Invariant sets for continuous QMF functions

A necessary condition for a periodic function $p(\xi)$ to be the squared modulus of a lowpass filter, that is, the generator of a scaling function for a multiresolution analysis (MRA) with respect to the usual dyadic dilation, is the cuadrature mirror filter (QMF) condition

$$p(\xi/2) + p(\xi/2 + 1/2) = 1 \text{ for all } \xi \in [0, 1],\ p(0) = 1.$$

Cohen (1990) gave a partial answer to the question of which QMF functions generate MRAs by providing necessary and sufficient restrictions on the nature of the roots of the function in question. From a probability perspective, a QMF function can be interpreted as the transition function for a Markov process on the unit interval, $\xi_{t+1} = \xi_t/2$ or $\xi_t/2 + 1/2$, having 0 and 1 as invariant sets. If p is smooth, Cohen's theorem applies and is equivalent to the statement that 0 and 1 are the only invariant sets. However there is a class of continuous QMF functions that generate MRAs, with multiple invariant sets, to which Cohen's theorem does not apply. First we characterize this class of functions in terms of their associated invariant sets. Then we describe the structure of those subsets of $[0, 1]$ that arise as invariant sets for continuous QMF functions that generate MRAs.

Work inspired by the article "Tilings, scaling functions, and a Markov process" in the October issue of Notices of the AMS. The general question concerning "inaccessible attractors" stated there will be settled during the talk.