# Almost periodic functionals

Matthew Daws

Leeds

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# Almost periodic functionals

Let $A$ be a Banach algebra and turn $A^*$ into an $A$-bimodule:

$$\langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle, \quad \langle \mu \cdot a, b \rangle = \langle \mu, ab \rangle \qquad (a, b \in A, \mu \in A^*).$$

Then consider the "orbit maps"

$$L_\mu : A \to A^*; a \mapsto a \cdot \mu, \quad R_\mu : A \to A^*; a \mapsto \mu \cdot a.$$

**Definition**

Say that $\mu \in A^*$ is (weakly) almost periodic, $\mu \in \mathrm{ap}(A)$ if $L_\mu$ is a (weakly) compact operator (i.e. $\{a \cdot \mu : \|a\| \leq 1\}$ is relatively (weakly) compact in $A^*$). (Equivalently can use $R_\mu$.)

# Turning into an algebra

Choose bounded nets $(a_i), (b_i)$ converging weak* to $\Phi, \Psi \in A^{**}$ respectively. Then we have two choices for a product:

$$\Phi \Box \Psi = \lim_i \lim_j a_i b_j, \quad \Phi \Diamond \Psi = \lim_j \lim_i a_i b_j.$$

These are the Arens products.
Consider $\mu \in \mathrm{ap}(A)$, so we may assume $b_j \cdot \mu \to \lambda$ *in norm*. Then:

$$\begin{aligned} \langle \Phi \Box \Psi, \mu \rangle &= \lim_i \lim_j \langle b_j \cdot \mu, a_i \rangle = \lim_i \langle \lambda, a_i \rangle = \langle \Phi, \lambda \rangle \\ &= \lim_j \langle \Phi, b_j \cdot \mu \rangle = \langle \Phi \Diamond \Psi, \mu \rangle. \end{aligned}$$

But more is true:

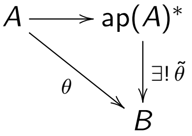
$$\lim_i \langle \mu, a_i b_i \rangle = \lim_i \langle b_i \cdot \mu, a_i \rangle = \lim_i \langle \lambda, a_i \rangle = \langle \Phi \Box \Psi, \mu \rangle.$$

# Universal property; link with dual Banach algebras

**Theorem (Lau, Loy, Runde, ...?)**

*By separate weak$^*$-continuity, the product on $A$ extends to* $\mathrm{ap}(A)^*$, *turning* $\mathrm{ap}(A)^*$ *into a dual Banach algebra. The product on* $\mathrm{ap}(A)^*$ *is jointly weak$^*$-continuous (on bounded set). This is the universal object for "jointly continuous" dual Banach algebras.*

That is, if $B$ is a dual Banach algebra with jointly continuous multiplication, and $\theta : A \to B$ a bounded homomorphism, then there is a unique $\tilde{\theta} : \mathrm{ap}(A)^* \to B$, a weak$^*$-continuous homomorphism, with:



# For group algebras

Let $G$ be a locally compact group and consider $A = L^1(G)$.

**Theorem (Lau, (Wong, Ulger))**

*$F \in L^\infty(G) = A^*$ with be almost periodic if and only if the set of (left, or right) translates of $F$ forms a relatively norm compact subset of $L^\infty(\mathbb{G})$. In this case, $F \in C^b(G)$.*

- Easy to see that $\mathrm{ap}(A) \subseteq C^b(G) \subseteq L^\infty(G)$ will be a unital (commutative) C*-algebra.
- Let $G^{\mathrm{ap}}$ be the (compact, Hausdorff) character space, so $\mathrm{ap}(A) = C(G^{\mathrm{ap}})$.

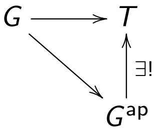
# The (semi)group $G^{\mathrm{ap}}$

$$\mathrm{ap}(L^1(G)) = C(G^{\mathrm{ap}}), \qquad \mathrm{ap}(L^1(G))^* = M(G^{\mathrm{ap}}).$$

- Point-evaluation gives a continuous dense-range map $G \to G^{\mathrm{ap}}$.
- Product on $M(G^{\mathrm{ap}})$ determined by map $L^1(G) \to M(G^{\mathrm{ap}})$.
- These maps are compatible.
- Can show that product of point masses in $M(G^{\mathrm{ap}})$ is again a point-mass.
- So $G^{\mathrm{ap}}$ is a semigroup and $G \to G^{\mathrm{ap}}$ a homomorphism.
- Product on $G^{\mathrm{ap}}$ is jointly continuous; contains dense subgroup; so is a group.

