (2.29) holds. But we cannot hope to verify (1.5) as easily as for α-stable processes for these more general processes. However, when (1.5) holds, then we have $H > 1$ in (1.2) because H only depends on α and was shown to satisfy $H > 1$ for totally skewed α-stable processes with $\mathfrak{m} = 0$ by Albin [1], Theorem 1.

Example 5.4. A totally skewed to the left 1-stable Lévy process ξ has charactersitic triple $(\nu, m, 0)$ given by

$$\frac{d\nu(x)}{dx} = \frac{2\gamma}{\pi(-x)^2} \quad \text{for } x<0, \text{ for some constants } \gamma>0 \text{ and } m\in\mathbb{R}.$$

By Albin [1], Theorem 2, (1.2) holds with $H=1$ in the case when $m=0$. We now extend Albin's result to a general $m\in\mathbb{R}$ by application of Theorem 1.1 and without using complicated results from the literature about tails of totally skewed 1-stable distributions, contrary to what did Albin.

With the functions w and q given by (2.8), Proposition 2.10 shows that (1.3) and (1.4) hold with $\zeta(a)=a$ in (1.3) for Lévy processes such that $\nu((0,\infty))=0$ and $\nu((-\infty,\cdot))\in\mathcal{R}_{0^-}(-1)$. We may derive (1.5) from Corollary 3.2 together with (2.55) for all values of m, because

$$\mathbf{P}\{\xi(t)>0\}\geq\mathbf{P}\{t\xi(1)-2\gamma t\ln(t)/\pi+mt>0\}\to 1 \quad \text{as } t\downarrow 0.$$

Now this implies that $\inf_{t\in[0,h]}\mathbf{P}\{\xi(t)\geq 0\}>0$ so that also $\inf_{t\in[0,h-q(u)T]}$ $\mathbf{P}\{\xi(t)\geq 0\}>0$. Hence Corollary 3.2 shows that

$$\mathbf{P}\Big\{\sup_{t\in[h-q(u)T]}\xi(t)>u\Big\}\leq K\,\mathbf{P}\{\xi(h-q(u)T)>u\}$$

for some constant K, so that

$$\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[h-q(u)T]}\xi(t)>u\Big\}\leq\limsup_{u\to\infty}\frac{K\mathbf{P}\{\xi(h-q(u)T)>u\}}{\mathbf{P}\{\xi(h)>u\}}=0.$$

Here we used (2.55) to get the last equality. Finally notice that an inspection of (4.7) gives that $H=1$.

The methodologies of Examples 5.3 and 5.4 readily carry over to, for example, the sum of two independent totally skewed stable Lévy processes with different stability indices.

Example 5.5. An unnamed superexponential Lévy process ξ is defined by Linnik and Ostrovskiĭ [18] pp. 52-53, see also Sato [23], Exercise 18.19, as having characteristics $(\nu,m,0)$, where

$$\frac{d\nu(x)}{dx}=\frac{\mathrm{e}^{bx}}{|x|(1-\mathrm{e}^{ax})} \quad \text{for } x<0, \quad \text{for some constants } a,b>0.$$

For a suitable constant $c=c(a,b,m)>0$ the corresponding Laplace transform is given by

$$\phi_1(\lambda) = \frac{\Gamma((b-\lambda)/a)c^{\lambda/a}}{\Gamma(b/a)} \quad \text{for } \lambda \le 0.$$

By Proposition 2.10, (1.3) and (1.4) hold with $\zeta_a = a$. Further, picking a function $g(t) > 0$ with $\lim_{t\downarrow 0} t\ln(1/g(t))/g(t) = 1$, Corollary 3.2 and (2.55) show that (1.5) holds, as by Stirling's formula (see e.g., Erdélyi, Magnus, Oberhettinger and Tricomi [14], Eq. 1.18.2) we have

$$\phi_1\Big(\frac{\lambda}{g(t)}\Big)^t \sim \frac{1}{(2\pi)^{t/2}} \exp\Big\{t\Big(\frac{bg(t)-\lambda}{ag(t)} - \frac{1}{2}\Big)\ln\Big(\frac{bg(t)-\lambda}{ag(t)}\Big) - \frac{bt}{a} + \frac{\ln(c/\mathrm{e})\lambda t}{ag(t)}\Big\} \to \mathrm{e}^{-\lambda/a}$$

as $t \downarrow 0$ for $\lambda \le 0$, so that

$$\mathbf{P}\{\xi(t) > 0\} = \mathbf{P}\{\xi(t)/g(t) > 0\} \to \mathbf{P}\{1/a > 0\} = 1 \quad \text{as } t \downarrow 0.$$

We finish by demonstrating how (2.10) gives the asymptotics of $\mathbf{P}\{\xi(h) > u\}$ as $u \to \infty$ in (1.2): Taking $a = 1$ for simplicity and denoting the polygamma function ψ (see e.g., Erdélyi, Magnus, Oberhettinger and Tricomi [14], Sections 1.16-1.17), we have

$$\begin{aligned}
\mu(\lambda) &= -\ln(c) + \psi(b-\lambda) = \ln((b-\lambda)/c) - \frac{1}{2(b-\lambda)} + O\Big(\frac{1}{\lambda^2}\Big) && \text{as } \lambda \to -\infty,\\
\sigma(\lambda)^2 &= \psi'(b-\lambda) = \frac{1}{b-\lambda} + O\Big(\frac{1}{\lambda^2}\Big) && \text{as } \lambda \to -\infty,\\
\mu^{\leftarrow}(x) &= b - \tfrac{1}{2} - c\mathrm{e}^x + O(\mathrm{e}^{-x}) && \text{as } x \to \infty.
\end{aligned}$$

Using this together with (2.10) and Stirling's formula, we get

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{(2\pi)^{(h-1)/2}}{\sqrt{hc}\,\Gamma(b)^h} \exp\Big\{-(cu + hc\ln(c/\mathrm{e}))\,\mathrm{e}^{u/h} - \frac{u}{2h}\Big\} \quad \text{as } u \to \infty.$$

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References

[1] Albin, J.M.P. (1993). Extremes of totally skewed stable motion. *Statistics & Probability Letters* **16** 219-224.

[2] Albin, J.M.P. (1998). A note on Rosenblatt distributions. *Statistics & Probability Letters* **40** 83-91.

[3] Albin, J.M.P. and Sundén, M. (2008). On the asymptotic behaviour of Lévy processes. Part I: Subexponential and exponential processes. *Stochastic Process. Appl.*, in press.

[4] Balkema, A.A., Klüppelberg, C. and Resnick, S.I. (1993). Densities with Gaussian tails. *Proc. London Math. Soc.* **66** 568-588.

[5] Balkema, A.A., Klüppelberg, C. and Resnick, S.I. (1999). Limit laws for exponential families. *Bernoulli* **5** 951-968.

[6] Balkema, A.A., Klüppelberg, C. and Resnick, S.I. (2003). Domains of attraction of exponential families. *Stochastic Process. Appl.* **107** 83-103.

[7] Balkema, A.A., Klüppelberg, C. and Stadtmüller, U. (1995). Tauberian results for densities with Gaussian tails. *J. London Math. Soc.* **51** 383-400.

[8] Bingham, N.H., Goldie, C.M. and Teugels J.L. (1987). *Regular Variation.* Cambridge University Press, Cambridge.

[9] Braverman, M. (1999). Remarks on suprema of Lévy processes with light tails. *Statist. Probab. Lett.* **43** 41-48.

[10] Braverman, M. (2000). Suprema of compound Poisson processes with light tails. *Stochastic Process. Appl.* **90** 145-156.

[11] Braverman, M. (2005). On a class of Lévy processes. *Statist. Probab. Lett.* **75** 179-189.

[12] Davis, R.A. and Resnick, S.I. (1991). Extremes of moving averages of random variables with finite endpoint. *Ann. Probab.* **19** 312-328.

[13] Doob, J.L. (1953). *Stochastic Processes* (Wiley, New York).

[14] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1953). *Higher Transcendental Functions, Vol. I.* McGraw-Hill, New York.

[15] Feigin, P.D. and Yashchin, E. (1983). On a strong Tauberian result. *Z. Wahrsch. Verw. Gebiete* **65** 35-48.

[16] Fristedt, B. (1974). Sample functions of stochastic processes with stationary independent increments. In *Advances in Probability, Vol. III* (P. Ney, S. Port Eds.) 241-396. Marcel Dekker, New York.

[17] Hoffmann-Jørgensen, J. (1994). *Probability with a View Towards Statistics.* Chapman and Hall, London.

[18] Linnik, Ju.V. and Ostrovskiĭ, I.V. (1977). *Decomposition of Random Variables and Vectors.* American Mathematical Society, Providence.

[19] Merton, R. (1976). *Option pricing when underlying stock returns are discontinuous.* J. Financial Economics **3** 125-144.

[20] Rootzén, H. (1986). Extreme value theory for moving average processes. *Ann. Probab.* **14** 612-652.

[21] Rootzén, H. (1987). A ratio limit theorem for the tails of weighted sums. *Ann. Probab.* **15** 728-747.

[22] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance.* Chapman and Hall, London.

[23] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge.

[24] Scheffé, H. (1947). A useful convergence theorem for probability distributions. *Ann. Math. Statist.* **18** 434-438.

The Kac Master Equation with Unbounded Collision Rate

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Abstract

The Kac model is a Markov jump process on the sphere $\sum_{i=1}^{N} v_i^2 = N$. The model was conceived as model for an N-particle system with pairwise interactions, and hence the jumps involve only pairs of coordinates, (v_i, v_j). This paper deals with Kac models with unbounded jump rates. We prove that the processes are Feller processes, and introduce a diffusion approximation that is useful for numerical simulation of the processes. We also study the spectral gap of the Markov generators, using methods from Carlen, Carvalho and Loss.

Short title: Brownian Approximation

Keywords: Brownian motion, Collision Kernel, Feller processes, Infinitesimal generator, Kac model, Laplace-Beltrami Operator, Markov Process, Semigroup, Spectral Gap.

1 Introduction

The Kac model, introduced by Mark Kac[18] in 1956, is a much simplified model of a dilute gas in which all interactions are binary collisions: for a real gas consisting of N particles, the phase space is $6N$-dimensional and involves the position and velocity in $\mathbb{R}^3$ for each particle; the Kac-model neglects the spatial variables and assumes velocities to be one-dimensonal, and hence the phase space is N-dimensional. And while the particles in a real gas evolve deterministically, the particles in the Kac model suffer random collisions. In spite of these simplifications, the model retains some of the key features of a real gas, and the simplifications made it possible to make a rigorous derivation of the limiting kinetic equation, the "Kac equation", which has attracted much interst in recent years, see for example Desvillettes [11, 12].

In this paper we are interested in the Kac model for a fixed but arbitrary number of particles, and the state of the system is thus given by a vector $\mathbf{v} = (v_1, v_2, ...v_N)$. The vector $\mathbf{v}$ evolves according to a jump process, involving only pairs of velocities:

$$(v_i, v_j) \quad \mapsto \quad (v_i \cos\theta + v_j \sin\theta, v_j \cos\theta - v_i \sin\theta)\,; \tag{1.1}$$

all jumps occur independently, and the rate of jumps with $\theta \in [a, b]$ is given by a density $h(\theta)$,

$$C_N \int_a^b h(\theta)\, d\theta.$$

The non-negative function $h(\theta)$ is usually taken to be even, and locally integrable (locally bounded) in $[-\pi, \pi] \setminus \{0\}$. Kac himself considered essentially a constant $h(\theta) = 1/(2\pi)$, but it is relevant to consider also functions $h(\theta)$ which have power law singularities near $\theta = 0$. This corresponds to the influence of grazing collisions in realistic models, and is often called the non-cutoff case, as opposed to the cutoff case where $h(\theta)$ is integrable. To make sense of the limit $N \to \infty$, one must take $C_N \sim 1/(N-1)$.

The jumps described in (1.1) conserve energy,

$$|\mathbf{v}|^2 = \sum_{i=1}^N v_i^2 = const\,,$$

and when considering the limit $N \to \infty$ it is convenient to assume that the constant is proportional to N. With this scaling, our models constitute Markov processes with state space $S^{N-1}(\sqrt{N})$. The infinitesimal generators of the processes considered, are integral operators defined by the right hand side of *Kac's master equation*

$$\frac{\partial}{\partial t} f(\mathbf{v},t) = \frac{2}{N-1} \sum_{1 \le i < j \le N} \int_{-\pi}^{\pi} [f(R_{ij}(\theta)\mathbf{v},t) - f(\mathbf{v},t)]\, \rho(\theta)\, d\theta.$$

Here $f \in C([0,\infty[, L^1(S^{N-1}(\sqrt{N}), \mu_N))$, or some subspace thereof depending on the choice of collision kernel ρ. The measure μ_N is the rotation invariant probability measure on $S^{N-1}(\sqrt{N})$ and

$$R_{ij}(\theta)\mathbf{v} = (v_1, \ldots, v_i \cos\theta + v_j \sin\theta, \ldots, -v_i \sin\theta + v_j \cos\theta, \ldots, v_N).$$

A subject of particular interest for a stochastic evolution of this kind is the rate of convergence to equilibrium, i.e. how fast is the relaxation $T_t f_0 \to \nu[f_0]$, for some initial data f_0, invariant measure ν and $\{T_t\}_{t \ge 0}$ the semigroup determined by the infinitesimal generator of the process. This rate of convergence is closely related to the spectral gap of the infinitesimal generator. Spectral gaps of Markov generators is a topic that has attracted much interest recently (see e.g. Chen [9] for more on this).

In this paper, the collision rate is given either by functions that behave like

$$\rho(\theta) = |\theta|^{-\alpha-1} \quad \text{for } \theta \in (-\pi, \pi]\,, \tag{1.2}$$

where $\alpha \in (0,2)$, by truncated forms,

$$\rho_\epsilon(\theta) = \min(\rho(\theta), \rho(\epsilon)), \tag{1.3}$$

or by step functions,

$$\rho(\theta) = \begin{cases} \varepsilon^{-3} & \text{when} \quad |\theta| < \varepsilon \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

These give rise to jump processes where small jumps are dominating, and the process obtained by taking the limit $\varepsilon \to 0$ in case (1.4) is a Brownian motion on the sphere, in which no jumps of finite size remain (this is proven in Section 2.2).

The inspiration for this paper comes from several sources. One example are the papers [20, 21] who take different diffusion processes on the sphere $S^{N-1}(\sqrt{N})$ as the starting point for obtaining kinetic equations as the limit when $N \to \infty$, much in the same way as Kac did for the original model.

Another source is a series of papers by Carlen et al [7, 8] who present a very detailed analysis of stochastic processes with a structure similar to the one in the Kac model, and obtain accurate estimates for the spectral gap of the generator of the corresponing model, and in some cases even explicit, exact expressions. In particular they proove a conjecture by Kac concerning the asymptotic behavior of the spectral gap in the limit $N \to \infty$.

We have also used some recent results concerning the approximation of Lévy processes (see Asmussen and Rosinski [4] or Cohen and Rosinski [10]), to develop an efficient method for simulating stationary state for a non-cutoff Kac-equation (see [5]). One can prove that methods based on truncation of the collision rate function converge to the correct solution (see e.g. Méléard et al [12], [14] and [15]), but adding a diffusion term to the truncated jump-operator may considerably improve the convergence rate. This is demonstrated by numerical example in Sundén and Wennberg [27], and analysed in more detail in this paper.

Subsection 2.4 deals with the approximation of the process corresponding to the unbounded collision kernel (1.2). We show that simple truncation of (1.2) corresponds to a Markov process which converges weakly to the process corresponding to the unbounded collision kernel. We also motivate that adding a Brownian part to mimic small angles gives a better (in terms of convergence of infinitesimal generators) approximation than just truncating the collision kernel. In section 4 we consider diffusion approximation in a three dimensional model, i.e. a model where velocities are elements of $\mathbb{R}^3$ and where also momentum is assumed constant.

The processes considered in the work at hand are so called Feller processes and in the Appendix below we give a brief review of the properties of such processes. The main reason for our interest in Feller processes are the available convergence result such as a.g., the Trotter-Kurtz theorem (i.e. Theorem A.3). We also state and prove to lemmas which are used to prove the convergence of some stochastic processes presented in this paper.

Throughout function spaces are to be taken as function spaces over $S^{N-1}(\sqrt{N})$ unless otherwise noted.

2 Convergence and approximation

2.1 The Kac model with truncated collision kernel

To approximate the Kac model corresponding to collision kernel (1.2) we will make extensive use of the Kac model corresponding to collision kernel (1.3). The inifinitesimal generator of the Kac model is defined by

$$\mathcal{L}f = \frac{2}{N-1} \sum_{1 \le i < j \le N} \int_{-\pi}^{\pi} [f(R_{ij}(\theta)\cdot) - f]\, \rho(\theta)\, d\theta. \tag{2.1}$$

When ρ is bounded, this is well-defined for $f \in L^p(S^{N-1}(\sqrt{N}))$ for any $p \ge 1$, and it also can be given a sense for measures on $S^{N-1}(\sqrt{N})$, and finally, for all the cases we consider here, it is well-defined as a principal value integral if f is sufficiently smooth.

We now give a general result on Kac models with bounded collision kernels.

