Ideals of operators on the Banach space of continuous functions on the first uncountable ordinal

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$C(K)$-spaces

For a compact Hausdorff space K, consider the Banach space

$$C(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\}.$$

Fact. *$C(K)$ separable $\iff$ K metrizable.*

Classification. *Let K be a compact metric space. Then:*

(i) *K has $n \in \mathbb{N}$ elements $\iff$ $C(K) \cong \ell_\infty^n$;*

(ii) (Milutin) *K is uncountable $\iff$ $C(K) \cong C[0,1]$;*

(iii) (Bessaga and Pełczyński) *K is countably infinite $\iff$ $C(K) \cong C[0, \omega^{\omega^\alpha}]$ for a unique countable ordinal α.*

Here, for an ordinal σ,

$$[0, \sigma] = \{\alpha \text{ ordinal} : \alpha \leqslant \sigma\}$$

is equipped with the *order topology*, which is determined by the basis

$$[0, \beta), \qquad (\alpha, \beta), \qquad (\alpha, \sigma] \qquad (0 \leqslant \alpha < \beta \leqslant \sigma).$$

The space $C[0, \omega_1]$

Let ω_1 be the first uncountable ordinal, so that $C[0, \omega_1]$ is the "next" $C(K)$-space after the separable ones $C[0, \omega^{\omega^\alpha}]$ for countable α.

Fact. *Suppose that $f : [0, \omega_1] \to \mathbb{C}$ is continuous at ω_1. Then f is eventually constant.*

Proof. For each $\varepsilon > 0$, there exists $\alpha < \omega_1$ such that $|f(\beta) - f(\omega_1)| < \varepsilon$ for each $\beta \in [\alpha, \omega_1]$.
For $n \in \mathbb{N}$, choose α_n corresponding to $\varepsilon = \frac{1}{n}$, and let

$$\alpha = \sup\{\alpha_n : n \in \mathbb{N}\} < \omega_1.$$

Then, for each $\beta \in [\alpha, \omega_1]$,

$$|f(\beta) - f(\omega_1)| < \frac{1}{n} \qquad (n \in \mathbb{N}),$$

so $f(\beta) = f(\omega_1)$. □

Introducing our main character: the Loy–Willis ideal

Theorem (B. E. Johnson 1967). *Let X be a Banach space with $X \cong X \oplus X$. Then each derivation from the Banach algebra $\mathscr{B}(X)$ of (bounded) operators on X into a Banach $\mathscr{B}(X)$-bimodule is automatically continuous.*

Question: what happens when $X \ncong X \oplus X$?

Theorem (Semadeni 1960). $C[0,\omega_1] \ncong C[0,\omega_1] \oplus C[0,\omega_1]$.

Theorem (Loy and Willis 1989). *Each derivation from the Banach algebra $\mathscr{B}(C[0,\omega_1])$ into a Banach $\mathscr{B}(C[0,\omega_1])$-bimodule is automatically continuous.*

Starting point: $\mathscr{B}(C[0,\omega_1])$ *contains a maximal ideal* $\mathscr{M}$ *of codimension one.*

Key step: $\mathscr{M}$ *has a bounded right approximate identity.*

We call $\mathscr{M}$ the *Loy–Willis ideal.*

It is defined using a representation of operators on $C[0,\omega_1]$ as scalar-valued $[0,\omega_1] \times [0,\omega_1]$-matrices; an operator belongs to $\mathscr{M}$ if and only if the final column of its matrix is continuous.

Uniqueness of the Loy–Willis ideal

Theorem (Kania and L 2012). *An operator T on $C[0,\omega_1]$ belongs to the Loy–Willis ideal if and only if the identity operator on $C[0,\omega_1]$ does not factor through T:*

$$\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : \forall\, R, S \in \mathscr{B}(C[0,\omega_1]) \colon I \neq STR\}.$$

Corollary. *The Loy–Willis ideal is the unique maximal ideal of $\mathscr{B}(C[0,\omega_1])$.*

Proof. Suppose that $T \notin \mathscr{M}$. Then $I = STR$ for some $R, S \in \mathscr{B}(C[0,\omega_1])$, so I belongs to the ideal generated by T. □

Remark. Many Banach spaces X share with $C[0,\omega_1]$ the property that

$$\mathscr{M}_X := \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) \colon I \neq STR\}$$

is the unique maximal ideal of $\mathscr{B}(X)$.

