Almost periodic functionals

Matthew Daws

Leeds

Sweden, July 2013
Almost periodic functionals

Let $A$ be a Banach algebra and turn $A^*$ into an $A$-bimodule:

$$\langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle, \quad \langle \mu \cdot a, b \rangle = \langle \mu, ab \rangle \quad (a, b \in A, \mu \in A^*).$$

Then consider the “orbit maps”

$$L_\mu : A \rightarrow A^*; \ a \mapsto a \cdot \mu, \quad R_\mu : A \rightarrow A^*; \ a \mapsto \mu \cdot a.$$ 

**Definition**

Say that $\mu \in A^*$ is (weakly) almost periodic, $\mu \in \text{ap}(A)$ if $L_\mu$ is a (weakly) compact operator (i.e. $\{ a \cdot \mu : \| a \| \leq 1 \}$ is relatively (weakly) compact in $A^*$). (Equivalently can use $R_\mu$.)
Turning into an algebra

Choose bounded nets $(a_i), (b_i)$ converging weak* to $\Phi, \Psi \in A^{**}$ respectively. Then we have two choices for a product:

$$\Phi \Box \Psi = \lim_i \lim_j a_i b_j, \quad \Phi \diamond \Psi = \lim_j \lim_i a_i b_j.$$  

These are the Arens products.
Consider $\mu \in \text{ap}(A)$, so we may assume $b_j \cdot \mu \to \lambda$ in norm. Then:

$$\langle \Phi \Box \Psi, \mu \rangle = \lim_i \lim_j \langle b_j \cdot \mu, a_i \rangle = \lim_i \langle \lambda, a_i \rangle = \langle \Phi, \lambda \rangle$$

$$= \lim_j \langle \Phi, b_j \cdot \mu \rangle = \langle \Phi \diamond \Psi, \mu \rangle.$$  

But more is true:

$$\lim_i \langle \mu, a_i b_i \rangle = \lim_i \langle b_i \cdot \mu, a_i \rangle = \lim_i \langle \lambda, a_i \rangle = \langle \Phi \Box \Psi, \mu \rangle.$$
Universal property; link with dual Banach algebras

**Theorem (Lau, Loy, Runde, ...?)**

*By separate weak*-continuity, the product on $A$ extends to $\text{ap}(A)^*$, turning $\text{ap}(A)^*$ into a dual Banach algebra. The product on $\text{ap}(A)^*$ is jointly weak*-continuous (on bounded set). This is the universal object for “jointly continuous” dual Banach algebras.*

That is, if $B$ is a dual Banach algebra with jointly continuous multiplication, and $\theta : A \to B$ a bounded homomorphism, then there is a unique $\tilde{\theta} : \text{ap}(A)^* \to B$, a weak*-continuous homomorphism, with:

$$
\begin{array}{ccc}
A & \longrightarrow & \text{ap}(A)^* \\
\theta & \downarrow & \exists! \tilde{\theta} \\
B & \downarrow
\end{array}
$$
For group algebras

Let $G$ be a locally compact group and consider $A = L^1(G)$.

**Theorem (Lau, (Wong, Ulger))**

$F \in L^\infty(G) = A^*$ with be almost periodic if and only if the set of (left, or right) translates of $F$ forms a relatively norm compact subset of $L^\infty(G)$. In this case, $F \in C^b(G)$.

- Easy to see that $\text{ap}(A) \subseteq C^b(G) \subseteq L^\infty(G)$ will be a unital (commutative) $C^*$-algebra.
- Let $G^{\text{ap}}$ be the (compact, Hausdorff) character space, so $\text{ap}(A) = C(G^{\text{ap}})$. 
The (semi)group $G^{ap}$

$$\text{ap}(L^1(G)) = C(G^{ap}), \quad \text{ap}(L^1(G))^* = M(G^{ap}).$$

- Point-evaluation gives a continuous dense-range map $G \rightarrow G^{ap}$.
- Product on $M(G^{ap})$ determined by map $L^1(G) \rightarrow M(G^{ap})$.
- These maps are compatible.
- Can show that product of point masses in $M(G^{ap})$ is again a point-mass.
- So $G^{ap}$ is a semigroup and $G \rightarrow G^{ap}$ a homomorphism.
- Product on $G^{ap}$ is jointly continuous; contains dense subgroup; so is a group.
Universal property

- Using some Banach algebra techniques, we started with a locally compact group $G$, and formed a compact group $G^{ap}$.
- $G^{ap}$ is universal in the sense that if $T$ is any compact topological (semi)group then have:

  \[
  G \twoheadrightarrow T \quad \exists! \quad G^{ap}
  \]

  $G^{ap}$ is the “Bohr compactification” of $G$.