Lemma 2.1. *Let ρ be a bounded function on $(-\pi, \pi]$. The Markov process defined by the Kac model, i.e. the process generated by the closure of $\mathcal{L}$ as in (2.1), is a Feller process.*

Proof. According to the Hille-Yosida theorem (See Theorem A.1 of the Appendix), it is enough to prove that (i) the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ is dense in C, (ii) if $f(\mathbf{v}^*) = \sup_{\mathbf{v} \in S^{N-1}(\sqrt{N})} f(\mathbf{v}) \ge 0$ then $\mathcal{L}f(\mathbf{v}^*) \le 0$ and (iii) the range of $\lambda - \mathcal{L}$ is dense in C for some $\lambda > 0$. Condition (i) is satisfied because $\mathcal{D}(\mathcal{L})$ is taken to be L^2 and (ii) holds by definition. To verify condition (iii), it suffices to show that for each $f \in C^\infty$ there is at least one $g \in C^\infty$ such that, for some λ it holds that

$$(\lambda - \mathcal{L})g = f. \tag{2.2}$$

We claim that the solution to (2.2) is given by $g = R_\lambda f$, where R_λ is the resolvent of the semigroup generated by $\mathcal{L}$ (see Appendix equation (A.2)). By boundedness of ρ, it follows that $\mathcal{L}$ is a bounded operator. Thus, we have that

$$\|R_\lambda f\| \le \int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{e}^{t\|\mathcal{L}\|}\|f\|\,dt \le \frac{\|f\|}{\lambda - \|\mathcal{L}\|},$$

so that R_λ is bounded for λ sufficiently large. Using this bound we see that differentiation and resolvent action commute so that $R_\lambda f \in C^\infty$. □

2.2 Brownian motion on $S^{N-1}(\sqrt{N})$ as a grazing collision limit of the Kac model

Here we consider families of collision kernels ρ_n which satisfy

$$\int_{|\theta|>n^{-1}} \rho_n(\theta)\,d\theta \to 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \theta^2 \rho_n(\theta)\,d\theta \to K \quad \text{as} \quad n \to \infty,$$

for some constant K. The collision kernels ρ_n may be of the form

$$\rho_n(\theta) = n^3 \psi(\theta/n),$$

where ψ has compact support. This is the "grazing collision limit" that has been considered for the Kac equation and the Boltzmann equation e.g. in Desvillettes [11] and Alexandre and Villani [1]. A particular example is

$$\rho_n(\theta) = n^3 \mathbf{1}_{[-1/n,1/n]}(\theta),$$

which yields the infinitesimal generator

$$\mathcal{L}_n f = \frac{2n^3}{N-1}\int_{-1/n}^{1/n} \sum_{1\le i<j\le N} [f(R_{ij}(\theta)\cdot) - f]\,d\theta, \quad \text{for} \quad f \in L^2. \tag{2.3}$$

Letting n grow, we will see that the limiting operator is a multiple of the Laplace-Beltrami operator in $S^{N-1}(\sqrt{N})$, i.e. the infinitesimal generator of a time-scaled Brownian motion on $S^{N-1}(\sqrt{N})$ (see e.g. Øksendal [24], p.149-150 or Stroock [26]). For processes related to Brownian motion on $S^{N-1}(\sqrt{N})$ in a kinetic setting, see Lancelotti and Kiessling [20] and [21].

Theorem 2.2. *The operator $\mathcal{L}_n$ as in (2.3) converges pointwise in C^∞ to the scaled Laplace-Beltrami operator*

$$\frac{2N}{3(N-1)}\Delta_{S^{N-1}(\sqrt{N})}$$

as $n \to \infty$.

Proof. Consider an arbitrary term $f(R_{ij}(\theta)\cdot) - f$ of the sum in (2.3) for some $f \in C^\infty$. Taylor expansion gives

$$
\begin{aligned}
& f(R_{ij}(\theta)\cdot) - f \\
=& f'_{v_i}(v_i(\cos\theta - 1) + v_j \sin\theta) + f'_{v_j}(v_j(\cos\theta - 1) - v_i \sin\theta) \\
&+ \frac{1}{2} f''_{v_i v_i}(v_i(\cos\theta - 1) + v_j \sin\theta)^2 + \frac{1}{2} f''_{v_j v_j}(v_j(\cos\theta - 1) - v_i \sin\theta)^2 \\
&+ \frac{1}{2} f''_{v_i v_j}(v_i(\cos\theta - 1) + v_j \sin\theta)(v_j(\cos\theta - 1) - v_i \sin\theta) \\
&+ \frac{1}{2} f''_{v_j v_i}(v_i(\cos\theta - 1) + v_j \sin\theta)(v_j(\cos\theta - 1) - v_i \sin\theta) \\
&+ \mathcal{O}\left(\left|[R_{ij}(\theta) - I]\cdot\right|^3\right).
\end{aligned}
\tag{2.4}
$$

Taylor expansion of the trigonometric functions in (2.4) yields

$$
\begin{aligned}
\cos\theta - 1 &= -\frac{\theta^2}{2} + \mathcal{O}(\theta^3) \\
(v_i(\cos\theta - 1) + v_j \sin\theta)^2 &= v_j^2\theta^2 + \mathcal{O}(\theta^3) \\
(v_i(\cos\theta - 1) + v_j \sin\theta)(v_j(\cos\theta - 1) - v_i \sin\theta) &= -v_i v_j \theta^2 + \mathcal{O}(\theta^3) \\
\text{and } \mathcal{O}\left(\left|[R_{ij}(\theta) - I]\cdot\right|^3\right) &= \mathcal{O}(\theta^3).
\end{aligned}
\tag{2.5}
$$

Replacing the expressions in (2.4) by its respective Taylor expansions from (2.5) and noting that $\sin(\theta)$ is an odd function, we obtain, after integration with respect to θ

$$
\begin{aligned}
& n^3 \int_{-1/n}^{1/n} [f(R_{ij}(\theta)\cdot) - f] \, d\theta \\
=& \frac{1}{3}\left(-v_i f'_{v_i} - v_j f'_{v_j}\right) \\
&+ \frac{1}{3}\left(v_j^2 f''_{v_i v_i} + v_i^2 f''_{v_j v_j} - v_i v_j f''_{v_i v_j} - v_i v_j f''_{v_j v_i}\right) + \mathcal{O}\left(\frac{1}{n}\right) \\
\to& \frac{1}{3}\left(-v_i f'_{v_i} - v_j f'_{v_j}\right) \\
&+ \frac{1}{3}\left(v_j^2 f''_{v_i v_i} + v_i^2 f''_{v_j v_j} - v_i v_j f''_{v_i v_j} - v_i v_j f''_{v_j v_i}\right),
\end{aligned}
$$

as $n \to \infty$. This means that the integral operator (2.3) converges to the diffusion operator

$$
\frac{2}{3(N-1)} \sum_{1 \le i < j \le N} \left(v_j^2 \partial_i^2 - v_i v_j \partial_i \partial_j - v_i v_j \partial_j \partial_i + v_i^2 \partial_j^2 - v_i \partial_i - v_j \partial_j\right), \tag{2.6}
$$

where ∂_k is to be interpreted as $\frac{\partial}{\partial v_k}$ and $\partial_i \partial_j$ as $\frac{\partial^2}{\partial v_i \partial v_j}$. Now the sum in (2.6) is precisely the Laplace-Beltrami operator $\Delta_{S^{N-1}}$, where S^{N-1} is the unit sphere in $\mathbb{R}^N$ (see e.g. Øksendal [24], Example 8.5.8). The relation

$$N\Delta_{S^{N-1}} = \Delta_{S^{N-1}(\sqrt{N})} \tag{2.7}$$

gives the desired result. □

Theorem 2.2 thus says, that the Markov process which has infinitesimal generator given by (2.6) is a (time-scaled) Brownian motion on $S^{N-1}(\sqrt{N})$. It is known from the literature (see e.g. Molchanov [23]) that diffusions such as Brownian motion on $S^{N-1}(\sqrt{N})$ are Feller processes. The following corollary deals with the weak convergence of the processes generated by (2.3) to Brownian motion.

Corollary 2.3. *The processes $\{X_t^n\}_{t\geq 0}$, generated by the closures of the operators $\mathcal{L}_n$ as in (2.3) converge weakly to a time scaled Brownian motion on $S^{N-1}(\sqrt{N})$ as $n \to \infty$, given that their initial distributions converge.*

Proof. We want to employ the equivalence of conditions (i) and (ii) of the Trotter-Kurtz theorem (see Appendix Theorem A.3). By Lemma A.4, C^∞ is a core for the Laplace-Beltrami operator (2.6). To prove (i) of Theorem A.3, we arbitrarily choose an $f \in C^\infty$, and as (2.3) is defined for $f \in L^2$, we may let $f_n = f$ for all n. By Lemma 2.1 the closures of the operators $\mathcal{L}_n$ are generators of Feller processes, and by Theorem 2.2 it holds that $\mathcal{L}_n f \to \mathcal{L} f$. Thus, the desired weak convergence follows, given convergence of initial distributions. □

2.3 The Kac model with unbounded collision kernel

In this subsection we extend our analysis to Kac models with certain unbounded collision kernels, namely those ρ for which $\rho(\theta)|\theta|^{\alpha+1} \to C$ as $\theta \to 0$ for some constants C and $\alpha \in (0, 2)$. The essential behaviour is captured by functions of the form

$$\rho(\theta) = \begin{cases} |\theta|^{-\alpha-1} & \text{for } |\theta| \leq \theta_0 \leq \pi \\ 0 & \text{otherwise}. \end{cases}$$

In this case, the infinitesimal generator of the process is the closure of the operator $\mathcal{L}$ given by

$$\mathcal{L}f = \frac{2}{N-1} \sum_{1 \le i < j \le N} \int_{-\pi}^{\pi} [f(R_{ij}(\theta)\cdot) - f] |\theta|^{-\alpha-1}\, d\theta, \tag{2.8}$$

where the integral is to be interpreted as a principal value for functions $f \in C^2$. To analyse the weak convergence of the process generated by (2.1) with collision kernel (1.3) to the process generated by (2.8), we need to prove that the closure of $\mathcal{L}$ as in (2.8) is indeed the generator of a Feller process:

Proposition 2.4. *Let $\mathcal{L}$ be the operator defined by (2.8) with domain C^2. The closure of $\mathcal{L}$ is the generator of a Feller process with values in $S^{N-1}(\sqrt{N})$.*

To prove this, we use Proposition A.2, which requires that there is a sequence of finite dimensional subspaces $L_n \subset C$ such that $\bigcup L_n$ is dense in $C = C(S^{N-1}(\sqrt{N}))$. Here we take L_n to be the space of spherical harmonics of degree less or equal to n, and recall that these are the restrictions of the harmonic, homogeneous polynomials in N real variables, of degree less or equal to n. For the following two lemmas, we note that the operator $\mathcal{L} : C^2(S^{N-1}(\sqrt{N})) \to C^2(S^{N-1}(\sqrt{N}))$ actually does not depend on the radius $\sqrt{N}$ of the sphere, and hence there is a natural definition of $\mathcal{L} : C^2(\mathbb{R}^N) \to C^2(\mathbb{R}^N)$.

Lemma 2.5. *The operator $\mathcal{L}$ defined by (2.8) and the Laplacian $\Delta = \sum_{\ell=1}^{N} \frac{\partial^2}{\partial \ell^2}$ commute for polynomials of N variables.*

Proof. Let $\mathcal{L}_{ij}$ denote the term corresponding to the pair ij of the sum defining $\frac{N-1}{2}\mathcal{L}$. Letting $\ell \neq i \neq j \neq \ell$, it is sufficient to check that

$$\mathcal{L}_{ij}(\partial_i^2 + \partial_j^2 + \partial_\ell^2)\mathbf{v}^k = (\partial_i^2 + \partial_j^2 + \partial_\ell^2)\mathcal{L}_{ij}\mathbf{v}^k,$$

where $\mathbf{v}^k = v_1^{k_1} \cdots v_N^{k_N}$ for non-negative integers $k_1, \ldots, k_N$ and where $\partial_m^2 = \frac{\partial^2}{\partial v_m^2}$ for $m = i, j, \ell$. It is obvious that $\mathcal{L}_{ij}\partial_\ell^2 \mathbf{v}^k = \partial_\ell^2 \mathcal{L}_{ij}\mathbf{v}^k$ for $\ell \neq i, j$. Letting

$$\mathbf{v}^{k\setminus ij} = v_1^{k_1} \cdots v_{i-1}^{k_{i-1}} v_{i+1}^{k_{\ell+1}} \cdots v_{j-1}^{k_{j-1}} v_{j+1}^{k_{j+1}} \cdots v_N^{k_N},$$

we have for the remaining part that

$$\begin{aligned} &\mathcal{L}_{ij}(\partial_i^2 + \partial_j^2)\mathbf{v}^k = \mathbf{v}^{k\setminus ij}\mathcal{L}_{ij}(\partial_i^2 + \partial_j^2) v_i^{k_i} v_j^{k_j} \\ =&\mathbf{v}^{k\setminus ij}\left(k_i(k_i-1)\mathcal{L}_{ij} v_i^{k_i-2} v_j^{k_j} + k_j(k_j-1)\mathcal{L}_{ij} v_i^{k_i} v_j^{k_j-2}\right). \end{aligned}$$

We also have that

$$
\begin{aligned}
&(\partial_i^2 + \partial_j^2)\mathcal{L}_{ij}\mathbf{v}^k = \mathbf{v}^{k\backslash ij}(\partial_i^2 + \partial_j^2)\mathcal{L}_{ij}v_i^{k_i}v_j^{k_j} \\
&= \mathbf{v}^{k\backslash ij}(\partial_i^2 + \partial_j^2)\int_{-\pi}^{\pi}\Big[(v_i\cos\theta + v_j\sin\theta)^{k_i}(v_j\cos\theta - v_i\sin\theta)^{k_j} \\
&- v_i^{k_i}v_j^{k_j}\Big]|\theta|^{-\alpha-1}\,d\theta \\
&= \mathbf{v}^{k\backslash ij}\int_{-\pi}^{\pi}\Big[k_i(k_i-1)\cos^2\theta(v_i\cos\theta + v_j\sin\theta)^{k_i-2}(v_j\cos\theta - v_i\sin\theta)^{k_j} \\
&+ k_j(k_j-1)\sin^2\theta(v_i\cos\theta + v_j\sin\theta)^{k_i}(v_j\cos\theta - v_i\sin\theta)^{k_j-2} \\
&- k_i(k_i-1)v_i^{k_i-2}v_j^{k_j} \\
&+ k_i(k_i-1)\sin^2\theta(v_i\cos\theta + v_j\sin\theta)^{k_i-2}(v_j\cos\theta - v_i\sin\theta)^{k_j} \\
&+ k_j(k_j-1)\cos^2\theta(v_i\cos\theta + v_j\sin\theta)^{k_i}(v_j\cos\theta - v_i\sin\theta)^{k_j-2} \\
&- k_j(k_j-1)v_i^{k_i}v_j^{k_j-2}\Big]|\theta|^{-\alpha-1}\,d\theta \\
&= \mathbf{v}^{k\backslash ij}\left(k_i(k_i-1)\mathcal{L}_{ij}v_i^{k_i-2}v_j^{k_j} + k_j(k_j-1)\mathcal{L}_{ij}v_i^{k_i}v_j^{k_j-2}\right).
\end{aligned}
$$

It follows that $\mathcal{L}\Delta p = \Delta\mathcal{L}p$ for all polynomials p. □

Lemma 2.6. *The operator $\mathcal{L}$ defined by (2.8) maps homogeneous harmonic polynomials of degree n to homogeneous harmonic polynomials of degree at most n.*

Proof. Let h_n be a homogeneous polynomial of degree n. We may write

$$\mathcal{L}h_n(\mathbf{v}) = \sum_{|k|=n} C_k\mathcal{L}\mathbf{v}^k$$

where $k = (k_1, \ldots, k_N)$, $C_k = C_{k_1\ldots k_N}$, $\mathbf{v}^k = v_1^{k_1}\cdots v_N^{k_N}$ and $|k| = \sum_{m=1}^N k_m$ for $k_m \geq 0$. Note that if $v_i \neq 0$, for $1 \leq i \leq N$ then

$$
\begin{aligned}
\mathcal{L}\mathbf{v}^k &= \frac{2}{N-1}\sum_{1\leq i<j\leq N} v_1^{k_1}\cdots v_{i-1}^{k_{i-1}}v_{i+1}^{k_{i+1}}\cdots v_{j-1}^{k_{j-1}}v_{j+1}^{k_{j+1}}\cdots v_N^{k_N}\times \\
&\int_{-\pi}^{\pi}\Big[(v_i\cos\theta + v_j\sin\theta)^{k_i}(v_j\cos\theta - v_i\sin\theta)^{k_j} - v_i^{k_i}v_j^{k_j}\Big]|\theta|^{-\alpha-1}\,d\theta \\
&= \frac{2\mathbf{v}^k}{N-1}\sum_{1\leq i<j\leq N}\int_{-\pi}^{\pi} g_{k_ik_j}(v_i, v_j, \theta)|\theta|^{-\alpha-1}\,d\theta,
\end{aligned}
$$

where

$$g_{k_ik_j}(v_i, v_j, \theta) =$$
$$-1 + \cos^{k_i+k_j}(\theta) \sum_{\ell=0}^{k_i} \sum_{m=0}^{k_j} \binom{k_i}{\ell} \binom{k_j}{m} (-1)^m \left(\frac{v_i}{v_j}\right)^{-\ell+m} \tan^{\ell+m}(\theta) .$$

Due to the presence of the factor $(v_i/v_j)^{-\ell+m}$ in each term of the double sum in $g_{k_ik_j}$, it follows that $\mathcal{L}h_n$ is homogeneous of degree n. By Lemma 2.5 it follows for a homogeneous harmonic polynomial H_n of degree n that $\mathcal{L}H_n$ is a homogeneous harmonic polynomial of degree n. □

Corollary 2.7. *The homogeneous polynomials $\mathcal{H}^N$ of N variables constitute a core for the operator $\mathcal{L}$ defined by (2.8).*

Proof. By Proposition 19.9 of Kallenberg [19] a sufficient condition for a dense subset D of $\mathcal{D}(\mathcal{L})$ to be a core for $\mathcal{L}$ is that $T_tD \subset D$. Let $D = \mathcal{H}^N$. It is known (see e.g. Vilenkin [28] p. 448) that the homogeneous harmonic polynomials are dense in $\mathcal{D}(\mathcal{L})$ (as they are dense in C and $\mathcal{D}(\mathcal{L}) \subset C$) and thus $\mathcal{H}^N$ is dense in $\mathcal{D}(\mathcal{L})$. Letting $\{T_t\}_{t\geq 0}$ be the semigroup generated by the closure of $\mathcal{L}$, we note that its restriction, $T_t|_{\mathcal{H}^N}$, to $\mathcal{H}^N$ can be written

$$T_t|_{\mathcal{H}^N} = \sum_{k\geq 0} \frac{t^k}{k!}\mathcal{L}^k.$$

By the same arguments as in the proof of Lemma 2.6 it follows that $T_th \in \mathcal{H}^N$ for $h \in \mathcal{H}^N$. □

Proof. (of Proposition 2.4) To prove the statement we use Proposition A.2, which tells us that the closure of $\mathcal{L}$ is the generator of a strongly continuous contraction semigroup on C if $\mathcal{L}$ is dissipative and if there are invariant finite dimensional subspaces $L_1, L_2, \ldots$ such that $\bigcup L_n$ is dense in C. It is known that the homogeneous harmonic polynomials of N variables are dense in C (see e.g. Vilenkin [28], p. 448), and according to Lemma 2.5 and 2.6 the space of spherical harmonics of degree less than n is invariant under $\mathcal{L}$. It is clear that $\mathcal{L}$ satisfies the maximum principle, i.e. condition (ii) of Theorem A.1, and thus it is dissipative. Proposition A.2 gives that the closure of $\mathcal{L}$ is the generator of a strongly continuous contraction semigroup. In order to check that the semigroup $T = \{T_t\}_{t\geq 0}$ generated by the closure of $\mathcal{L}$ is a Feller semigroup we have ensure that T is positive, but this follows from Theorem 1.6 p. 125 of Arendt et al [3], which states that a semigroup on $C(K)$, K compact, will be positive, given that its generator satisfies the positive maximum principle. $\square$

2.4 Approximation of unbounded collision kernel model

2.4.1 Truncated collision kernel approximation

A natural way of approximating the process generated by the closure of (2.8) is to just truncate the collision kernel, i.e. to replace the collision kernel in (2.8) by the one given by (1.3). That this approximation method actually works is proven in Proposition 2.8 and Corollary 2.9 below, which deal with the generator of the process, and the process itself, respectively.