Examples: $X = \ell_p$ and $L_p[0,1]$ for $1 \leqslant p \leqslant \infty$; c_0; $C[0,1]$; ℓ_∞/c_0; $C[0,\omega^\omega]$; the Lorentz sequence spaces; and many more!

A generalization

Theorem (Kania, Koszmider and L). *Let* $T \in \mathscr{B}(C_0[0,\omega_1))$. *Then TFAE:*

(a) $T \in \mathscr{M}$;

(b) T *is a* Semadeni operator, *in the sense that* T^{**} *maps the subspace*

$$\{\Lambda \in C_0[0,\omega_1)^{**} : \langle \lambda_n, \Lambda\rangle \to 0 \text{ as } n \to \infty \text{ for each weakly}^* \text{ null sequence } (\lambda_n) \text{ in } C_0[0,\omega_1)^*\}$$

into the canonical copy of $C_0[0,\omega_1)$ *in its bidual;*

(c) *there is a closed, unbounded subset* D *of* $[0,\omega_1)$ *such that*

$$(Tf)(\alpha) = 0 \qquad (f \in C_0[0,\omega_1),\ \alpha \in D);$$

(d) T *factors through the Banach space* $C_0(L_0)$, *where* $L_0 = \bigsqcup_{\sigma<\omega_1}[0,\sigma]$;

(e) *the range of* T *is contained in a* Hilbert-generated subspace *of* $C_0[0,\omega_1)$; *that is, there exist a Hilbert space* H *and an operator* $U\colon H \to C_0[0,\omega_1)$ *such that* $T(C_0[0,\omega_1)) \subseteq \overline{U(H)}$;

(f) *the range of* T *is contained in a* weakly compactly generated subspace *of* $C_0[0,\omega_1)$; *that is, there exist a reflexive Banach space* X *and an operator* $V\colon X \to C_0[0,\omega_1)$ *such that* $T(C_0[0,\omega_1)) \subseteq \overline{V(X)}$;

(g) T *does not fix a copy of* $C_0[0,\omega_1)$;

(h) *the identity operator on* $C_0[0,\omega_1)$ *does not factor through* T.

The Loy–Willis ideal has a bounded left approximate identity

Recall: $\mathscr{M}$ has a bounded right approximate identity (Loy and Willis 1989).

Question: does $\mathscr{M}$ also have a bounded left approximate identity?

Answer: yes!

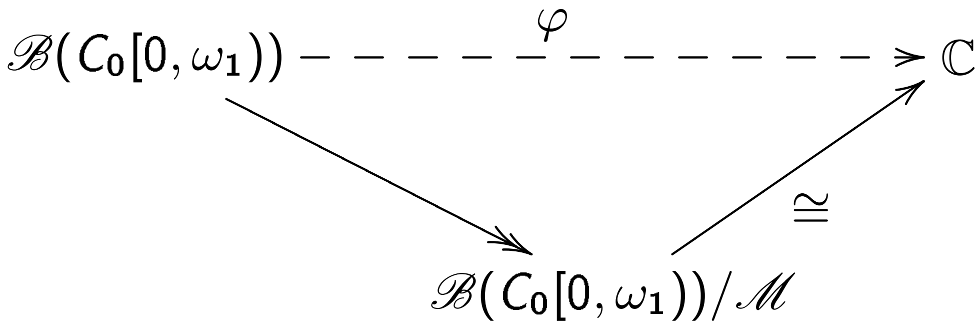
Theorem (Kania, Koszmider and L). *$\mathscr{M}$ contains a net (Q_j) of projections with $\|Q_j\| \leqslant 2$ such that*

$$\forall\, T \in \mathscr{M}\; \exists\, j_0\; \forall\, j \geqslant j_0 : \;\; Q_j T = T.$$

Corollary (using Dixon 1973). *$\mathscr{M}$ has a bounded two-sided approximate identity.*

Other consequences: traces and K-theory

Since $\mathscr{M}$ has codimension one, we have a *character* φ on $\mathscr{B}(C_0[0,\omega_1))$:



Theorem (Kania, Koszmider and L). *Let* $\tau\colon \mathscr{B}(C_0[0,\omega_1)) \to \mathbb{C}$ *be linear. Then:* τ *is a trace, in the sense that* $\tau(ST) = \tau(TS)$ *for all* $S, T \in \mathscr{B}(C_0[0,\omega_1))$, *if and only if* $\tau = \tau(I)\varphi$.