- As $G^{ap}$ is a compact group, Peter-Weyl tells us that functions of the form

  \[
  G^{ap} \to \mathbb{C}; \quad s \mapsto (\pi(s)\xi|\eta),
  \]

  are dense in $C(G^{ap})$. Here $\pi : G^{ap} \to U(n)$ is a finite-dimensional unitary representation, and $\xi, \eta \in \mathbb{C}^n$.

- Such $\pi$ are in 1-1 correspondence with finite-dimensional unitary representations of $G$. So such continuous functions are dense in $ap(L^1(G))$. Not clear how to see this directly...
Non-commutative world

Let $G$ be a locally compact group, and let $\pi : G \to U(H)$ be the universal, strongly continuous, unitary representation of $G$ (direct sum over “all” such representations).

- We can “integrate” this to a map $\pi : L^1(G) \to B(H)$

\[ \pi(f)\xi = \int_G f(s)\pi(s)\xi \, ds \quad (f \in L^1(G), \xi \in H). \]

- This is a $\ast$-homomorphism of $L^1(G)$; it’s the universal one.
- The closure of $\pi(L^1(G))$ is $C^*(G)$ the universal group $C^*$-algebra.
- If we replace $\pi$ by $\lambda$ the left-regular representation on $L^2(G)$, we get the reduced group $C^*$-algebra $C^*_r(G)$.
- $C^*(G) \to C^*_r(G)$ is an isomorphism precisely when $G$ is amenable.
- Finally, define $VN(G) = C^*_r(G)^\prime\prime$ the group von Neumann algebra.
Fourier theory

If $G$ is an abelian group, then we have the Pontryagin dual $\hat{G}$. The Fourier transform gives a unitary map

$$\mathcal{F} : L^2(G) \to L^2(\hat{G}).$$

- Let $C_0(G)$ act on $L^2(G)$ by multiplication, say $f \leftrightarrow M_f$.
- The conjugation map $M_f \mapsto \mathcal{F} M_f \mathcal{F}^{-1}$ gives a $\ast$-isomorphism $C_0(G) \to C_r^*(\hat{G})$.
- It also gives a normal $\ast$-isomorphism $L^\infty(G) \to VN(\hat{G})$.
- So the predual $VN(\hat{G})_\ast$ is isomorphic to the algebra $L^1(G)$.
- By biduality, $VN(G)_\ast \cong L^1(\hat{G})$.
- What happens when $G$ is not abelian?
Hopf von Neumann algebras

There is a normal $\ast$-homomorphism

$$\Delta : \mathcal{VN}(G) \to \mathcal{VN}(G) \bar{\otimes} \mathcal{VN}(G); \quad \lambda(s) \mapsto \lambda(s) \otimes \lambda(s).$$

- That this exists is most easily seen by finding a unitary operator $W$ on $L^2(G \times G)$ with $\Delta(x) = W^*(1 \otimes x)W$.
- $\Delta$ is coassociative: $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$.
- Let the predual of $\mathcal{VN}(G)$ by $A(G)$; the predual of $\Delta$ gives an associative product

$$\Delta_* : A(G) \times A(G) \to A(G).$$

- This is the Fourier algebra; the map $A(G) \to C_0(G); \omega \mapsto (\langle \lambda(s), \omega \rangle)$ is a contractive algebra homomorphism.
- So $A(G)$ is a commutative Banach algebra; it is semisimple (and regular, Tauberian) with character space $G$. 
Almost periodic for the Fourier algebra

**Question**

What is \( \text{ap}(A(G)) \)? What relation does it have to a “compactification”?

Let \( C^*_\delta(G) \) be the C*-algebra generated by \( \{ \lambda(s) : s \in G \} \) inside \( VN(G) \).

- If \( G \) is discrete, then \( C^*_\delta(G) = C^*_r(G) \).
- If \( G \) is discrete and amenable, or \( G \) is abelian, then \( \text{ap}(A(G)) = C^*_\delta(G) \).
Compact quantum groups

Unital C*-algebra $A$ and coassociative $\Delta : A \to A \otimes A$ with “cancellation”:

$$\text{lin}\{\Delta(a)(1 \otimes b) : a, b \in A\}, \quad \text{lin}\{\Delta(a)(b \otimes 1) : a, b \in A\}$$

are dense in $A \otimes A$.

- $G$ compact gives $C(G)$ with $\Delta(f)(s, t) = f(st)$;
- $G$ discrete gives $C_r^*(G)$ with $\Delta$ as before.