Proposition 2.8. *Define the operator* $\mathcal{L}_\varepsilon$ *by*

$$\mathcal{L}_\varepsilon f = \frac{2}{N-1} \sum_{1\leq i<j\leq N} \int_{-\pi}^{\pi} [f(R_{ij}(\theta)\,\cdot\,) - f]\rho_\varepsilon(\theta)\, d\theta \qquad (2.9)$$

with ρ_ε *as in (1.3). For all* $f \in C^\infty$ *we have* $\mathcal{L}_\varepsilon f \to \mathcal{L} f$*, where the operator* $\mathcal{L}$ *is defined by (2.8).*

Proof. We want to show that for any $f \in C^\infty$

$$\lim_{\varepsilon\downarrow 0} \sup_{\mathbf{v}\in S^{N-1}(\sqrt{N})} |\mathcal{L}_\varepsilon f(\mathbf{v}) - \mathcal{L} f(\mathbf{v})| = 0.$$

To this end let $\varepsilon > 0$ and pick an arbitrary $f \in C^\infty$. Note that

$$\rho(\theta) - \rho_\varepsilon(\theta) = \mathbf{1}_{\{|\theta|\leq\varepsilon\}} \left(|\theta|^{-\alpha-1} - \varepsilon^{-\alpha-1}\right),$$

so that for an arbitary $\mathbf{v} \in S^{N-1}(\sqrt{N})$

$$(\mathcal{L} - \mathcal{L}_\varepsilon) f(\mathbf{v}) = \frac{2}{N-1} \sum_{1\leq i<j\leq N} \int_{-\varepsilon}^{\varepsilon} [f(R_{ij}(\theta)\mathbf{v}) - f(\mathbf{v})] \left[|\theta|^{-\alpha-1} - \varepsilon^{-\alpha-1}\right] d\theta.$$

Using the same Taylor expansion arguments as in (2.5) we get that

$$|\mathcal{L} f(\mathbf{v}) - \mathcal{L}_\varepsilon f(\mathbf{v})| \leq K \sum_{1\leq i<j\leq N} \int_{-\varepsilon}^{\varepsilon} \Big[\Big(\sup_{\mathbf{v}} \sup_{\ell\in\{i,j\}} |f'_{v_\ell}(\mathbf{v})|$$
$$+ \sup_{\mathbf{v}} \sup_{\ell_1,\ell_2\in\{i,j\}} |f''_{v_{\ell_1} v_{\ell_2}}(\mathbf{v})|\Big)\theta^2 + \mathcal{O}(\theta^3)\Big] \left(|\theta|^{-\alpha-1} - \varepsilon^{-\alpha-1}\right) d\theta$$

for some constant K depending on N. By compactness of $S^{N-1}(\sqrt{N})$ we get that

$$\sup_{\mathbf{v}} |\mathcal{L}_\varepsilon f(\mathbf{v}) - \mathcal{L} f(\mathbf{v})| \leq K^1 \int_{-\varepsilon}^{\varepsilon} \left(\theta^2 + \mathcal{O}(\theta^3)\right) \left(|\theta|^{-\alpha-1} - \varepsilon^{-\alpha-1}\right) d\theta$$

for some constant K^1 depending on N. Letting $\varepsilon \downarrow 0$ we have the desired convergence. □

Corollary 2.9. *Given that initial distributions converge, the process $X^\varepsilon = \{X_t^\varepsilon\}_{t\geq 0}$ generated by the operator $\mathcal{L}_\varepsilon$ as in (2.9) converges weakly to the process $X = \{X_t\}_{t\geq 0}$ generated by the closure of $\mathcal{L}$ in (2.8)*

Proof. Given that initial distributions converge, the Trotter-Kurtz theorem (i.e. Theorem A.3) gives that the convergence $X^\varepsilon \xrightarrow{\mathrm{d}} X$ is equivalent to $\mathcal{L}_\varepsilon f \to \mathcal{L} f$ for f in a core for $\mathcal{L}$. By Lemma A.4 C^∞ is a core for $\mathcal{L}$. Thus the statement follows from Proposition 2.8. □

2.4.2 Brownian approximation

The operator defined by equation (2.8) is unbounded and this makes the process generated by it hard to simulate. The idea of this subsection is to replace small jumps (angles) by a continuous process in order to get a better approximation of the process generated by (2.8) than the approximation given by just truncating the collision kernel as in (1.3). This idea originates from Asmussen and Rosinski [4] and Rosinksi and Cohen [4], where Brownian approximation schemes for Lévy Processes in $\mathbb{R}$ and $\mathbb{R}^n$, respectively, are proposed and analysed. Replacing small jumps by an appropriately scaled Brownian motion gives processes which can be simulated at moderate computational cost and some numerical results are presented in Sundén and Wennberg [27]. In terms of infinitesimal generators our approximation is given by the operator $\mathcal{L}_{b,\varepsilon}$ which is defined by

$$\mathcal{L}_{b,\varepsilon} f = \frac{2}{N-1} \left(\sum_{i<j} \int_{-\pi}^{\pi} \left(f(R_{ij}(\theta)\theta) - f\right) \rho_\varepsilon(\theta)\, d\theta + \frac{N K_{\varepsilon,\alpha}}{3} \Delta_{S^{N-1}(\sqrt{N})} f \right), \tag{2.10}$$

where ρ_ε is as in (1.3) and $f \in C^2$. It remains to choose $K_{\varepsilon,\alpha}$ in (2.10) appropriately. This choice is made by equating the second moments of the truncated and the unbounded collision kernels with respect to small θ such that $|\theta| \le \varepsilon$. For the unbounded collision kernel ρ in (2.8) the second moment of angles θ such that $|\theta| \le \varepsilon$ is given by

$$\int_{-\varepsilon}^{\varepsilon} \theta^2 \rho(\theta)\, d\theta = \int_{-\varepsilon}^{\varepsilon} \theta^2 |\theta|^{-\alpha-1}\, d\theta = \frac{2\varepsilon^{2-\alpha}}{2-\alpha},$$

and for the truncated collision kernel ρ_ε, we have that

$$\int_{-\varepsilon}^{\varepsilon} \theta^2 \rho_\varepsilon(\theta)\, d\theta = \varepsilon^{-\alpha-1} \int_{-\varepsilon}^{\varepsilon} \theta^2\, d\theta = \frac{2\varepsilon^{2-\alpha}}{3}.$$

What is lost by truncation in terms of second moments is to be compensated for by the Brownian part. This means that the right choice of $K_{\varepsilon,\alpha}$ is given by

$$K_{\varepsilon,\alpha} := \varepsilon^{2-\alpha} \left(\frac{3}{2-\alpha} - 1 \right). \tag{2.11}$$

Note that, given convergence of initial distributions $X_0^{b,\varepsilon} \xrightarrow{\mathrm{d}} X_0$, an argument analoguous to Corollary 2.9 implies that $X^{b,\varepsilon} \xrightarrow{\mathrm{d}} X$, as $\varepsilon \downarrow 0$, for $X^{b,\varepsilon} = \{X_t^{b,\varepsilon}\}_{t\ge 0}$ and $X = \{X_t\}_{t\ge 0}$ the processes with infinitesimal generators given by (2.10) and (2.8), respectively. To further justify the idea of replacing small angles by a time-scaled Brownian motion, we have the following proposition.

Proposition 2.10. *Let $\mathcal{L}_{b,\varepsilon}$, $\mathcal{L}_\varepsilon$ and $\mathcal{L}$ be the operators defined in (2.10), (2.9) and (2.8), respectively. For an arbitrary function $f \in C^\infty$, the rate of the convergence $\mathcal{L}_{b,\varepsilon} f \to \mathcal{L} f$ is of the order $\mathcal{O}(\varepsilon^{3-\alpha})$, whereas the rate of the convergence $\mathcal{L}_\varepsilon f \to \mathcal{L} f$ is of the order $\mathcal{O}(\varepsilon^{2-\alpha})$.*

Proof. Let $\mathcal{L}_{b,\varepsilon}^{ij}$, $\mathcal{L}_\varepsilon^{ij}$ and $\mathcal{L}^{ij}$ be the terms of $\mathcal{L}_{b,\varepsilon}$, $\mathcal{L}_\varepsilon$ and $\mathcal{L}$ corresponding to the velocity pair (v_i, v_j). To prove the assertion of the lemma, it clearly suffices to check that the convergence $\mathcal{L}_{b,\varepsilon}^{ij} f \to \mathcal{L}^{ij} f$ is of the order $\varepsilon^{3-\alpha}$, whereas the convergence $\mathcal{L}_\varepsilon^{ij} f \to \mathcal{L}^{ij} f$ is of the order $\varepsilon^{2-\alpha}$.

Pick $f \in C^\infty$. A Taylor expansion as in the proof of Theorem 2.2 leads to

$$\begin{aligned}
&(\mathcal{L}_\varepsilon^{ij} - \mathcal{L}^{ij}) f = \int_{-\varepsilon}^{\varepsilon} \left(f(R_{ij}(\theta)\cdot) - f\right) \left(\varepsilon^{-\alpha-1} - |\theta|^{-\alpha-1}\right) d\theta \\
&= \frac{1}{2} \int_{-\varepsilon}^{\varepsilon} \Big((-v_i f'_{v_i} - v_j f'_{v_j} + v_j^2 f''_{v_i v_i} + v_i^2 f''_{v_j v_j} - 2 v_i v_j f''_{v_i v_j}) \theta^2 \\
&+ \mathcal{O}(\theta^3) \Big) \left(\varepsilon^{-\alpha-1} - |\theta|^{-\alpha-1}\right) d\theta \\
&= \varepsilon^{2-\alpha} \left(\frac{1}{3} - \frac{1}{2-\alpha} \right) \left(v_j^2 f''_{v_i v_i} + v_i^2 f''_{v_j v_j} - 2 v_i v_j f''_{v_i v_j} - v_i f'_{v_i} - v_j f'_{v_j} \right) + \mathcal{O}(\varepsilon^{3-\alpha}).
\end{aligned} \tag{2.12}$$

Note that

$$\mathcal{L}^{ij}_{b,\varepsilon} = \mathcal{L}^{ij}_{\varepsilon} + \varepsilon^{2-\alpha}\left(\frac{1}{2-\alpha} - \frac{1}{3}\right)(v_j^2\partial_i^2 + v_i^2\partial_j^2 - 2v_iv_j\partial_i\partial_j - v_i\partial_i - v_j\partial_j).$$

This means that when we add the diffusion part to $\mathcal{L}^{ij}_{\varepsilon}$, the $\varepsilon^{2-\alpha}$ term in (2.12) will cancel so that

$$(\mathcal{L}^{ij}_{b,\varepsilon} - \mathcal{L}^{ij})f = \mathcal{O}(\varepsilon^{3-\alpha}).$$

Since $S^{N-1}(\sqrt{N})$ is compact

$$\sup_{\mathbf{v}\in S^{N-1}(\sqrt{N})} |(\mathcal{L} - \mathcal{L}_{\varepsilon})f(\mathbf{v})| = \mathcal{O}(\varepsilon^{2-\alpha}),$$

whereas

$$\sup_{\mathbf{v}\in S^{N-1}(\sqrt{N})} |(\mathcal{L} - \mathcal{L}_{b,\varepsilon})f(\mathbf{v})| = \mathcal{O}(\varepsilon^{3-\alpha})$$

and we are done. □

We would like to be able to draw conclusions about the speed of convergence of the processes invlolved as did Asmussen and Rosinski [4] for $\mathbb{R}$-valued Lévy Processes, but as of now we are in no position to do this. We are not aware of any general results on how the rate of convergence of processes can be determined by the rate of convergence of a family of generators. Asmussen and Rosinski established bounds of the type

$$\sup_{x\in\mathbb{R}} |\mathbf{P}\{X_1^{b,\varepsilon} \le x\} - \mathbf{P}\{X_1 \le x\}| \le K_{\varepsilon},$$

where the constant K_{ε} depends on the Lévy measure of the process X. Their proof uses the Berry-Esseen Theorem (see e.g. Gut [16] p.355), which is not available in our setting. In the paper on $\mathbb{R}^n$-valued Lévy processes by Cohen and Rosinski [10] there are no explicit bounds for the rate of convergence of distributions. This may indicate that it is hard to draw conclusions about the rate of convergence of the distributions in our case.

3 Invariant measures and spectral gaps

Recall that a measure ν is *invariant* for the Markov process with generator $\mathcal{L}$ and state space S if

$$\int_S \mathcal{L}f\,d\nu = 0 \tag{3.1}$$

for f in the domain of $\mathcal{L}$. If ν has a density g, (3.1) is equivalent to $\mathcal{L}^* g = 0$ where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$.

Due to compactness of the state space $S^{N-1}(\sqrt{N})$, all processes considered in this work enjoy invariant measures. Kac proved that the uniform measure, μ_N, is the unique invariant measure for the processes that he considered, i.e. the ones with bounded jump rate. It is almost obvious that the same holds for the processes with unbounded jump rate that are considered here, but below we will nontheless give an argument for making this rigorous.

For a measure ν and a function $f \in L^2(\nu)$ we define

$$\nu[f] := \int_{S^{N-1}(\sqrt{N})} f d\nu$$

and

$$\nu[f\,;f] := \nu[f^2] - (\nu[f])^2.$$

Consider now an arbitrary Markov process with generator $\mathcal{L}$ and unique invariant measure ν. The *spectral gap* of $\mathcal{L}$ with domain $\mathcal{D}(\mathcal{L})$ is the largest constant k such that

$$k\nu[f;f] \leq -\nu[f\mathcal{L}f] \quad \text{for} \quad f \in \mathcal{D}(\mathcal{L}),$$

or equiavalently

$$k = \inf\left\{-\frac{\nu[f\mathcal{L}f]}{\nu[f\,;f]}, f \in \mathcal{D}(\mathcal{L})\right\}.$$

Letting $\{T_t\}_{t\geq 0}$ be the semigroup generated by $\mathcal{L}$ it holds for an initial density f_0 that

$$\nu\left[(T_t f_0 - \nu[f_0])^2\right] \leq \mathrm{e}^{-2kt}\nu[f_0\,;f_0]. \tag{3.2}$$

For the Kac model with bounded jump rate, (3.2) takes the form

$$\mu_N\left[(T_t f_0 - 1)^2\right] \leq \mathrm{e}^{-2kt}\mu_N[f_0;f_0]$$

which makes sense for all $f_0 \in L^2(\mu_N)$.

With the collision kernel $\rho(\theta) = (2\pi)^{-1}$ Carlen, Carvalho and Loss [7] prove that the spectral gap is

$$k = \frac{1}{2}\frac{N+2}{N-1}.$$

This is a consequence of Theorem 1.3 of Carlen et al [7], which can be stated as follows:

Theorem 3.1 (Carlen, Carvalho and Loss). *Let ρ be a bounded collision kernel. The spectral gap k of the collision operator defined by*

$$\mathcal{L}f = \frac{2}{N-1}\sum_{i<j}\int_{-\pi}^{\pi}[f(R_{ij}(\theta)\cdot) - f]\,\rho(\theta)\,d\theta,$$

for $f \in L^2$, is bounded from below and the following estimate holds

$$k \geq \frac{N+2}{N-1}\inf_{m\neq 0}\int_{-\pi}^{\pi}\sin^2\left(\frac{m\theta}{2}\right)\rho(\theta)\,d\theta. \tag{3.3}$$

The following proposition concerning the spectral gap of the Kac model with a truncated collision kernel ρ_ε as in (1.3) is a simple consequence of Theorem 3.1.

Corollary 3.2. *The spectral gap k_ε of the operator $\mathcal{L}_\varepsilon$ as in (2.9) is bounded from below and the following estimate holds,*

$$k_\varepsilon \geq \frac{(N+2)}{(N-1)}\int_{-\pi}^{\pi}\sin^2\left(\frac{\theta}{2}\right)\rho_\varepsilon(\theta)\,d\theta. \tag{3.4}$$

Proof. For ε sufficiently small the infimum in (3.3) is attained at $m = 1$ and the claim follows from Theorem 3.1. □

Note that the the integral is (3.4) is bounded, uniformly in $\varepsilon \to 0$, and increasing with decreasing ε, and hence one is lead to believe that the rate of convergence for the process given by (2.8) can be determined immediately by letting $\varepsilon \to 0$; the first step is to establish the uniqueness of the invariant measure.

Proposition 3.3. *Let $\mathcal{L}$ be as in (2.8). The rotation invariant probability meausure μ_N on $S^{N-1}(\sqrt{N})$ is the unique normalized invariant measure for the process generated by the closure of $\mathcal{L}$.*

Proof. First we note that $\mathcal{L}$ (and its closure) commutes with convolution in the following way: Let g be a radial, smooth, function, which for clarity will be written $g(|\mathbf{v}|)$, and let

$$g * f(\mathbf{v}) \quad = \quad \mu_N[g(|\mathbf{v}-\cdot|)f].$$

Then

$$\mathcal{L}(g*f) \quad = \quad g*(\mathcal{L}f).$$

For $f \in C^2$, this follows by a direct calculation:

$$\begin{aligned} g * \mathcal{L}f(\mathbf{v}) &= \mu_N\Big[\sum_{1\le i<j\le N}\int_{-\pi}^{\pi} g(|\mathbf{v}-\cdot|)\rho(\theta)\,(f(R_{ij}(\theta)\cdot)-f)\;d\theta\Big] \\ &= \mu_N\Big[\sum_{1\le i<j\le N}\int_{-\pi}^{\pi} g(|R_{ij}(\theta)\mathbf{v}-R_{ij}(\theta)\cdot|)\rho(\theta)f(R_{ij}(\theta)\cdot) - \\ &\qquad\qquad g(|\mathbf{v}-\cdot|)\rho(\theta)f\;d\theta\Big] \\ &= \mathcal{L}(g*f)(\mathbf{v}), \end{aligned}$$

which holds because μ_N is invariant under rotations. For less regular functions f, or measures, the same holds, because $< g*f, h >=< f, g*h >$.