With any ring $\mathscr{A}$, one can associate an abelian group $K_0(\mathscr{A})$, which reflects the structure of the idempotent matrices over $\mathscr{A}$. Since $C_0[0,\omega_1)^n$ does not embed in $C_0[0,\omega_1)^m$ for $m < n$, the element $[I]_0 \in K_0(\mathscr{B}(C_0[0,\omega_1)))$ corresponding to the identity operator has infinite order. Now:

Theorem (Kania, Koszmider and L). $\quad K_0(\mathscr{B}(C_0[0,\omega_1))) = \mathbb{Z}[I]_0.$

Remark. Work of Edelstein and Mityagin (1970) implies that

$$K_1(\mathscr{B}(C_0[0,\omega_1))) = \{0\}.$$

Theorem (Kania and L). *Let $T \in \mathscr{B}(C_0[0,\omega_1))$. Then TFAE:*

(a) *T fixes a copy of $C_0(L_0)$, where $L_0 = \bigsqcup_{\sigma<\omega_1}[0,\sigma]$;*

(b) *the identity operator on $C_0(L_0)$ factors through T;*

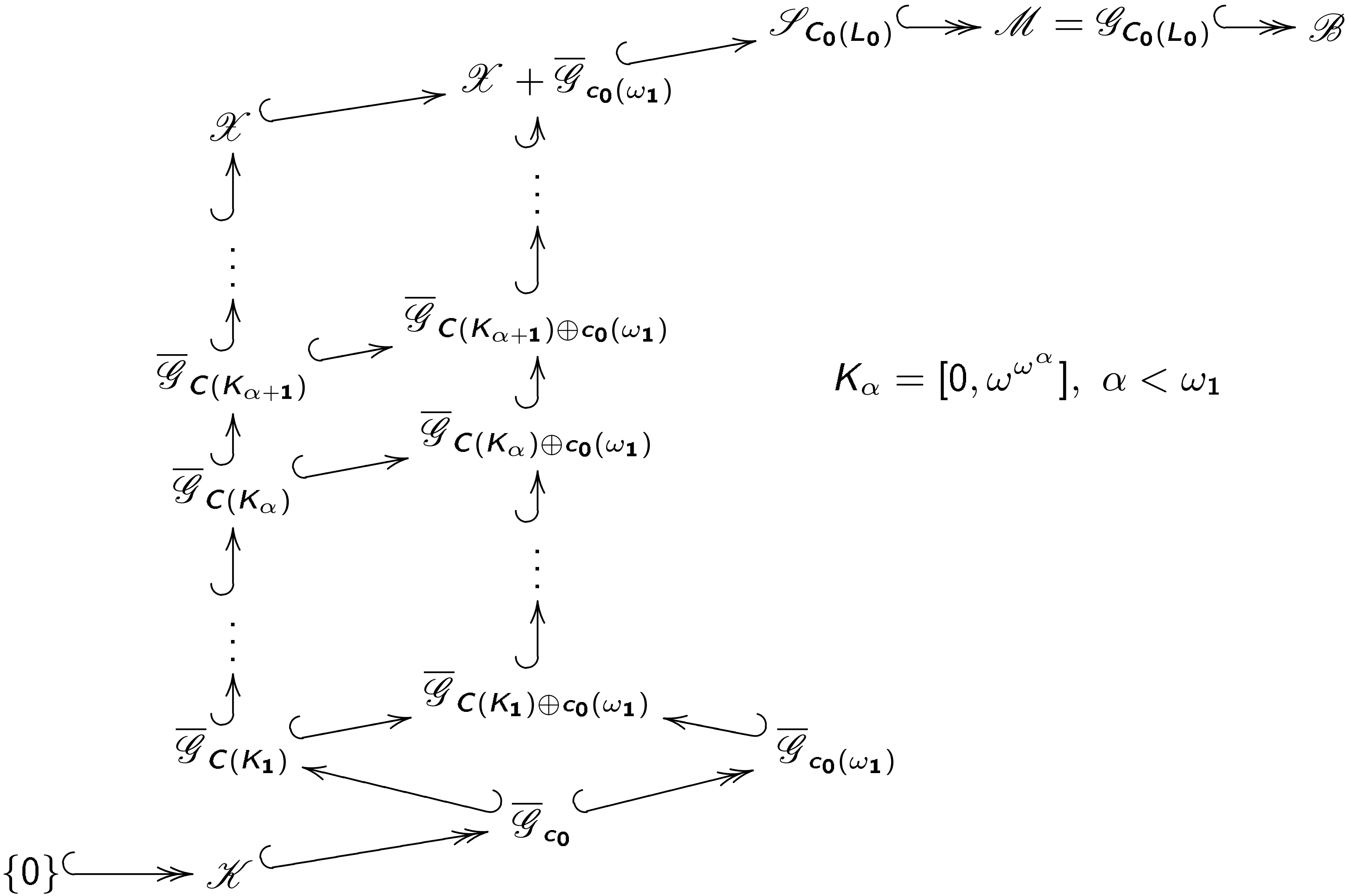
(c) *the Szlenk index of T is uncountable.*