# Universal property

- Using some Banach algebra techniques, we started with a locally compact group $G$, and formed a compact group $G^{\mathrm{ap}}$.
- $G^{\mathrm{ap}}$ is universal in the sense that if $T$ is any compact topological (semi)group then have:




$G^{\mathrm{ap}}$ is the "Bohr compactification" of $G$

- As $G^{\mathrm{ap}}$ is a compact group, Peter-Weyl tells us that functions of the form

$$G^{\mathrm{ap}} \to \mathbb{C}; \quad s \mapsto (\pi(s)\xi|\eta),$$

are dense in $C(G^{\mathrm{ap}})$. Here $\pi : G^{\mathrm{ap}} \to U(n)$ is a finite-dimensional unitary representation, and $\xi, \eta \in \mathbb{C}^n$.
- Such $\pi$ are in 1-1 correspondence with finite-dimensional unitary representations of $G$. So such continuous functions are dense in $\mathrm{ap}(L^1(G))$. Not clear how to see this directly...

# Non-commutative world

Let $G$ be a locally compact group, and let $\pi : G \to \mathcal{U}(H)$ be the universal, strongly continuous, unitary representation of $G$ (direct sum over "all" such representations).

- We can "integrate" this to a map $\pi : L^1(G) \to \mathcal{B}(H)$

$$\pi(f)\xi = \int_G f(s)\pi(s)\xi \, ds \qquad (f \in L^1(G), \xi \in H).$$

- This is a $*$-homomorphism of $L^1(G)$; it's the universal one.
- The closure of $\pi(L^1(G))$ is $C^*(G)$ the universal group C$^*$-algebra.
- If we replace $\pi$ by $\lambda$ the left-regular representation on $L^2(G)$, we get the reduced group C$^*$-algebra $C_r^*(G)$.
- $C^*(G) \to C_r^*(G)$ is an isomorphism precisely when $G$ is amenable.
- Finally, define $VN(G) = C_r^*(G)''$ the group von Neumann algebra.

# Fourier theory

If $G$ is an abelian group, then we have the Pontryagin dual $\widehat{G}$. The Fourier transform gives a unitary map

$$\mathcal{F} : L^2(G) \to L^2(\widehat{G}).$$

- Let $C_0(G)$ act on $L^2(G)$ by multiplication, say $f \leftrightarrow M_f$.
- The conjugation map $M_f \mapsto \mathcal{F} M_f \mathcal{F}^{-1}$ gives a $*$-isomorphism $C_0(G) \to C_r^*(\widehat{G})$.
- It also gives a normal $*$-isomorphism $L^\infty(G) \to VN(\widehat{G})$.
- So the predual $VN(\widehat{G})_*$ is isomorphic to the *algebra* $L^1(G)$.
- By biduality, $VN(G)_* \cong L^1(\widehat{G})$.
- What happens when $G$ is not abelian?

# Hopf von Neumann algebras

There is a normal $*$-homomorphism

$$\Delta : VN(G) \to VN(G)\overline{\otimes}VN(G); \quad \lambda(s) \mapsto \lambda(s) \otimes \lambda(s).$$

- That this exists is most easily seen by finding a unitary operator $W$ on $L^2(G \times G)$ with $\Delta(x) = W^*(1 \otimes x)W$.
- $\Delta$ is coassociative: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.
- Let the predual of $VN(G)$ by $A(G)$; the predual of $\Delta$ gives an associative product

$$\Delta_* : A(G) \times A(G) \to A(G).$$

- This is the Fourier algebra; the map $A(G) \to C_0(G); \omega \mapsto (\langle \lambda(s), \omega \rangle)$ is a contractive algebra homomorphism.
- So $A(G)$ is a commutative Banach algebra; it is semisimple (and regular, Tauberian) with character space $G$.

# Almost periodic for the Fourier algebra

**Question**

What is $\mathrm{ap}(A(G))$? What relation does it have to a "compactification"?

Let $C^*_\delta(G)$ be the C$^*$-algebra generated by $\{\lambda(s) : s \in G\}$ inside $VN(G)$.

- If $G$ is discrete, then $C^*_\delta(G) = C^*_r(G)$.
- If $G$ is discrete and amenable, or $G$ is abelian, then $\mathrm{ap}(A(G)) = C^*_\delta(G)$.

# Compact quantum groups

Unital C$^*$-algebra $A$ and coassociative $\Delta : A \to A \otimes A$ with "cancellation":

$$\mathrm{lin}\{\Delta(a)(1 \otimes b) : a, b \in A\}, \quad \mathrm{lin}\{\Delta(a)(b \otimes 1) : a, b \in A\}$$

are dense in $A \otimes A$.

- $G$ compact gives $C(G)$ with $\Delta(f)(s,t) = f(st)$;
- $G$ discrete gives $C_r^*(G)$ with $\Delta$ as before.