If ν is an invariant measure for the process generated by the closure of $\mathcal{L}$, then for all $\phi \in C$, and all $t>0$, $< T_t\phi-\phi, \nu >= 0$, where $\{T_t\}_{t\ge 0}$ is the semigroup generated by the closure of $\mathcal{L}$. Take a smooth function $g(|\mathbf{v}|)$ with compact support, and let $g_\gamma = c_\gamma g(|\mathbf{v}|\gamma^{-1})$, and let c_γ be such that

$$\mu_N[\,g_\gamma(|\mathbf{v}-\cdot|)] \;=\; 1\,,$$

where $|\mathbf{w}| = \sqrt{N}$ (the result only depends on $|\mathbf{w}|$), and let $\phi = g_\gamma * \psi$ for some function $\psi \in C$. Then, for an invarant measure ν,

$$\begin{aligned} 0 = \nu[T_t\phi-\phi] = \nu[T_t(g_\gamma*\psi) - g_\gamma*\psi] = \nu[g_\gamma*(T_t\psi-\psi)] \\ =< T_t\psi-\psi, g_\gamma*\nu >=< \psi, T_t(g_\gamma*\nu) - g_\gamma*\nu > \end{aligned}$$

But $g_\gamma*\nu$ is a smooth function, and hence the last member of this equation can be zero for all t only if $g_\gamma*\nu = 1$ for all γ, and hence $\nu = \mu_N$. □

For the rate of convergence to the invariant measure we have the following proposition.

Proposition 3.4. *The spectral gap k for the process given by (2.8) exists and is bounded below by*

$$k \ge \frac{(N+2)}{(N-1)}\int_{-\pi}^{\pi}\sin^2\left(\frac{\theta}{2}\right)|\theta|^{-\alpha-1}\,d\theta$$

Proof. As seen in Corollary 3.2 the spectral gap for the Kac model with truncated collision kernel ρ_ε is bounded from below by

$$\frac{(N+2)}{(N-1)}\int_{-\pi}^{\pi}\sin^2\left(\frac{\theta}{2}\right)\rho_\varepsilon(\theta)\,d\theta.$$

Letting $\mathcal{L}$ be the generator of the Kac model with unbounded collision kernel, as in (2.8), and $\mathcal{L}_\varepsilon$ be the generator of the Kac model with truncated collision kernel ρ_ε. Let $f \in \mathcal{D}(\mathcal{L})$ and $f_n \in C^\infty$ such that $f_n \to f$. By Proposition 2.8 we have

$$-\frac{\mu_N[f_n \mathcal{L} f_n]}{\mu_N[f_n; f_n]} = -\lim_{\varepsilon \downarrow 0} \frac{\mu_N[f_n \mathcal{L}_\varepsilon f_n]}{\mu_N[f_n; f_n]}.$$

Note also that

$$-\frac{\mu_N[f_n \mathcal{L}_\varepsilon f_n]}{\mu_N[f_n; f_n]} \geq \inf\left\{-\frac{\mu_N[f \mathcal{L}_\varepsilon f]}{\mu_N[f; f]}; f \in L^2\right\} \geq \frac{(N+2)}{(N-1)} \int_{-\pi}^{\pi} \sin^2\left(\frac{\theta}{2}\right) \rho_\varepsilon(\theta)\, d\theta,$$

Letting $\varepsilon \downarrow 0$ it follows that

$$-\frac{\mu_N[f_n \mathcal{L} f_n]}{\mu_N[f; f]} \geq \frac{(N+2)}{(N-1)} \int_{-\pi}^{\pi} \sin^2\left(\frac{\theta}{2}\right) |\theta|^{-\alpha-1}\, d\theta.$$

Letting $n \to \infty$ we get

$$-\frac{\mu_N[f \mathcal{L} f]}{\mu_N[f; f]} \geq \frac{(N+2)}{(N-1)} \int_{-\pi}^{\pi} \sin^2\left(\frac{\theta}{2}\right) |\theta|^{-\alpha-1}\, d\theta$$

and consequently

$$k = \inf\left\{-\frac{\mu_N[f \mathcal{L} f]}{\mu_N[f; f]}; f \in \mathcal{D}(\mathcal{L})\right\} \geq \frac{(N+2)}{(N-1)} \int_{-\pi}^{\pi} \sin^2\left(\frac{\theta}{2}\right) |\theta|^{-\alpha-1}\, d\theta.$$

□

The rotation invariant probability measure μ_N on $S^{N-1}(\sqrt{N})$ is the unique invariant measure for the process generated by the closure of $\mathcal{L}_{b,\varepsilon}$ as in (2.10) and the rate of convergence to equilibrium is quantified by the following theorem.

Theorem 3.5. *The operator $\mathcal{L}_{b,\varepsilon}$ defined in (2.10) enjoys a spectral gap $k_{b,\varepsilon}$ for which it holds that*

$$k_{b,\varepsilon} \geq k_\varepsilon + \frac{2}{3} K_{\varepsilon,\alpha},$$

with k_ε and $K_{\varepsilon,\alpha}$ as in (3.4) and (2.11), respectively.

Proof. As $\Delta_{S^{N-1}(\sqrt{N})}$ is self-adjoint and $\Delta_{S^{N-1}(\sqrt{N})} 1 = 0$, it follows from Proposition 4.5 of Ikeda and Watanabe [17] that the rotation invariant probability measure μ_N is the unique invariant measure for Brownian motion on $S^{N-1}(\sqrt{N})$. This means that μ_N is the unique invariant measure for the process generated by the closure of $\mathcal{L}_{b,\varepsilon}$. As $\Delta_{S^{N-1}}$ enjoys a spectral gap which is $N-1$ (see e.g. Ledoux [22], p.18) it follows from (2.7) that the spectral gap for

$$\frac{2NK_{\varepsilon,\alpha}}{3(N-1)}\Delta_{S^{N-1}(\sqrt{N})} \quad \text{is} \quad \frac{2}{3}K_{\varepsilon,\alpha}.$$

By Corollary 3.2 the truncated collision kernel ρ_ε gives a collision operator $\mathcal{L}_\varepsilon$ with domain L^2 that enjoys a spectral gap k_ε. Now if the domain of $\mathcal{L}_\varepsilon$ is restricted to C^2 we have that

$$k_\varepsilon = \inf\left\{-\frac{\mu_N[f\mathcal{L}_\varepsilon f]}{\mu_N[f;f]}; f\in L^2\right\} \le \inf\left\{-\frac{\mu_N[f\mathcal{L}_\varepsilon f]}{\mu_N[f;f]}; f\in C^2\right\} := k_\varepsilon^*.$$

For the spectral gap $k_{b,\varepsilon}$ of $\mathcal{L}_{b,\varepsilon}$ we have, by linearity of $\mu_N[\cdot]$, that

$$\begin{aligned}
k_{b,\varepsilon} &= \inf\left\{-\frac{\mu_N[f\mathcal{L}_{b,\varepsilon} f]}{\mu_N[f;f]}; f\in C^2\right\} \\
&\ge \inf\left\{-\frac{\mu_N[f\mathcal{L}_\varepsilon f]}{\mu_N[f;f]}; f\in C^2\right\} \\
&+ \inf\left\{-\frac{2NK_{\varepsilon,\alpha}}{3(N-1)}\frac{\mu_N[f\Delta_{S^{N-1}(\sqrt{N})}f]}{\mu_N[f;f]}; f\in C^2\right\} \\
&= k_\varepsilon^* + \frac{2}{3}K_{\varepsilon,\alpha} \ge k_\varepsilon + \frac{2}{3}K_{\varepsilon,\alpha}.
\end{aligned}$$

□

The following corollary is an immediate consequence of Theorem 3.1 and Theorem 3.5.

Corollary 3.6. *The spectral gap $k_{b,\varepsilon}$ for the operator $\mathcal{L}_{b,\varepsilon}$ defined in equation (2.10) satisfies the following lowerbound:*

$$k_{b,\varepsilon} \ge \frac{N+2}{N-1}\int_{-\pi}^{\pi}\sin^2\left(\frac{\theta}{2}\right)\rho_\varepsilon(\theta)\,d\theta + \frac{2}{3}K_{\varepsilon,\alpha},$$

where $K_{\varepsilon,\alpha}$ is given by (2.11).

4 Diffusion approximation in the three-dimensional model

In this section we consider velocity vectors $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_N)$, where $\mathbf{v}_k \in \mathbb{R}^3$ and a jump process that corresponds to physically realistic collisions between two point particles moving in $\mathbb{R}^3$. Such collisions conserve momentum and energy and hence

$$\sum_{k=1}^{N} |\mathbf{v}_k|^2 = N \quad \text{and} \quad \sum_{k=1}^{N} \mathbf{v}_k = 0.$$

As in the original Kac model, a jump involves only two components of the velocity vector; in this case the components are three-dimensional and they are updated by the formula

$$\begin{aligned} \mathbf{v}_i^*(\omega) &= \mathbf{v}_i + (\omega \cdot (\mathbf{v}_j - \mathbf{v}_i))\omega \\ \mathbf{v}_j^*(\omega) &= \mathbf{v}_j - (\omega \cdot (\mathbf{v}_j - \mathbf{v}_i))\omega, \end{aligned}$$

where $\omega \in S^2$ is picked at random. The generator of the Markov process is in this case given by

$$\mathcal{L}f(\mathbf{v}) = \frac{2}{N-1} \sum_{1 \le i < j \le N} \int_{S^2} \left[f(\mathbf{v}_{ij}^*(\omega)) - f(\mathbf{v}) \right] B(\omega \cdot (\mathbf{v}_i - \mathbf{v}_j)/|\mathbf{v}_i - \mathbf{v}_j|) d\omega, \tag{4.1}$$

where

$$\mathbf{v}_{ij}^*(\omega) = (\mathbf{v}_1, \dots, \mathbf{v}_i^*(\omega), \dots, \mathbf{v}_j^*(\omega), \dots, \mathbf{v}_N),$$

and the *cross-section* B maps $[-1, 1]$ to $\mathbb{R}^+$. The integral in (4.1) is to be interpreted as a principal value if B is non-integrable. If we define θ through

$$\cos\theta = \omega \cdot (\mathbf{v}_i - \mathbf{v}_j)/|\mathbf{v}_i - \mathbf{v}_j|$$

we may rewrite an arbitrary term of the sum (4.1) as (see e.g. Desvillettes [11] p.261)

$$\begin{aligned} &\int_0^{\pi} \int_0^{2\pi} [f(\mathbf{v}_1, \dots, \mathbf{v}_i - \frac{1}{2}(\mathbf{v}_i - \mathbf{v}_j)(1 - \cos\theta) \qquad (4.2) \\ &+ \frac{1}{2}|\mathbf{v}_i - \mathbf{v}_j|(\cos(\phi) h_{\mathbf{v}_i - \mathbf{v}_j} + \sin(\phi) i_{\mathbf{v}_i - \mathbf{v}_j}) \sin\theta, \dots \mathbf{v}_j + \frac{1}{2}(\mathbf{v}_i - \mathbf{v}_j)(1 - \cos\theta) \\ &- \frac{1}{2}|\mathbf{v}_i - \mathbf{v}_j|(\cos(\phi) h_{\mathbf{v}_i - \mathbf{v}_j} + \sin(\phi) i_{\mathbf{v}_i - \mathbf{v}_j}) \sin\theta, \dots, \mathbf{v}_N) - f \big] D(\theta) d\phi \, d\theta, \end{aligned}$$

where $D : [0, \pi] \to \mathbb{R}^+$ and where

$$\left(\frac{\mathbf{v}_i - \mathbf{v}_j}{|\mathbf{v}_i - \mathbf{v}_j|}, h_{\mathbf{v}_i - \mathbf{v}_j}, i_{\mathbf{v}_i - \mathbf{v}_j} \right)$$

is an orthonormal basis of $\mathbb{R}^3$. The relationship between the cross-sections B in (4.1) and D in (4.2) is stated explicitely in Desvillettes [11] (eq. (7) and (10)).

In order to find the generator of the approximating diffusion for the process generated by (4.2), we carry out a grazing collision limit similar to the one in

Subsection 2.2. To this end we let

$$D^\delta(\theta) = \frac{1}{\delta^3} D(\delta^{-1}\theta),$$

and let f be an arbitrary C^2-function. Replacing D with D^δ in (4.1), we note that the upper integration limit of the outer integral becomes $\delta\pi$. A formal Taylor expansion gives

$$\int_0^{\delta\pi}\int_0^{2\pi}\Bigg[f(\mathbf{v}_1,\dots,\mathbf{v}_i - \frac{1}{2}(\mathbf{v}_i - \mathbf{v}_j)(1-\cos\theta)$$
$$+\frac{1}{2}|\mathbf{v}_i - \mathbf{v}_j|(\cos(\phi)h_{\mathbf{v}_i-\mathbf{v}_j} + \sin(\phi)i_{\mathbf{v}_i-\mathbf{v}_j})\sin\theta,\dots \mathbf{v}_j + \frac{1}{2}(\mathbf{v}_i - \mathbf{v}_j)(1-\cos\theta)$$
$$-\frac{1}{2}|\mathbf{v}_i - \mathbf{v}_j|(\cos(\phi)h_{\mathbf{v}_i-\mathbf{v}_j} + \sin(\phi)i_{\mathbf{v}_i-\mathbf{v}_j})\sin\theta,\dots,\mathbf{v}_N) - f\Bigg] D^\delta(\theta)d\phi\, d\theta$$
$$\to C\left[(\mathbf{v}_i - \mathbf{v}_j)\cdot(\nabla_j - \nabla_i) + \frac{1}{2}|\mathbf{v}_i - \mathbf{v}_j|^2(\nabla_j - \nabla_i)^2\right] f,$$

as $\delta \downarrow 0$, where

$$C = \pi \lim_{\delta\downarrow 0}\int_0^{\delta\pi}(1-\cos\theta)D^\delta(\theta)\, d\theta.$$

Taking the limit in each of the terms and summing we obtain a diffusion operator known as the *Balescu-Prigogine operator*, which was introduced in [6] (See also Lancelotti and Kiessling [20]). This operator may be written

$$\mathcal{L}_{d,\varepsilon} = \frac{2K_\varepsilon^1}{N-1}\sum_{1\le i<j\le N}(\mathbf{v}_i - \mathbf{v}_j)\cdot(\nabla_j - \nabla_i) + \frac{1}{2}|\mathbf{v}_i - \mathbf{v}_j|^2(\nabla_j - \nabla_i)^2$$
$$= \frac{2K_\varepsilon^1}{N-1}\sum_{1\le i<j\le N}(\nabla_j - \nabla_i)|\mathbf{v}_i - \mathbf{v}_j|^2(\nabla_i - \nabla_j),$$

Here we have introduced the notation with ε because, as in subsection 2.4.2, the diffusion operator will be used to approximate the small jumps in 4.2: We can then write

$$\mathcal{L} \approx \mathcal{L}_\varepsilon + \mathcal{L}_{d,\varepsilon},$$

where $\mathcal{L}_\varepsilon$ is obtained by restricting D to the interval $[\varepsilon,\pi]$ and where K_ε^1 is chosen to equate the difference in second moments of D and its restriction to $[\varepsilon,\pi]$.

For simulation purposes the following proposition is relevant.

Proposition 4.1. *Let $\{W_t\}_{t\geq 0}$ be a $3N$-dimensional standard Brownian motion. The $S^{3N-4}(\sqrt{3N})$-valued process $\{V_t\}_{t\geq 0}$ generated by $\frac{N-1}{2K_\varepsilon^1}\mathcal{L}_{d,\varepsilon}$ satisfies the SDE*

$$dV_t = -NV_t dt + \sigma(V_t)dW_t,$$

where the elements of the $3N \times 3N$ matrix σ have to be chosen so that

$$\sigma\sigma^T = \begin{pmatrix} S_{11} & S_{12} & \dots & S_{1N} \\ S_{21} & S_{22} & \dots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \dots & S_{NN} \end{pmatrix},$$

where S_{ij} are 3×3 matrices such that

$$S_{ij} = \begin{cases} N(1+|\mathbf{v}_i|^2)\, I & \text{if } i = j \\ -|\mathbf{v}_i - \mathbf{v}_j|^2\, I & \text{if } i \neq j, \end{cases}$$

and where I is the 3×3 identity matrix.

Proof. It is well known from the literature, see e.g. Øksendal [24], Theorem 7.3.3, that the generator $\mathcal{A}$ of the $\mathbb{R}^{3N}$-valued process $\{X_t\}_{t\geq 0}$ satisfying

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

is given by

$$\mathcal{A}f = \sum_{k=1}^{3N} b_k(x)\frac{\partial f}{\partial x_k} + \frac{1}{2}\sum_{k,\ell=1}^{3N}(\sigma\sigma^T)_{k\ell}(x)\frac{\partial^2 f}{\partial x_k \partial x_\ell},$$

for $f \in C_0^2(\mathbb{R}^{3N})$. Letting $\mathbf{v}_i = (v_{i1}, v_{i2}, v_{i3})^T$, the drift coefficients b_k are determined by

$$\sum_{1\leq i<j\leq N}(\mathbf{v}_i - \mathbf{v}_j)\cdot(\nabla_j - \nabla_i) = \sum_{1\leq i<j\leq N}\sum_{\ell=1}^{3}(v_{i\ell} - v_{j\ell})(\partial_{j\ell} - \partial_{i\ell}). \tag{4.3}$$

It is an exercise to check that (4.3) equals

$$\sum_{k=1}^{N}\sum_{\ell=0}^{3}\left(\sum_{m=0}^{N} v_{m\ell} - Nv_{k\ell}\right)\partial_{k\ell} = -N\sum_{k=1}^{N}\sum_{\ell=1}^{3} v_{k\ell}\partial_{k\ell},$$

where the last equality is due to the assumption of zero momentum. Re-indexing according to $(v_{11}, v_{12}, v_{13}, v_{21}, \dots v_{NN-1}, v_{NN}) = (v_1, v_2, v_3, v_4, \dots v_{3N-1}, v_{3N})$, we get

$$b_k(\mathbf{v}) = -Nv_k.$$

The diffusion coefficients $(\sigma\sigma^T)_{k,\ell}$ are determined by

$$\sum_{1\le i<j\le N} |\mathbf{v}_i - \mathbf{v}_j|^2(\nabla_j - \nabla_i)^2 = \sum_{1\le i<j\le N} |\mathbf{v}_i - \mathbf{v}_j|^2 \sum_{m=1}^{3} (\partial_{im}^2 - 2\partial_{im}\partial_{jm} + \partial_{jm}^2), \tag{4.4}$$

so that the coefficient of each ∂_{im}^2 is $\sum_{j=1}^N |\mathbf{v}_i - \mathbf{v}_j|^2$ for $1 \le i \le N$. Now

$$\begin{aligned}\sum_{j=1}^{N} |\mathbf{v}_i - \mathbf{v}_j|^2 =&(N-1)|\mathbf{v}_i|^2 - 2\mathbf{v}_i \cdot \sum_{j\neq i} \mathbf{v}_j + \sum_{j\neq i} |\mathbf{v}_j|^2 \\ =&(N-1)|\mathbf{v}_i|^2 - 2\mathbf{v}_i \cdot (-\mathbf{v}_i) + N - |\mathbf{v}_i|^2 \\ =&N(1+|\mathbf{v}_i|^2),\end{aligned}$$

where the second equality is due to zero momentum and constant and equal to N kinetic energy. Thus, the diagonal elements $(\sigma\sigma^T)_{kk}$ are $N(1+|\mathbf{v}_i|^2)$ for $k = 3i-2, 3i-1$ and $3i$. For the off-diagonal elements we have that $\partial_{im}\partial_{jm} = \partial_{jm}\partial_{im}$, since we assume that the domain of the operator $\mathcal{L}_{d,\varepsilon}$ is C^2. From (4.4) it is clear that that the coefficient of $\partial_{im}\partial_{jm}$ is given by $-2|\mathbf{v}_i - \mathbf{v}_j|^2$ for $i \neq j$. This means that the elements $(\sigma\sigma^T)_{k\ell}$ of the diffusion matrix are $-|\mathbf{v}_i - \mathbf{v}_j|^2$ for $(k,\ell) = (3i-2, 3j-2), (3i-1, 3j-1)$ and $(3i, 3j)$ for $i \neq j$. It follows that the matrix $\sigma\sigma^T$ consists of the 3×3 matrices S_{ij} as claimed in the proposition.