Corollary. *The set*

$$\begin{aligned}\mathscr{S}_{C_0(L_0)}(C_0[0,\omega_1)) &= \{T \in \mathscr{B}(C_0[0,\omega_1)) : T \text{ does not fix a copy of } C_0(L_0)\} \\ &= \{T \in \mathscr{B}(C_0[0,\omega_1)) : \text{the identity operator on } C_0(L_0) \\ &\qquad\qquad \text{does not factor through } T\} \\ &= \{T \in \mathscr{B}(C_0[0,\omega_1)) : \text{the Szlenk index of } T \text{ is countable}\}\end{aligned}$$

is the second-largest proper closed ideal of $\mathscr{B}(C_0[0,\omega_1))$; that is, for each proper ideal $\mathscr{I}$ of $\mathscr{B}(C_0[0,\omega_1))$, either $\mathscr{I} = \mathscr{M}$ or $\mathscr{I} \subseteq \mathscr{S}_{C_0(L_0)}(C_0[0,\omega_1))$.

Partial structure of the lattice of closed ideals of $\mathscr{B} = \mathscr{B}(C_0[0, \omega_1))$



$$K_\alpha = [0, \omega^{\omega^\alpha}], \ \alpha < \omega_1$$

Conventions

(i) We suppress $C_0[0,\omega_1)$ everywhere, thus writing $\mathscr{K}$ instead of $\mathscr{K}(C_0[0,\omega_1))$ for the ideal of compact operators on $C_0[0,\omega_1)$, *etc.*;

(ii) $\mathscr{I} \hookrightarrow \mathscr{J}$ means that the ideal $\mathscr{I}$ is properly contained in the ideal $\mathscr{J}$;

(iii) a double-headed arrow indicates that there are no closed ideals between $\mathscr{I}$ and $\mathscr{J}$;

(iv) $\mathscr{G}_{C(K)}$ denotes the set of operators that factor through $C(K)$, and $\overline{\mathscr{G}}_{C(K)}$ its closure;

(v) $\mathscr{X}$ denotes the set of operators with separable range.

A view behind the scenes: the topological dichotomy

Theorem (Kania, Koszmider and L). *Let K be a weakly* compact subset of $C_0[0,\omega_1)^*$. Then:*

- *either K is homeomorphic to a weakly compact subset of a Hilbert space* (K *is* uniformly Eberlein compact);
- *or K contains a homeomorphic copy of $[0,\omega_1]$ of the form*

$$\{\rho + s\delta_\alpha : \alpha \in D\} \cup \{\rho\},$$

where $\rho \in C_0[0,\omega_1)^*$, $s \in \mathbb{C} \setminus \{0\}$, δ_α is the point evaluation at α, and D is a closed and unbounded subset of $[0,\omega_1)$.

Note: $[0,\omega_1]$ is not uniformly Eberlein compact.

Some references (in chronological order)

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- T. Kania, P. Koszmider and N. J. Laustsen, A weak*-topological dichotomy with applications in operator theory; arXiv:1303.0020.
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