Natural morphisms are the "Hopf $*$-homomorphisms"; a morphism $(A, \Delta_A)$ to $(B, \Delta_B)$ is a $*$-homomorphism $\theta : B \to A$ with $\Delta_A \circ \theta = (\theta \otimes \theta) \circ \Delta_B$.

- If $\phi : G \to H$ is a continuous group homomorphism, then may define $\theta : C(H) \to C(G)$ by $\theta(f) = f \circ \phi$.

Can extend this to the non-compact world by considering multiplier algebras.

# Quantum Bohr compactification

Sołtan (2005) considered "compactifications" in this category. In particular, $C_\delta^*(G)$ is the universal object for $C_r^*(G)$. For any compact quantum group $(A, \Delta_A)$, we have:

$$
\begin{array}{ccc}
A & \longrightarrow & MC_r^*(G) \\
& \searrow^{\exists!} & \uparrow \\
& & C_\delta^*(G)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{G} & \longleftarrow & \widehat{G} \\
& \nwarrow^{\exists!} & \downarrow \\
& & (\widehat{G})^{\mathrm{ap}}
\end{array}
$$

This gives a justification for looking at $C_\delta^*(G)$.

# Counter-example

It's easy to see that always $C^*_\delta(G) \subseteq \text{ap}(A(G))$.

**Theorem (Chou ('90), Rindler ('92))**

*There are compact (connected, if you wish) groups $G$ such that $\text{ap}(A(G)) \neq C^*_\delta(G)$.*

As $G$ is compact, the constant functions are members of $L^2(G)$. Let $E$ be the orthogonal projection onto the constants; then $E = \lambda(1_G) \in MC^*_r(G)$.

- $E \in \text{ap}(A(G))$ if and only if $G$ is tall.
- $E \in C^*_\delta(G)$ if and only if $G$ does not have the weak-mean-zero containment property: there is a net of unit vectors $(\xi_i)$ in $\ker E$ with $\|\lambda(s)\xi_i - \xi_i\|_2 \to 0$ for each $s \in G$.
- [Rindler] Clever choice of $G$...

# Stronger forms of "compact"

$VN(G)$ is naturally an *operator space*: we have a family of norms on $M_n(VN(G))$. Then $A(G)$ is also an operator space. The natural morphisms are the *completely bounded* maps: those whose matrix dilations are uniformly bounded.

- There are various notions of being "completely compact"; they do not interact well with taking adjoints.
- [Runde, 2011] defined $x \in A(G)^*$ to be "completely almost periodic" if both orbit maps $L_x$ and $R_x$ are completely compact.
- If $G$ is amenable, or connected, then

  $$\mathrm{cap}(A(G)) = \{x \in VN(G) : \Delta(x) \in VN(G) \otimes VN(G)\},$$

  here $\otimes$ is the C$^*$-spatial product.
- So $x \in \mathrm{cap}(A(G))$ if and only if $\Delta(x)$ can be norm approximated by a *finite* sum $\sum_{i=1}^n a_i \otimes b_i$.

# Stronger forms of "compact" cont.

**Theorem (D.)**

*Let $G$ be discrete. Then $\Delta(x) \in VN(G) \otimes VN(G)$ if and only if $x \in C^*_\delta(G)$.*

**Theorem (D.)**

*Let $G$ be a [SIN] group (compact, discrete, abelian. . . ). Then $\Delta^2(x) \in VN(G) \otimes VN(G) \otimes VN(G)$ if and only if $x \in C^*_\delta(G)$.*

**Theorem (Woronowicz, 92)**

*Let $\mathbb{G}$ be quantum $E(2)$ (for $\mu \in (0,1)$). Then $\Delta(C_0(\mathbb{G})) \subseteq C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$, and so $C_0(\mathbb{G}) \subseteq \cap(L^\infty(\mathbb{G}))$. However, the quantum Bohr compactification of $\mathbb{G}$ is $\mathbb{T}$. So $\mathrm{cap}(L^1(\mathbb{G}))$ is (far) too large.*

# Even stronger forms of "compact"

**Definition**

Say that $\mu \in A^*$ is "periodic" if $L_\mu : A \to A^*$ is a finite-rank operator. Say that $\mu$ is "strongly almost periodic" if $L_\mu$ can be cb-norm approximated by operators of the form $L_{\mu'}$ with $\mu'$ periodic.

For $A(G)$, equivalently, $x$ is strongly almost periodic if $\Delta(x - x')$ can be made arbitrarily small with $\Delta(x')$ finite-rank.

**Theorem (Chou, D.)**

*$x \in VN(G)$ is strongly almost periodic if and only if $x \in C^*_\delta(G)$.*

- (D.) An analogous result holds for all Kac algebras.
- For a locally compact quantum group, also need to assume things are in $D(S) \cap D(S^*)$, which is rather messy...