□

To sum things up, we get that the generator of the approximating process to the process generated by the closure of the unbounded operator (4.2) is the closure of $\mathcal{L}_{d,\varepsilon} + \mathcal{L}_\varepsilon$, where $\mathcal{L}_{d,\varepsilon}$ is the appropriately scaled Balescu-Prigogine operator and $\mathcal{L}_\varepsilon$ is (4.2) with the unbounded cross-section D replaced by its restriction to $[\varepsilon, \pi]$. A rather detailed numerical investigation of this approximation is presented in Sundén and Wennberg [27].

A Prerequisities on Feller processes

This appendix contains a few results on Feller processes that are used in the paper. Standard references are e.g. Ethier and Kurtz [13] or Kallenberg [19].

Let S be a compact metric space and let $C(S)$ be the Banach space of continuous functions with norm $||f|| = \sup_{x \in S} |f(x)|$. In our work $S = S^{N-1}(\sqrt{N})$. A time-homogeneous Markov process $\{X_t\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in S is *associated* with a semigroup $T = \{T_t\}_{t \geq 0}$ on $C(S)$ if

$$\mathbf{E}\left\{ f(X_{t+s}) \middle| \mathcal{F}_t^X \right\} = T_s f(X_t),$$

for all $s, t \geq 0$ and $f \in C(S)$. The semigroup T is said to be *strongly continuous* if $\lim_{t \to 0} T_t f = f$ for all $f \in C(S)$ and it is said to be *positive* if it maps non-negative functions to non-negative functions. The semigroup T is a *contraction* if $||T_t|| \leq 1$ for all $t \geq 0$. A strongly continuouos positive contraction semigroup is called a *Feller semigroup*. It can be shown, see e.g. Ethier and Kurtz [13], chapter 4, that every Feller semigroup is associated with a Markov process with sample paths in $D_S[0, \infty)$, the space of right continous with left limits functions with values in S. Such a Markov process is called a *Feller process*.

The *infinitesimal generator* $\mathcal{L}$ of the semigroup is the linear operator defined by the strong limit

$$\mathcal{L}f = \lim_{t \to 0} \frac{T_t f - f}{t}. \tag{A.1}$$

The domain, $\mathcal{D}(\mathcal{L}) \subset C(S)$, of the infinitesimal generator is the space of all functions $f \in C(S)$ for which (A.1) exists. The infinitesimal generator of a Feller semigroup uniquely determines the Feller semigroup (see e.g. Kallenberg [19] Lemma 19.5). If $\mathcal{L}$ is closed (i.e. if the graph $G_{\mathcal{L}} = \{(f, \mathcal{L}f); f \in \mathcal{D}(\mathcal{L})\}$ is a closed subspace of $C(S) \times C(S)$) a subspace D of $\mathcal{D}(\mathcal{L})$ is a *core* for $\mathcal{L}$ if the closure of the restriction $\mathcal{L}|_D$ of $\mathcal{L}$ to D is $\mathcal{L}$ and in this case $\mathcal{L}$ is uniquely determined by $\mathcal{L}|_D$. An operator $\mathcal{L}$ is called *dissipative* if for all $f \in \mathcal{D}(\mathcal{L})$ and $\lambda > 0$ it holds that $||(\lambda - \mathcal{L})f|| \geq \lambda ||f||$.

The *resolvent* R_λ of a semigroup T may be defined for each $\lambda > 0$ and $f \in C(S)$ as

$$R_\lambda f = \int_0^\infty \mathrm{e}^{-\lambda t} (T_t f) dt, \tag{A.2}$$

It can be shown for the resolvent of a strongly continuous contraction semigroup with generator $\mathcal{L}$ (e.g. Ethier and Kurtz [13], Proposition 2.1, p.10) that $R_\lambda = (\lambda - \mathcal{L})^{-1}$.

In order to determine which linear operators that are infinitesimal generators of Feller semigroups, the Hille-Yosida theorem (see e.g. Kallenberg [19] Theorem 19.11) is very important.

Theorem A.1. (**Hille-Yosida**) *Let $\mathcal{L}$ be a linear operator on $C(S)$ with domain $\mathcal{D}(\mathcal{L})$. Then $\mathcal{L}$ is closable and its closure $\bar{\mathcal{L}}$ is the generator of a Feller semigroup on $C(S)$ if and only if the following three conditions hold:*

$$\begin{aligned} &(i)\ \mathcal{D} \text{ is dense in } C(S) \\ &(ii)\ \text{if } f(x_0) = \sup_{x \in S} f(x) \geq 0 \text{ then } \mathcal{L}f(x_0) \leq 0 \\ &(iii)\ \text{the range of } \lambda_0 - \mathcal{L} \text{ is dense in } C(S) \text{ for some } \lambda_0 > 0. \end{aligned} \tag{A.3}$$

One should note that if, for an operator $\mathcal{L}$, condition (ii) of the Hilla-Yosida theorem is satisfied then $\mathcal{L}$ is dissipative.

The following proposition, which can be found in Ethier and Kurtz [13], will also be used to prove that certain operators are generators of Feller processes.

Proposition A.2. *Let $\mathcal{L}$ be a dissipative linear operator on a space L with domain $\mathcal{D}(\mathcal{L})$ and $L_1, L_2 \ldots$ finite dimensional subspaces of $\mathcal{D}(\mathcal{L})$ such that $\overline{\cup_{n=1}^{\infty} L_n} = L$. If it holds that $\mathcal{L}L_n \subset L_n$ then $\mathcal{L}$ is closable and its closure is the generator of a strongly continuous contraction semigroup on L.*

The following theorem relates convergence of generators to weak convergence of Feller Processes.

Theorem A.3. **(Trotter-Kurtz)** *Let $\{X_t\}_{t\geq 0}$, $\{X_t^1\}_{t\geq 0}$, $\{X_t^2\}_{t\geq 0}, \dots$ be Feller processes taking values in a compact space S. Let the corresponding Feller semigroups $\{T_t\}_{t\geq 0}$, $\{T_t^1\}_{t\geq 0}$, $\{T_t^2\}_{t\geq 0}, \dots$ have infinitesimal generators $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2 \dots$ with domains $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2 \dots$, respectively. Let D be a core for $\mathcal{D}$. Then the following three conditions are equivalent:*

(i) If $f \in D$, there exist $f_n \in \mathcal{D}_n$ with $f_n \to f$ and $\mathcal{L}_n f_n \to \mathcal{L} f$.

(ii) If $X^n(0) \stackrel{d}{\to} X(0)$ then $X^n \stackrel{d}{\to} X$ in $D_S[0, \infty)$.

(iii) For all $f \in C$, $T_t^n f \to T_t f$ for $t > 0$.

To use the Trotter-Kurtz theorem in our setting, we need to verify that Kac models with bounded collision kernels constitute Feller processes. This will be done using the Hille-Yosida theorem (Theorem A.1). It is also of importance to find cores of infinitesimal generators and the following lemma will be employed in doing so.

Lemma A.4. *The class C^∞ is a core for the operator $\mathcal{L}$ defined by (2.8) and for the Laplace-Beltrami operator $\Delta_{S^{N-1}(\sqrt{N})}$.*

Proof. By Lemma 19.8 of Kallenberg [19], a subset D of the domain $\mathcal{D}(\mathcal{A})$ of the generator $\mathcal{A}$ of a Feller semigroup on C, is a core for $\mathcal{A}$ if and only if for any $\lambda > 0$ it holds that $(\lambda - \mathcal{A})D$ is dense in C. By Corollary 2.7, $\mathcal{H}^N$ is a core for $\mathcal{L}$ so for any $\lambda > 0$ we have that $(\lambda - \mathcal{L})\mathcal{H}^N$ is dense in C and thus $(\lambda - \mathcal{L})C^\infty$ is dense in C and C^∞ is a core for $\mathcal{L}$.

Note that that $\Delta_{S^{N-1}(\sqrt{N})} h \in \mathcal{H}^N$ for $h \in \mathcal{H}^N$. Letting $\{T_t^\Delta\}_{t\geq 0}$ be the semigroup generated by the closure of $\Delta_{S^{N-1}(\sqrt{N})}$, we note that its restriction $T_t^\Delta|_{\mathcal{H}^N}$ to $\mathcal{H}^N$ can be written

$$T_t^\Delta|_{\mathcal{H}^N} = \sum_{k\geq 0} \frac{t^k}{k!} \Delta^k_{S^{N-1}(\sqrt{N})}.$$

It follows that $T_t^\Delta h \in \mathcal{H}^N$ for $h \in \mathcal{H}^N$. By Proposition 19.9 of Kallenberg [19] $\mathcal{H}^N$ is a core for $\Delta_{S^{N-1}(\sqrt{N})}$ and thus $(\lambda - \Delta_{S^{N-1}(\sqrt{N})})\mathcal{H}^N$ is dense in C. Consequently $(\lambda - \Delta_{S^{N-1}(\sqrt{N})})C^\infty$ is dense in C and thus C^∞ is a core for $\Delta_{S^{N-1}(\sqrt{N})}$.

□

References

[1] R. Alexandre, C. Villani (2002). On The Boltzmann Equation For Long-Range Interactions *Comm. Pure Appl. Math.* **55**, No 1, 30-70.

[2] G.E Andrews, R. Askey, R, Roy (1999). Special Functions. *Cambridge University Press*

[3] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H.P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, U. Schlotterbeck (1986). One-parameter Semigroups of Positive Operators. *Springer Lecture Notes in Mathematics*

[4] S. Asmussen, J. Rosinski (2001). Approximations of small jumps of Lévy processes with a view towards simulation. *J. Appl. Probab.* **38**, No 2, 482-493.

[5] V. Bagland, B. Wennberg and Y. Wondmagegne (2007). Stationary states for the non-cutoff Kac equation with a Gaussian thermostat, *Nonlinearity* **20**, No 3, 583-604.

[6] R. Balescu, I. Prigogine (1959). Irreversible Processes in Gases, II. *Physica* **25**, 302-323.

[7] E.A. Carlen, M.C. Carvalho, M. Loss (2003). Determination of the Spectral Gap for Kacs Master Equation and Related Stochastic Evolutions. *Acta Mathematica* **191**, No 1, 1-54.

[8] E.A. Carlen, J.S. Geronimo, M. Loss (2008). Determination of the Spectral Gap in the Kac Model for Physical Momentum and Energy Conserving Collisions. *SIAM J. Math. Anal.* **40**, No 1, 327-364.

[9] M-F. Chen (2005). Eigenvalues, Inequalities, and Ergodic Theory. *Springer, Probability and Its Applications.*

[10] S. Cohen, J. Rosinski (2007). Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes. *Bernoulli* **13**, No 1, 195-210.

[11] L. Desvillettes (1992). On Asymptotics of the Boltzmann Equation. When the Collisions Become Grazing. *Transport Theory and Statistical Physics* **21**, No 3, 259-276.

[12] L. Desvillettes, C. Graham, S. Méléard (1999). Probabilistic Interpretation and Numerical Approximation of a Kac Equation without Cutoff. *Stoch. Proc. Appl.* **84**, No 1, 115-135.

[13] S.N. Ethier, T.G. Kurtz (1986). Markov Processes, Characterization and Convergence. *Wiley Series in Probability and Mathematical Statistics*

[14] N. Fournier, S. Méléard (2001). Monte Carlo Approximations for 2D Boltzmann Equations without Cutoff. *Markov Process. Related Fields* **7**, 159-191.

[15] N. Fournier, S. Méléard (2002). A Stochastic Particle Numerical Method for 3D Boltzmann Equations without Cutoff. *Math. Comp.* **71**, 583-604.

[16] A. Gut (2005). Probability: A Graduate Course. *Springer Texts in Statistics*

[17] N. Ikeda, S. Watanabe (1989). Stochastic Differential Equations and Diffusion Processes. *North-Holland.*

[18] M. Kac (1956). Foundations of Kinetic Theory. *Third Berkeley Symposium on Mathematical Statistics and Probability. Edited by Neyman, J*, 171-197.

[19] O. Kallenberg (2002). Foundations of Modern Probability, Second Edition. *Springer Probability and Its Applications.*

[20] M. Kiessling, C. Lancelotti (2004) On the Master-Equation Approach to Kinetic Theory: Linear and Nonlinear Fokker-Planck Equations. *Transport Theory and Statistical Physics* **33**, 379-401.

[21] M. Kiessling, C. Lancelotti (2006) The Linear Fokker-Planck Equation for the Ornstein-Uhlenbeck Process as an (Almost) Nonlinear Kinetic Equation for an Isolated N-Particle System. *J Stat. Phys.* **123**, No 3, 525-546.

[22] M. Ledoux (1998). The Geometry of Markov Diffusion Generators. *Lecture notes, ETH Zürich.*

[23] S.A. Molchanov (1968). Strong Feller Property of Diffusion Processes on Smooth Manifolds. *Theor.Probab.Appl* **13**, No 3, 471-475.

[24] B. Øksendal (2000). Stochastic Differential Equations, 5th ed. 2nd printing. *Springer Universitext.*

[25] L.C.G. Rogers, D. Williams (1994). Diffusions, Markov Processes and Martingales, Volume one: Foundations, Second Edition *Wiley Series in Probability and Mathematical Statistics.*

[26] D.W. Stroock (1971). On the growth of stochastic integrals. *Z. Wahrscheinlichkeitsthorie verw. Geb.* **18**, 340-344.

[27] M. Sundén, B. Wennberg, B. (2008). Brownian approximation and Monte Carlo simulation for the Kac Equation without Cutoff. *J Stat.Phys.* **130**, No 2, 295-312.

[28] N.J. Vilenkin (1968). Special Functions and the Theory of Group Representations. *AMS Translations of Mathematical Monograps, Volume 22.*

[29] Y. Wondmagegne. (2005). Kinetic Equations with a Gaussian Thermostat, Doctoral Thesis. *Chalmers University of Technology, Göteborg University.*

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Brownian Approximation and Monte Carlo Simulation of the Non-Cutoff Kac Equation

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Abstract The non-cutoff Boltzmann equation can be simulated using the DSMC method, by a truncation of the collision term. However, even for computing stationary solutions this may be very time consuming, in particular in situations far from equilibrium. By adding an appropriate diffusion, to the DSMC-method, the rate of convergence when the truncation is removed, may be greatly improved. We illustrate the technique on a toy model, the Kac equation, as well as on the full Boltzmann equation in a special case.

Keywords Kac equation · Direct simulation Monte Carlo · Diffusion approximation · Thermostat · Non-equilibrium stationary state · Markov jump process

The Boltzmann equation describes the evolution of the phase space density of a gas. It is a nonlinear equation in many dimensions, which makes it difficult to treat by e.g. finite difference methods. The classical way of solving the Boltzmann equation numerically is by means of Monte Carlo simulation. The method was first described by Bird (see the book [4]), but since then many variations on the theme have been published [2, 23, 25, 29].

Very briefly, the DSMC-method can be described as follows: The gas is represented by a finite (although sometimes rather large) number of particles. The time evolution is then carried out by alternating a transport step, in which the particles move independently with their own velocity, and a collision step. The spatial domain of calculation is divided into cells which should be large enough that it typically contains a not too small number of particles, but still small enough to take into account the spatial gradients in the problem.

The collision step is then carried out in each cell separately as a jump process in $\mathbb{R}^{3n}$, where n is the number of particles, and the jumps occur as the velocity change in classical collisions between randomly chosen pairs of particles. The collisions conserve en-

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ergy and momentum, so the jump process actually takes place on $\{(v_1,\ldots,v_n)\in\mathbb{R}^{3n} \mid \sum v_k = u\in\mathbb{R}^3, \sum v_k^2 = W\}$.

The simulation can be understood as a sampling of the solution of a Poisson driven stochastic differential equation. In the original DSMC methods, the rate of the underlying Poisson process was always taken to be finite, but there are at least two reasons for considering infinite (or very large) collision rates. The first one occurs when the density of the gas is very large in a cell, i.e. when n is very large. One way of handling that situation is to sample the velocity distribution at the end of a collision step as a suitable mixture of a distribution as the one that results from a moderate number of collisions and the equilibrium distribution, which is a Maxwellian [25–27].

The other case derives from the fact that many realistic collision models correspond to long-range potentials, which effectively gives rise to an infinite collision rate. The vast majority of collisions only change the velocities marginally, and so the rate of change of momentum due to collisions is finite.

To carry out a Monte Carlo simulation in this situation, one may truncate the jump process so as to obtain a finite collision rate. It has been proven [10, 13, 14] that truncated Monte Carlo methods converge as the truncation is lifted, and this has also been illustrated by numerical experiments. However, we have found that in certain cases, in particular when the stationary solutions are far from equilibrium, the jump rate must be truncated at a very high (and so, costly) level to get an acceptable accuracy.