Natural morphisms are the “Hopf ∗-homo-morphisms”; a morphism $(A, \Delta_A)$ to $(B, \Delta_B)$ is a ∗-homo-morphism $\theta : B \to A$ with $\Delta_A \circ \theta = (\theta \otimes \theta) \circ \Delta_B$.

- If $\phi : G \to H$ is a continuous group homo-morphism, then may define $\theta : C(H) \to C(G)$ by $\theta(f) = f \circ \phi$.

Can extend this to the non-compact world by considering multiplier algebras.
Quantum Bohr compactification

Sołtan (2005) considered “compactifications” in this category. In particular, $C_\delta^*(G)$ is the universal object for $C_r^*(G)$. For any compact quantum group $(A, \Delta_A)$, we have:

\[
\begin{array}{ccc}
A & \rightarrow & MC_r^*(G) \\
\exists! & \downarrow & \uparrow \\
C_\delta^*(G) & \leftarrow & (\hat{G})^{ap}
\end{array}
\]

This gives a justification for looking at $C_\delta^*(G)$.
Counter-example

It’s easy to see that always $C_\delta^*(G) \subseteq \text{ap}(A(G))$.

**Theorem (Chou ('90), Rindler ('92))**

*There are compact (connected, if you wish) groups $G$ such that $\text{ap}(A(G)) \neq C_\delta^*(G)$.*

As $G$ is compact, the constant functions are members of $L^2(G)$. Let $E$ be the orthogonal projection onto the constants; then $E = \lambda(1_G) \in MC_r^*(G)$.

- $E \in \text{ap}(A(G))$ if and only if $G$ is tall.
- $E \in C_\delta^*(G)$ if and only if $G$ does not have the weak-mean-zero containment property: there is a net of unit vectors $(\xi_i)$ in ker $E$ with $\|\lambda(s)\xi_i - \xi_i\|_2 \to 0$ for each $s \in G$.
- [Rindler] Clever choice of $G$...
Stronger forms of “compact”

$VN(G)$ is naturally an operator space: we have a family of norms on $M_n(VN(G))$. Then $A(G)$ is also an operator space. The natural morphisms are the completely bounded maps: those whose matrix dilations are uniformly bounded.

- There are various notions of being “completely compact”; they do not interact well with taking adjoints.
- [Runde, 2011] defined $x \in A(G)^*$ to be “completely almost periodic” if both orbit maps $L_x$ and $R_x$ are completely compact.
- If $G$ is amenable, or connected, then

$$\text{cap}(A(G)) = \{ x \in VN(G) : \Delta(x) \in VN(G) \otimes VN(G) \},$$

here $\otimes$ is the C*-spatial product.
- So $x \in \text{cap}(A(G))$ if and only if $\Delta(x)$ can be norm approximated by a finite sum $\sum_{i=1}^n a_i \otimes b_i$. 
Stronger forms of “compact” cont.

**Theorem (D.)**

Let $G$ be discrete. Then $\Delta(x) \in \text{VN}(G) \otimes \text{VN}(G)$ if and only if $x \in C^*_\delta(G)$.

**Theorem (D.)**

Let $G$ be a [SIN] group (compact, discrete, abelian...). Then $\Delta^2(x) \in \text{VN}(G) \otimes \text{VN}(G) \otimes \text{VN}(G)$ if and only if $x \in C^*_\delta(G)$.

**Theorem (Woronowicz, 92)**

Let $\mathbb{G}$ be quantum $E(2)$ (for $\mu \in (0, 1)$). Then $\Delta(C_0(\mathbb{G})) \subseteq C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$, and so $C_0(\mathbb{G}) \subseteq \cap(L^\infty(\mathbb{G}))$. However, the quantum Bohr compactification of $\mathbb{G}$ is $\mathbb{T}$. So $\text{cap}(L^1(\mathbb{G}))$ is (far) too large.
Even stronger forms of “compact”

Definition
Say that $\mu \in A^*$ is “periodic” if $L_\mu : A \to A^*$ is a finite-rank operator. Say that $\mu$ is “strongly almost periodic” if $L_\mu$ can be cb-norm approximated by operators of the form $L_{\mu'}$ with $\mu'$ periodic.

For $A(G)$, equivalently, $x$ is strongly almost periodic if $\Delta(x - x')$ can be made arbitrarily small with $\Delta(x')$ finite-rank.

Theorem (Chou, D.)
$x \in VN(G)$ is strongly almost periodic if and only if $x \in C^*_\delta(G)$.

(D.) An analogous result holds for all Kac algebras.

For a locally compact quantum group, also need to assume things are in $D(S) \cap D(S^*)$, which is rather messy...