We propose here a method to replace the small jumps by an appropriate diffusion, and show by example that this gives an important improvement of the accuracy compared to just ignoring the small jumps. This has been inspired by the works [1, 30], where this technique is proposed for simulating Lévy processes in $\mathbb{R}^n$.

There are, of course, other methods than the Monte Carlo methods, for solving the Boltzmann equation, and in particular there are fast methods based on the Fourier transform, that have been used successfully for the non-cutoff situation [11, 12, 22].

This study was motivated by the difficulty of obtaining accurate estimations of the non-equilibrium stationary state for a non-cutoff collision kernel, and of the theoretical results in [5], where the one-dimensional Kac equation with a Gaussian thermostat was studied. Also in this paper, the main part is devoted to the Kac equation. The diffusion term is then a Brownian motion on the sphere S^{n-1}, and the approximation is very straight forward. We describe the method in some detail for this case. However, from a conceptual point of view, the method is not restricted to one-dimensional models, and we have also carried out numerical calculations for the Boltzmann equation with a thermostat and Maxwellian non-cutoff collisions. In that case, the diffusion model is more complicated (it is essentially the Balescu-Prigogine model for Maxwellian molecules [18, 33]), and the actual calculation is carried out somewhat differently, as described in Sect. 3.

In [34, 36], it is shown that, contrary to the Kac equation, the Boltzmann equation with a thermostatted force field only has trivial stationary states, and hence it is only interesting to compare the evolution of the solutions. At the end of the paper, we present some numerical results for this model.

We note, finally, that for non-Maxwellian molecules the situation is rather different, and then it may in some cases be more appropriate to go the other way around and to approximate the diffusion process by a non-cutoff collision process [7].

1 The Kac Equation, the Master Equation and Monte Carlo Simulations

We consider a system of n particles that are entirely characterized by their one-dimensional velocities v_i, $i=1,\ldots,n$. These velocities undergo random jumps,

$$\begin{aligned}&(v_1,\dots,v_j,\dots,v_k,\dots,v_n)\\&\quad\mapsto R_{j,k,\theta}(v_1,\dots,v_j,\dots,v_k,\dots,v_n)\\&\qquad=(v_1,\dots,v_j\cos\theta-v_k\sin\theta,\dots,v_j\sin\theta+v_k\cos\theta,\dots,v_n),\\&\qquad\quad 1\le j,k\le n,\theta\in]-\pi,\pi].\end{aligned}\tag{1}$$

These jumps occur independently with a rate proportional to

$$n^{-1}b(\theta)d\theta.\tag{2}$$

This is the Kac model of a dilute gas [17]. In the original paper, $b(\theta)=(2\pi)^{-1}$, i.e. all rotation angles θ are equally probable.

We note that the jumps are rigid rotations in a plane spanned by a pair of velocities, and hence it is clear that the kinetic energy, $W=\frac{1}{2}\sum_{j=1}^{n}v_j^2$ is preserved, and therefore this describes a jump process with values in $S^{n-1}(\sqrt{2W})$. It is convenient to choose $2W=n$.

Another important thing to notice is the factor n^{-1} in (2); this implies that for the rate of jumps that involve a particular velocity, e.g. v_1, asymptotically does not depend on the number of velocities, n. However, in many physically realistic cases, the rate of jumps depend on the rotation angle, approximately as $|\theta|^{-(1+\alpha)}$, where $\alpha\in[0,2[$. This implies that the total jump rate for the vector $\mathbf{v}=(v_1,\dots,v_j,\dots,v_k,\dots,v_n)$ is infinite. We speak about non-cutoff models as opposed to the cutoff models where b is replaced by some function $\tilde{b}$ such that $\int_{-\pi}^{\pi}\tilde{b}(\theta)\,d\theta<\infty$.

This jump process may equally well be defined by the master equation, which describes the evolution of a phase space density under the process. We let $\Psi(t,\cdot)\in C([0,\infty[,L^1(S^{n-1}(\sqrt{n})))$, and assume that $\Psi(0,\cdot)$ is a non negative density on $S^{n-1}(\sqrt{n})$. Then Ψ satisfies the equation

$$\partial_t\Psi^n(t,\mathbf{v})=\frac{2}{n}\sum_{1\le j<k\le n}\int_{-\pi}^{\pi}(\Psi^n(t,R_{j,k,-\theta}\mathbf{v})-\Psi^n(t,\mathbf{v}))b(\theta)\,d\theta,\tag{3}$$

which is known as Kac's master equation. The superscript n denotes the number of variables in the model, and this corresponds to the number of particles in a cell. Kac proved that if one considers the family of master equations for Ψ_n, $n=0,\dots,\infty$, with initial data Ψ_0^n that is symmetric with respect to permutation of the variables, and such that the marginal densities $\Psi_{k,0}^n(v,\dots,v_k)$ satisfy

$$\lim_{n\to\infty}\Psi_{k,0}^n(v_1,\dots,v_k)=\lim_{n\to\infty}\prod_{j=1,\dots,k}\Psi_{1,0}^n(v_j).\tag{4}$$

Then also the time evolved density $\Psi^n(t,\mathbf{v})$ factories into a product of one-particle marginals $f(t,v)\equiv\Psi_1^n(t,v)$, and that $f(t,v)$ satisfies the so-called Kac equation

$$\partial_t f=Q(f,f),\tag{5}$$

where the collision operator Q

$$Q(g,g)(v)=\int_{\mathbb{R}}\int_{-\pi}^{\pi}(g(v')g(v'_*)-g(v)g(v_*))b(\theta)\,d\theta\,dv_*,\tag{6}$$

and where

$$(v', v'_*) = (v\cos\theta - v_*\sin\theta, v\sin\theta + v_*\cos\theta). \tag{7}$$

Compared to the real Boltzmann equation, the Kac equation is very easy to analyze mathematically, and we refer to [9, 21] for some of the basic results concerning existence and uniqueness of solutions, trend to equilibrium, etc.

From a numerical point of view, the connection between the jump process described in (1) and (2) and the Kac equation, is that one may regard

$$\frac{1}{n}\sum_{j=1}^{n}\delta_{v_j(t)} \tag{8}$$

as an approximation of the probability density $f(t, v)$, and it has been proven for many different cases that (8) does indeed converge to $f(t, v)$ (see [3, 28]).

The solutions of (5) converge to equilibrium exponentially as t increases to infinity, the equilibrium solution being a Gaussian function with mean zero. It is of interest to study situations where the stationary solution is not an equilibrium state. This is the case e.g. in kinetic models for dissipative systems (see e.g. [6, 8, 15]).

Another example comes from molecular dynamics and the introduction of thermostats, and which is used here as a test case for the Monte Carlo method with Brownian approximation.

The basic model is the one described in the beginning of this section, with a vector $\mathbf{v} \in \mathbb{R}^n$, that jumps according to (1). The difference in the thermostat model is that the velocity field is accelerated by a constant force field $\mathbf{E} = E(1, \dots, 1)$, which is projected on the tangent plane to the sphere $S^{n-1}(\sqrt{n})$. This means that between jumps, the a velocity component v_j satisfies

$$\frac{d}{dt}v_j(t) = E\left(1 - \frac{\sum v_k}{\sum v_k^2}v_j\right). \tag{9}$$

The physical interpretation is that each particle is accelerated by the same, constant, force field of strength E, but that a force field depending on the whole system of particles keeps the total kinetic energy fixed. The model is described more in details in [34, 36], where also the corresponding Kac equation is derived and analyzed. This Kac equation takes the form

$$\frac{\partial}{\partial t}f + \mathrm{E}\frac{\partial}{\partial v}((1-\zeta(t)v)f) = Q(f, f), \tag{10}$$

where

$$\zeta(t) = \int_{\mathbb{R}} vf(v, t)\,dv. \tag{11}$$

The collision operator, Q is defined as before, in (6). The stationary states to this equation are far from Gaussian. One can show that for an integrable kernel, $\int_{-\pi}^{\pi} b(\theta)\,d\theta$, the stationary state becomes singular for sufficiently large values of E (see [35]), whereas in the non-cutoff case, the stationary state is C^∞ (see [5]). The latter case is really the motivation for the work presented in this paper, because it proved very difficult to get accurate numerical results using a truncated kernel.

2 Brownian Approximation

The evolution of the n-particle system (including a thermostatted force field of strength E) can be described by a stochastic differential equation driven by a Poisson random measure:

$$\mathbf{v}(t) = \mathbf{v}(0) + E\int_0^t \left(\mathbf{e} - \frac{\mathbf{e}\cdot\mathbf{v}(s)}{|\mathbf{v}(s)|^2}\mathbf{v}(s)\right) ds + \sum_{1\le j<k\le n} \int_0^t\int_{-\pi}^{\pi} A_{j,k}(\theta_{j,k})\mathbf{v}(s_-)N(ds, d\theta_{j,k}). \tag{12}$$

Here $A_{j,k}(\theta)$ is the $n\times n$-matrix

$$A_{j,k}(\theta) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \cos\theta - 1 & \cdots & -\sin\theta & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \sin\theta & \cdots & \cos\theta - 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{13}$$

$\mathbf{e} = (1, 1, \ldots, 1) \in \mathbb{R}^n$, and $N(ds, d\theta)$ is a Poisson random measure with intensity measure $n^{-1}b(\theta)\,d\theta\,dt$. In the non-cutoff case,

$$b(\theta) \sim |\theta|^{-(1+\alpha)} \tag{14}$$

near $\theta = 0$, and with $0 < \alpha < 2$, this implies that the total jump rate is infinite.

The Brownian approximation consists in replacing $N(ds, d\theta)$ in (12) by a truncated measure $\tilde{N}_\epsilon(ds, d\theta)$ with intensity measure $n^{-1}\tilde{b}_\epsilon(\theta)\,d\theta\,dt$, where $\tilde{b}$ is defined by

$$\tilde{b}_\epsilon(\theta) = \min(b(\theta), b(\epsilon)), \tag{15}$$

and adding a Brownian term to compensate for the truncated part.

The explicit form of a stochastic differential equation whose solution is a Brownian motion on an n-dimensional sphere $S^{n-1}(r)$ can be found e.g. in [31], or in Øksendal's book [24]:

$$X(t) = X(0) + \int_0^t \lambda r_W(X(s))\,ds + \int_0^t \sqrt{\lambda}\sigma(X(s))\,dW(s), \tag{16}$$

where $\{W(t)\}$ is a standard Wiener process in $\mathbb{R}^n$ with mean zero and whose covariance is the identity matrix. The matrix σ projects the dW onto the tangent plane to $S^{n-1}(\sqrt{n})$ at X, its elements being given by

$$\sigma_{j,k}(\mathbf{x}) = \delta_{j,k} - \frac{x_j x_k}{|\mathbf{x}|^2}. \tag{17}$$

The drift term r_W is

$$r_W(x) = -\frac{n-1}{2}\frac{x}{|x|^2}. \tag{18}$$

The diffusion rate μ is computed so as to match the second moment of the truncated part of the jump measure:

$$\mu = \mu_\epsilon = \frac{2n}{n-1}\frac{1}{2}\int_{-\epsilon}^{\epsilon}(b(\theta) - b(\epsilon))\,\theta^2\,d\theta. \tag{19}$$

When $b(\theta) = |\theta|^{-(1+\alpha)}$, we find

$$\mu_\epsilon = \frac{2n}{n-1}\frac{1+\alpha}{3(2-\alpha)}\epsilon^{2-\alpha}. \tag{20}$$

Details of this calculation will be found in [32], where also the rate of convergence is analysed. One result is that while the generators of the processes corresponding to (14) with a truncated kernel converges with rate $\epsilon^{2-\alpha}$, the convergence rate is $\epsilon^{3-\alpha}$ when the Brownian term is added. When α is close to 2, the improvement is significant.

Adding the force term, like in (12), gives

$$\begin{aligned}\mathbf{v}(t) = \mathbf{v}(0) &+ \int_0^t (r_{Th}(\mathbf{v}(s)) + r_W(\mathbf{v}(s)))\,ds + \int_0^t \sigma(\mathbf{v}(s))\,dW(s)\\ &+ \sum_{1\le j<k\le N}\int_0^t\int_{-\pi}^{\pi} A_{j,k}(\theta_{j,k})\mathbf{v}(s_-)N_\epsilon(ds, d\theta_{j,k}),\end{aligned} \tag{21}$$

where $r_{Th}(\mathbf{v}(t)) = \mathbf{e} - \frac{\mathbf{e}\cdot\mathbf{v}(s)}{|\mathbf{v}(s)|^2}\mathbf{v}(t) = \Gamma(\mathbf{v})\mathbf{e}$, Γ being the matrix with elements $\Gamma_{j,k} = \delta_{j,k} - v_j(t)v_k(t)/|\mathbf{v}|^2$, and where N_ϵ is a Poisson random measure with intensity measure $n^{-1}\tilde{b}_\epsilon(\theta)\,d\theta\,dt$.

3 The 3*n* Dimensional Master Equation and the Boltzmann Equation

The master equation that corresponds to the full Boltzmann equation and with an added thermostatted force term is [36]

$$\begin{aligned}\partial_t\Psi^n(t,\mathbf{v}) &+ \sum_{i=1}^{n}\frac{\partial}{\partial v_i}([F - F\cdot\mathbf{j}\quad v_i]\Psi^n(t,\mathbf{v}))\\ &= \frac{2}{n}\sum_{1\le j<k\le n}\int_{S^2}(\Psi^n(t, R_{j,k,\omega}\mathbf{v}) - \Psi^n(t,\mathbf{v}))b(\theta)\,d\omega,\end{aligned} \tag{22}$$

where in this case, $\mathbf{v}\in\mathbb{R}^{3n}$, and where $R_{j,k,\omega}$ is an operator that models the collision of two particles,

$$(v_j, v_k)\mapsto\left(\frac{v_j+v_k}{2} + \frac{|v_j - v_k|}{2}\omega, \frac{v_j+v_k}{2} - \frac{|v_j-v_k|}{2}\omega\right).$$

In $b(\theta)$, $\theta = \arccos(\omega \cdot \frac{v_j - v_k}{|v_j - v_k|})$. To keep the notation similar to the one-dimensional case, we let $\mathbf{v} = (v_1, \dots, v_n)$, where $v_j \in \mathbb{R}^3$, $j = 1, \dots, n$. The non-cutoff case again corresponds to allowing $\int_{S^2} b(\theta)\, d\omega$ to diverge. The force field is $F = (E, 0, 0)$, and $\mathbf{j} = \frac{1}{n}\sum_{j=1}^n v_j$. At the level of the master equation, the Brownian approximation corresponds to truncating the non-cutoff jump rate, and replacing the truncated part by a suitable diffusion term. For the Kac equation, the diffusion is just the one given by the Laplace-Beltrami operator on the sphere S^{n-1}, but here it is rather the Balescu-Prigogine operator for Maxwellian interactions (see [18, 33]),

$$\frac{\mu}{n}\sum_{j\neq k}(\partial_{v_j} - \partial_{v_k})|v_j - v_k|^2 \mathrm{P}^{\perp}_{v_j - v_k}(\partial_{v_j} - \partial_{v_k}), \tag{23}$$

where $\mathrm{P}^{\perp}_z$ is the 3×3-matrix $\mathrm{I} - \frac{zz^{\mathrm{tr}}}{|z|^2}$, and μ is a constant depending on the level of truncation.

Given this expression, one can write a stochastic equation much like (21). However, from a computational point of view, it does not seem to be efficient, because of the effort needed to compute $\sigma\, dW$. An alternative, that gives good results at much lower computational cost, is to replace (23) by

$$\frac{\mu}{n}\sum_{j\neq k}\partial_{v_j}|v_j - v_k|^2 \mathrm{P}^{\perp}_{v_j - v_k}\partial_{v_j}, \tag{24}$$

which corresponds to adding a Brownian motion $R_j\, dW_j \in \mathbb{R}^3$ to each velocity, where $R_j^{\mathrm{tr}} R_j = \mu\sum_{j\neq k}(|v_j - v_k|^2 \mathrm{I} - (v_j - v_k)(v_j - v_k)^{\mathrm{tr}})$. The matrices R_j can be expressed in terms of moments and v_j, and hence the computational cost is proportional to the number of particles. The increments $(R_1\, dW_1, \dots, R_n\, dW_n)$ still need to be projected onto the tangent space of the manifold of constant energy and momentum, but that is an operation that can be carried out in time proportional to the number of particles. Some numerical results are given in the following section.

4 Numerical Experiments

The numerical experiments have been carried out in the most direct way, with no large effort to make the code efficient. The large jumps have been simulated by computing exponentially distributed time intervals with rate proportional to $n\int_{-\pi}^{\pi}\tilde{b}(\theta)d\theta$. At the end of such an interval, a random pair (j, k) is chosen and the jump is effectuated by rotating the vector (v_j, v_k) by a random angle θ distributed according to $\tilde{b}(\theta)/\int_{-\pi}^{\pi}\tilde{b}(\theta)d\theta$.

In the intervals between the large jumps, we solve (21) using a simple explicit Euler method with a step size that depends on μ_ϵ. The step size is taken to be a given fraction of the typical rate for the truncated jump process. We have not made a rigorous analysis that would help in choosing the step size, rather we have numerically tested that the choice gives relevant answers.

We have computed uniformly distributed pseudo-random variables using the routine DLARAN from the LAPACK package [19] and to compute normally distributed random variables, we have used the ziggurat method of Marsaglia and Tsang [20].

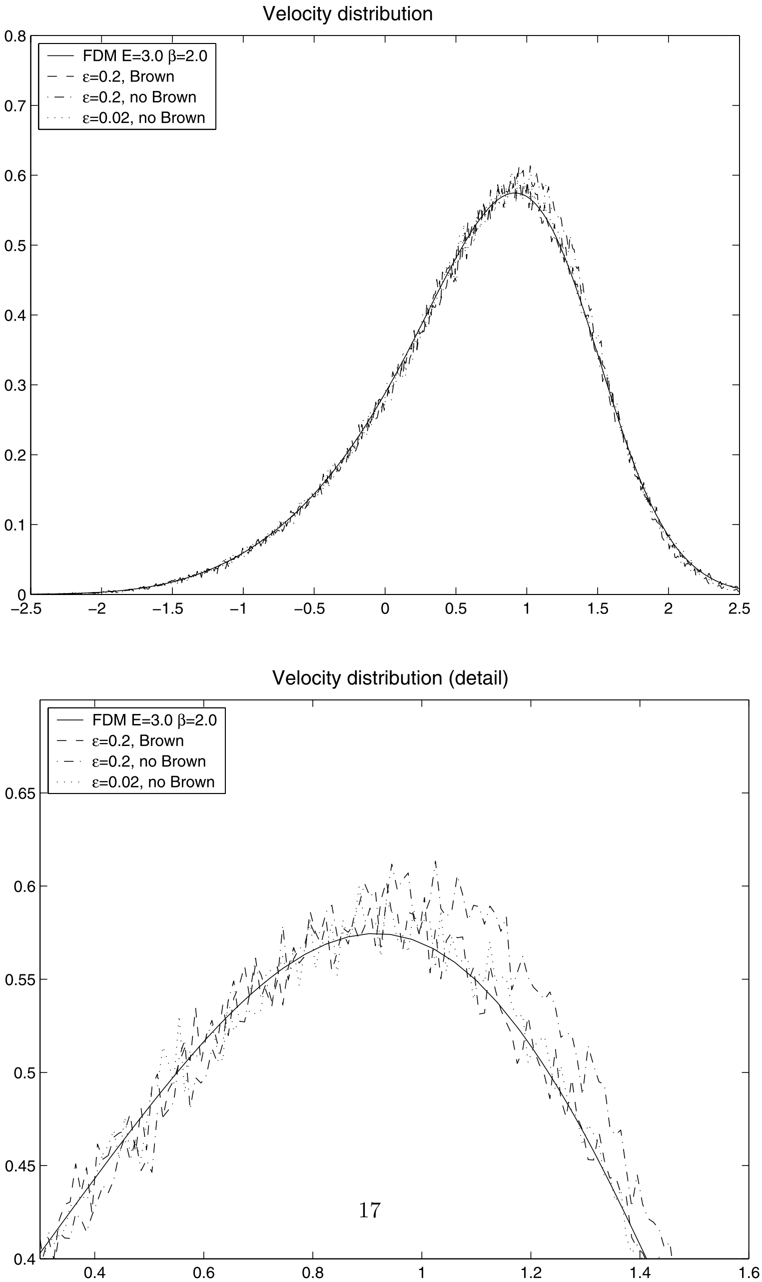

Fig. 1 Simulation results using different values of truncation: the velocity distribution, and an enlargement

As initial data, we have taken $\frac{1}{2}(\delta_{v=1} + \delta_{v=-1})$.

We have no exact solutions to compare the results with. However, taking the Fourier transform of the time independent Kac equation, i.e. (10) considered without t, gives

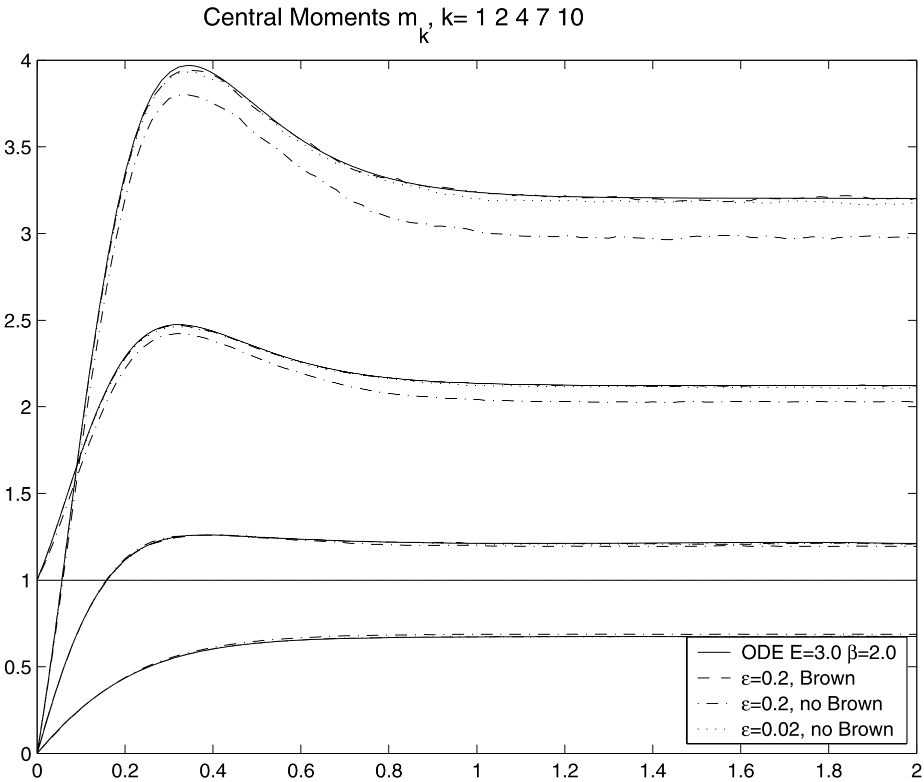

Fig. 2 Simulation results using different values of truncation: time evolution of some of the moments

equation

$$\hat{f}'(\xi) + \frac{i}{\gamma\xi}\hat{f}(\xi) = \frac{1}{E\gamma\xi}\int_{-\pi}^{\pi}(\hat{f}(\xi\cos\theta)\hat{f}(\xi\sin\theta) - \hat{f}(0)\hat{f}(\xi))b(\theta)\,d\theta, \tag{25}$$

where γ is the stationary current, which can be explicitly computed. Equation (25) can be solved accurately numerically using a finite difference method, combined with the built-in ODE-solvers of Matlab™. Numerical results obtained in this way were presented in [5], where also a detailed mathematical analysis of the non-cutoff Kac equation with a thermostat can be found. Although no rigorous error analysis has been carried out, we consider this finite-difference solution to be an accurate solution to the stationary problem: a fine discretization was used, and the method was found to converge well.

Another test for the accuracy is to compare the evolution of moments. Also here we do not have any exact results to compare with, but as with other Boltzmann like equations of Maxwell type (i.e. models where the collision rate does not depend on the relative velocity of the colliding particles), one can write a closed system of ordinary differential equations for the first moments $m_k = \int f(v,t)v^k\,dv$, and this system can then be solved accurately with a numerical ODE-solver. We have used Matlab™ to solve the following

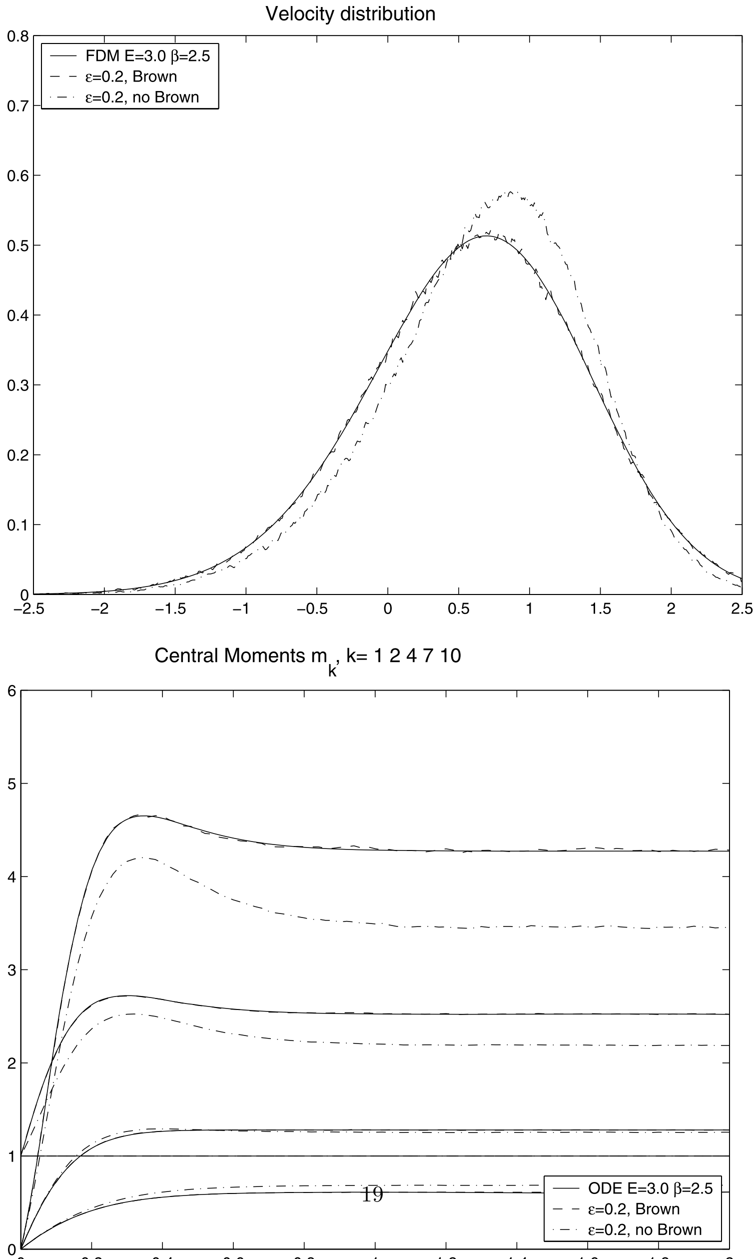

Fig. 3 Simulation results using different values of truncation for a stronger singularity: the velocity distribution, and the evolution of moments. Here the effect of the Brownian correction is more evident

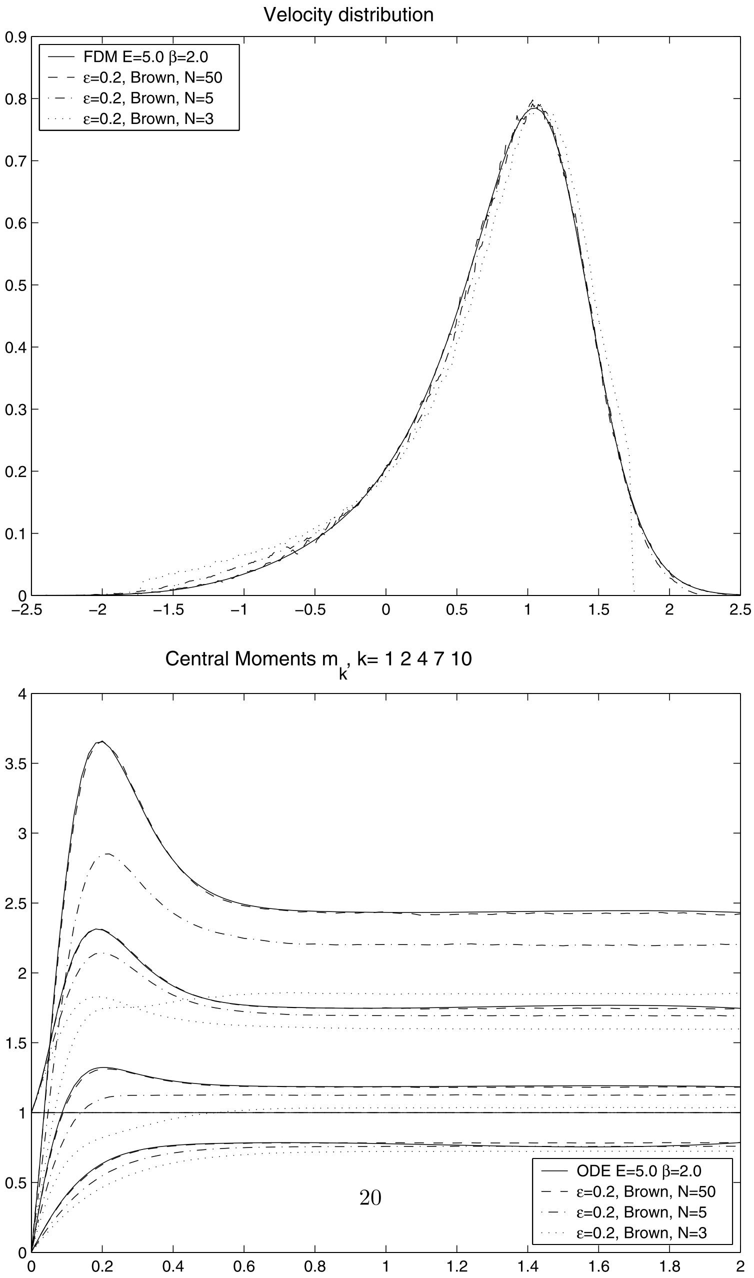

Fig. 4 Simulation results using different values of the number of particles. The simulation is repeated more times to get comparable results. As few as 50 particles gives a rather good agreement, but as few as five or even three particles are clearly not enough

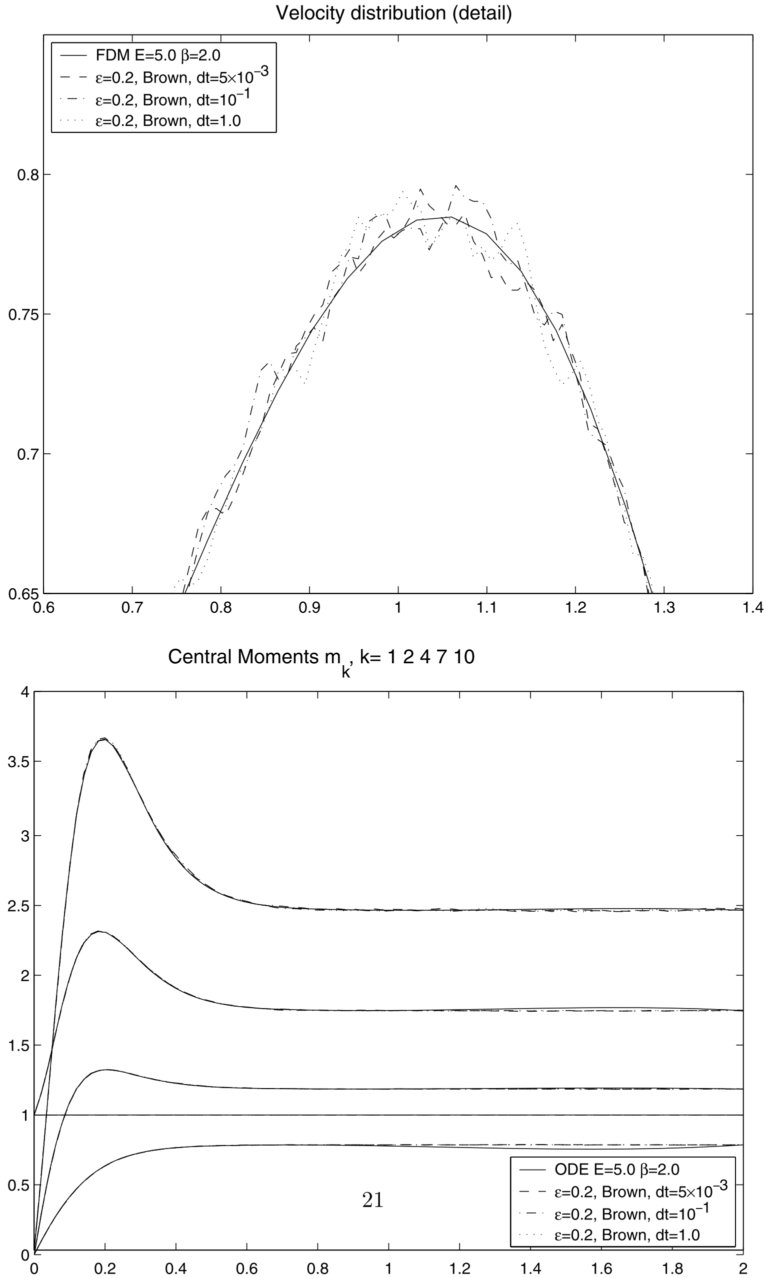

Fig. 5 The evolution between jumps is computed using a simple forward Euler method. The results here show that the time step is not critical

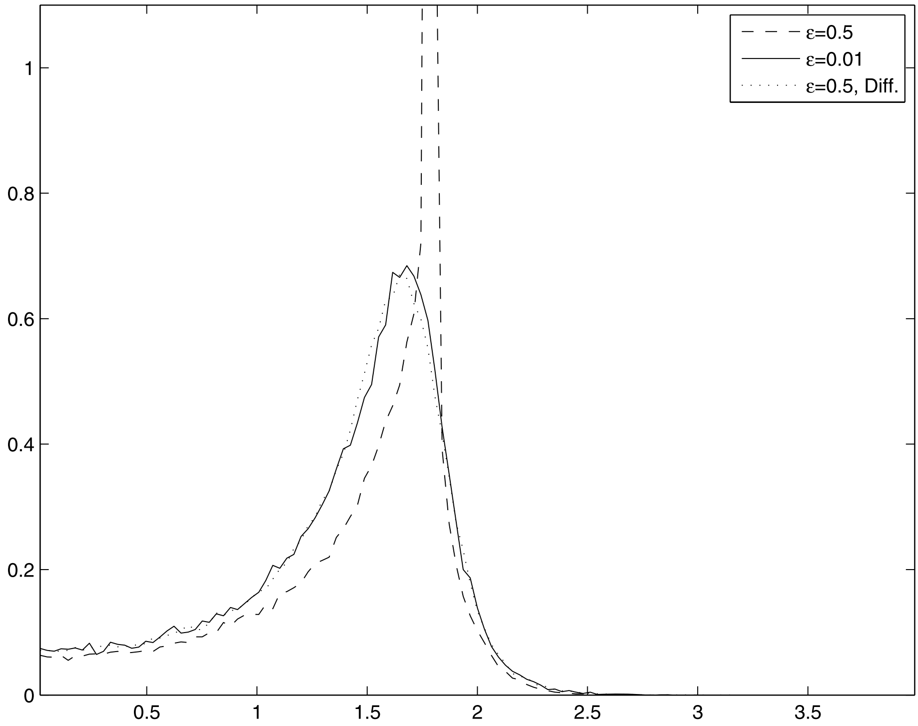

Fig. 6 Simulation of the Boltzmann equation: the distribution of the first velocity component v_x, i.e. an approximation of $\int_{\mathbb{R}^2} f(v,t)\,dv_y\,dv_z$ at $t = 0.06$

system:

$$\dot{m}_k = Ekm_{k-1} - Ekm_1 m_k - A_k m_k + \sum_{j=1}^{k-1} B_{kj} m_{k-j} m_j,$$

$$A_k = \int_{-\pi}^{\pi} (1 - \cos^k\theta - \sin^k\theta) b(\theta)\,d\theta,$$

$$B_{kj} = \binom{k}{j} \int_{-\pi}^{\pi} \cos^{k-j}\theta \sin^j\theta b(\theta)\,d\theta,$$

$$m_k(0) = \frac{1}{2}(1 + (-1)^k). \tag{26}$$

In the numerical calculations we have used $b(\theta) = |\theta|^{-1-\alpha}$ for different values of $\beta = \alpha + 1 \in]1, 3[$, and different values of the force parameter $E \in [2, 5]$. We have also varied n, the number of particles, and the time step used in solving the SDE, (21).

The first series of results, presented in Figs. 1, 2 (an enlargement), and Fig. 3 (for a stronger singularity) shows how the results depend on the level of truncation, and compares this with the result from using a much truncated model but with a Brownian correction. The parameters used were $E = 3.0$, $\beta = 2.0$, $n = 2000$, and $\epsilon = 0.2, 0.02, 0.002$. The time dt used in the Euler method for approximating the SDE was here taken to be $0.0001 \times dt_0$,

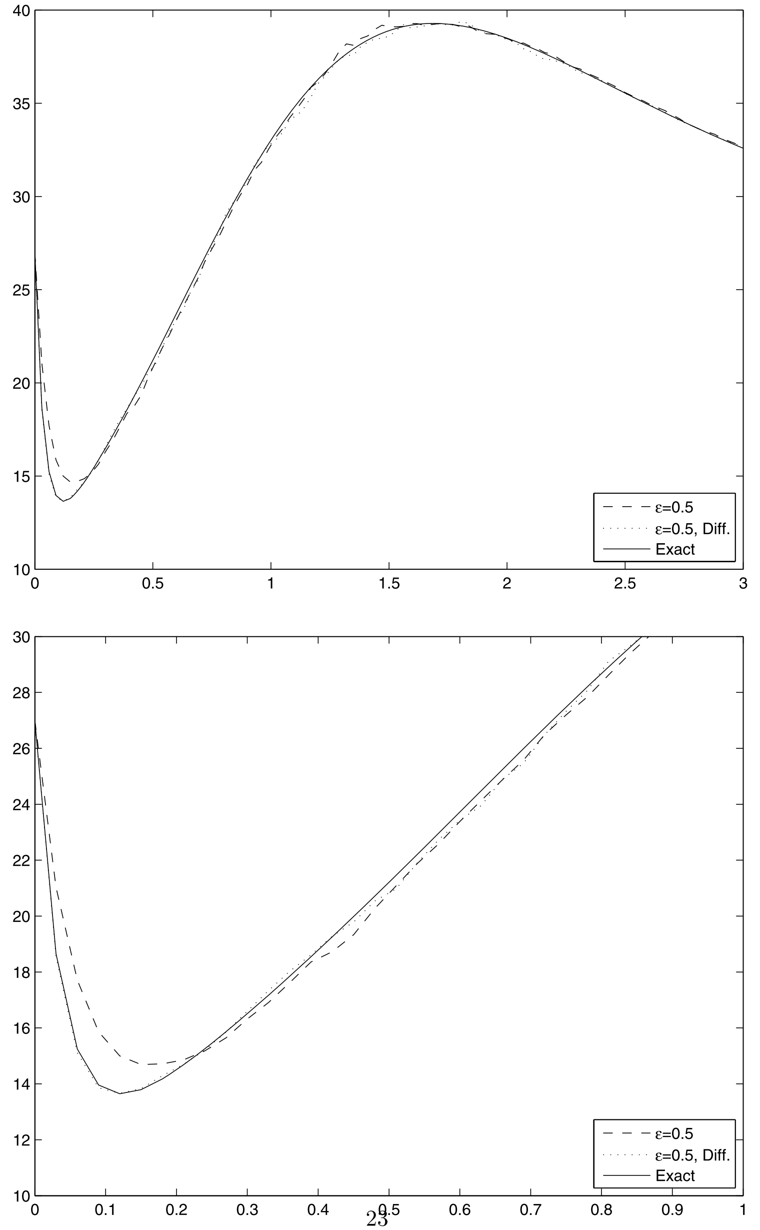

Fig. 7 Evolution of the moment $M_{6,0,0}$ with truncation at $\epsilon = 0.5$, with and without the diffusion correction. The moments for the Boltzmann equation are given as a reference. The enlargement shows that although the reference curve and the curve with the diffusion correction almost coincide, they are not identical

where dt_0 corresponds to a displacement of order ϵ from the Brownian motion. This is excessively small, and we will see below that it is far from necessary to get an accurate result. The calculation was then repeated 200 times to reduce noise. We see that $\epsilon = 0.2$ together with a Brownian approximation compares very well with a simulation using $\epsilon = 0.002$ with

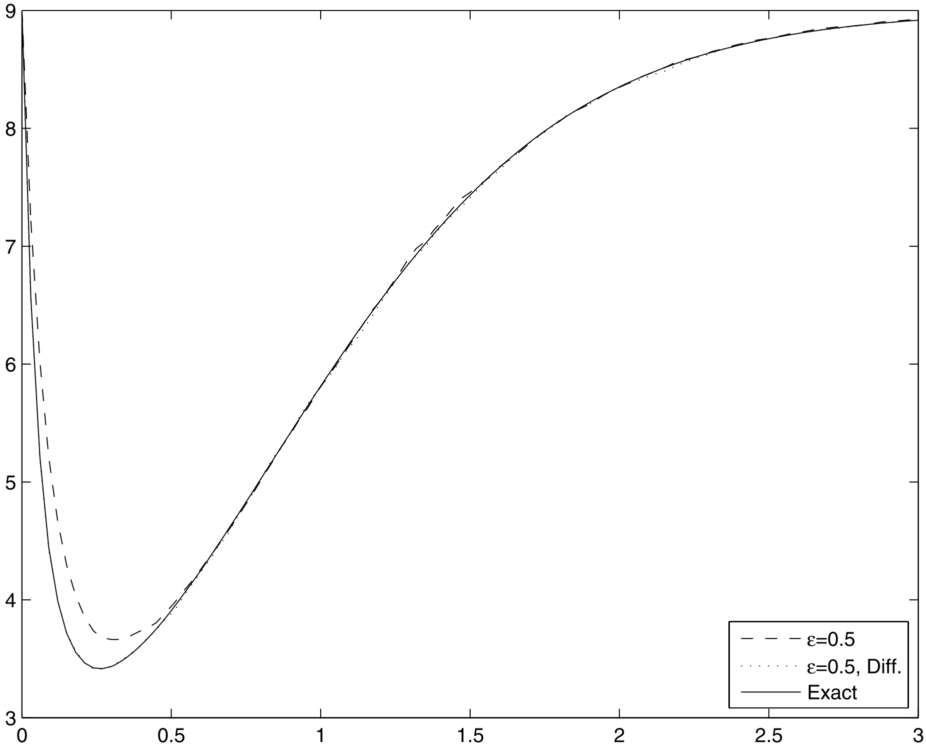

Fig. 8 Evolution of the moment $M_{4,0,0}$ with truncation at $\epsilon = 0.5$, with and without the diffusion correction. The moments for the Boltzmann equation are given as a reference

no approximation. The estimates in [32] give a convergence rate of $\epsilon^{3-\beta}$ without the Brownian correction, and a rate of $\epsilon^{4-\beta}$ with the Brownian term added. Hence it is not surprising to see that, when the singularity in the crossection $b(\theta)$ is stronger, the influence of the truncation and of the Brownian approximation is much more important.

Figure 4 shows results for different values of n, the number of particles used in the simulation. There are at least two reasons for using a large value of n, when simulating kinetic equations: first, the Boltzmann equation itself assumes a limit of infinitely many particles, and secondly, a large value of n reduces the noise when computing moments or other functions. In this series we have taken $n \times$ number of simulated trajectories $\simeq$ *const*, and very large in order to obtain a noiseless result. The calculations show that in fact it is not necessary to use a very large number of particles to find a good agreement, $n = 50$ is quite enough, both to get a reasonable agreement of the distribution functions and of the evolution of moments. However, with a small number of particles, it is necessary to repeat the calculations many times to avoid excessive noise in the result.

The last example for the Kac model, Fig. 5, shows some simulations that illustrate the influence of the time-step in the Euler method for solving the SDE (21). The reference time step is so large that the mean step is of the order ϵ, and the figure shows that both for computing the distribution function and the evolution of moments, it is not necessary to decrease the step size much below the reference value to get a good result.

The simulation is carried out in very much the same way for velocities in $\mathbb{R}^3$, the main difference being the in which the diffusion is added. To compute the matrices R_j, we have

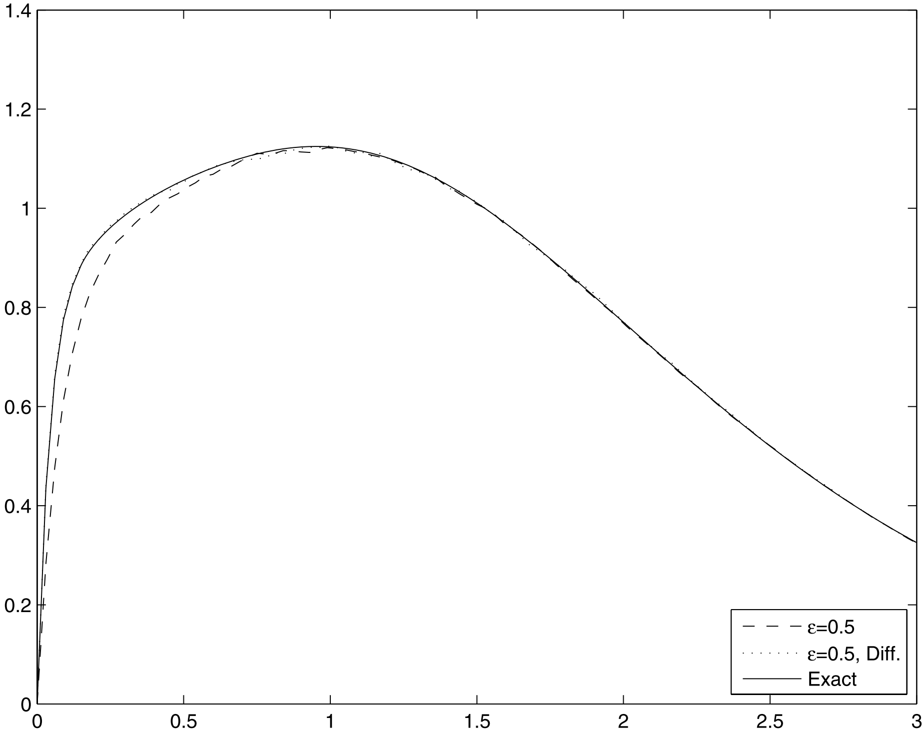

Fig. 9 Evolution of the moment $M_{2,0,2}$ with truncation at $\epsilon = 0.5$, with and without the diffusion correction. The moments for the Boltzmann equation are given as a reference

used routines from the Gnu Scientific Library [16] to evaluate the square root. This is rather time consuming, and the code spends a major part of the time doing this; still the method is faster than just using a smaller truncation, also before any attempts have been made to making the code efficient.

In this case we do not have an alternative method for computing the velocity distribution, but rather we compare the velocity distribution with a calculation with very small truncation. Because we are dealing with the case of Maxwellian interactions, there is a closed set of equations that describe the evolution of moments for the limiting Boltzmann equation also here.

All calculations were carried out with a constant force field $F = (1, 0, 0)$, and the number of particles was $n = 500$. The initial data was $\frac{1}{2}(\delta_{(1,0,0)} + \delta_{(-1,0,0)})$.

The first graph, Fig. 6 compares the approximations of

$$\int_{\mathbb{R}^2} f(v_x, v_y, v_z, t)\, dv_y\, dv_z,$$

with and without the diffusion correction, and a reference solution obtained by carrying out a simulation with a very small truncation of the crossection.

We also compare the evolution of moments of the form

$$M_{j_x, j_y, j_z}(t) = \frac{1}{n} \sum_{k=1}^{n} v_{kx}^{j_x} v_{ky}^{j_y} v_{kz}^{j_z},$$

with the moments (computed as the solution of the closed ODE-system) for solutions to the limiting Boltzmann equation,

$$\int_{\mathbb{R}^3} f(v_x, v_y, v_z, t) v_{0x}^{j_x} v_{0y}^{j_y} v_{0z}^{j_z} \, dv_x \, dv_y \, dv_z.$$

We consider the solutions to the ode's to be exact. Hence Figs. 7, 8 and 9 show the evolution of $M_{6,0,0}(t)$, $M_{4,0,0}(t)$, and $M_{2,0,2}(t)$, respectively, for a strong truncation of the collision term $\varepsilon = 0.5$, with and without the diffusion correction, and compared with the moments for the limiting Boltzmann equation.

5 Conclusions

The paper presents a method to compute accurate solutions to non-cutoff Boltzmann equations in the non-cutoff case, by approximating the small jumps by a diffusion. We have presented numerical examples showing that it works well, and gives accurate results.

There are several open issues that merit being studied. From a numerical point of view, of course one would have to find good means of choosing n, the level of truncation, and an efficient method for solving (21).

Some results that aim at putting the method on a solid theoretical ground will be presented in [32].

References

1. Asmussen, S., Rosiński, J.: Approximations of small jumps of Lévy processes with a view towards simulation. J. Appl. Probab. **38**, 482–493 (2001)
2. Babovsky, H.: On a simulation scheme for the Boltzmann equation. Math. Methods Appl. Sci. **8**, 223–233 (1986)
3. Babovsky, H., Illner, R.: A convergence proof for Nanbu's simulation method for the full Boltzmann equation. SIAM J. Numer. Anal. **26**(1), 45–65 (1989)
4. Bird, G.A.: Molecular Gas Dynamics. Oxford University Press, London (1976)
5. Bagland, V., Wennberg, B., Wondmagegne, Y.: Stationary states for the non-cutoff Kac equation with a Gaussian thermostat. Nonlinearity **20**(3), 583–604 (2007)
6. Bobylev, A.V., Gamba, I.M., Panferov, V.A.: Moment inequalities and high-energy tails for Boltzmann equations with inelastic interactions. J. Stat. Phys. **116**(5, 6), 1651–1682 (2004)
7. Bobylev, A.V., Mossberg, E., Potapenko, I.F.: A DSMC method for the Landau-Fokker-Planck equation. In: Proc. 25th International Symposium on Rarefied Gas Dynamics. St. Petersburg, Russia, July (2006)
8. Brilliantov, N.V., Pöschel, T.: Kinetic Theory of Granular Gases. Oxford University Press, Oxford (2004)
9. Desvillettes, L.: About the regularizing properties of the non-cut-off Kac equation. Commun. Math. Phys. **168**(2), 417–440 (1995)
10. Desvillettes, L., Graham, C., Méléard, S.: Probabilistic interpretation and numerical approximation of a Kac equation without cutoff. Stoch. Process. Appl. **84**(1), 115–135 (1999)
11. Filbet, F., Pareschi, L.: Numerical solution of the Fokker-Planck-Landau equation by spectral methods. Commun. Math. Sci. **1**(1), 206–207 (2003)
12. Filbet, F., Mouhot, C., Pareschi, L.: Solving the Boltzmann equation in $N \log N$. SIAM J. Sci. Comput. **28**(3), 1029–1053 (2006)
13. Fournier, N., Méléard, S.: Monte-Carlo approximations and fluctuations for 2D Boltzmann equations without cutoff. Markov Process. Relat. Fields **7**, 159–191 (2001)
14. Fournier, N., Méléard, S.: A stochastic particle numerical method for 3D Boltzmann equations without cutoff. Math. Comput. **71**, 583–604 (2002)
15. Goldhirsch, I.: Inelastic kinetic theory: the granular gas. In: Topics in Kinetic Theory, pp. 289–312. AMS, Providence (2005)
16. GNU scientific library. See http://directory.fsf.org/GNU/gsl.html

17. Kac, M.: Foundations of kinetic theory. In: Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, pp. 171–197. University of California Press, Berkeley (1956)
18. Kiessling, M., Lancellotti, C.: On the Master-Equation approach to kinetic theory: linear and nonlinear Fokker-Planck equations. Transp. Theory Stat. Phys. **33**, 379–401 (2004)
19. LAPACK: Linear Algebra PACKage. Available at http://www.netlib.org/lapack/
20. Marsaglia, G., Tsang, W.W.: The ziggurat method for generating random variables. J. Stat. Softw. **5**(8), 1–7 (2000)
21. McKean, H.P.: Speed of approach to equilibrium for Kac's caricature of a Maxwellian gas. Arch. Ration. Mech. Anal. **21**, 343–367 (1966)
22. Mouhot, C., Pareschi, L.: Fast algorithms for computing the Boltzmann collision operator. Math. Comput. **75**(256), 1833–1852 (2006)
23. Nanbu, K.: Direct simulation scheme derived from the Boltzmann equation. J. Phys. Soc. Jpn. **49**, 2042–2049 (1980)
24. Øksendal, B.: Stochastic Differential Equations, 6th edn. Springer, Berlin (2003)
25. Pareschi, L., Russo, G.: Time relaxed Monet Carlo methods for the Boltzmann equation. SIAM J. Sci. Comput. **23**, 1253–1273 (2001)
26. Pareschi, L., Trazzi, S.: Numerical solution of the Boltzmann equation by time relaxed Monte Carlo (TRMC) methods. Int. J. Numer. Methods Fluids **48**, 947–983 (2005)
27. Pareschi, L., Wennberg, B.: A recursive Monte Carlo method for the Boltzmann equation in the Maxwellian case. Monte Carlo Methods Appl. **7**, 349–357 (2001)
28. Pulvirenti, M., Wagner, W., Zavelani Rossi, M.B.: Convergence of particle schemes for the Boltzmann equation. Eur. J. Mech. B Fluids **13**(3), 339–351 (1994)
29. Rjasanow, S., Wagner, W.: Stochastic Numerics for the Boltzmann Equation. Springer Series in Computational Mathematics, vol. 37. Springer, Berlin (2005)
30. Rosiński, J., Cohen, S.: Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered and operator stable processes. Preprint
31. Stroock, D.: On the growth of stochastic integrals. Z. Wahrscheinlichkeitstheorie Verw. Gebiete **18**, 340–344 (1971)
32. Sundén, M., Wennberg, B.: The Kac master equation with unbounded collision rate. In preparation
33. Villani, C.: A review of mathematical topics in collisional kinetic theory. In: Friedlander, S., Serre, D. (eds.) Handbook of Mathematical Fluid Mechanics. North-Holland, Amsterdam (2002)
34. Wennberg, B., Wondmagegne, Y.: The Kac equation with a thermostatted force field. J. Stat. Phys. **124**(2–4), 859–880 (2006)
35. Wennberg, B., Wondmagegne, Y.: Stationary states for the Kac equation with a Gaussian thermostat. Nonlinearity **17**, 633–648 (2004)
36. Wondmagegne, Y.: Kinetic equations with a Gaussian thermostat. Doctoral thesis, Department of Mathematical Sciences, Chalmers University of Technology and Göteborg University, Göteborg (2005)