Ideals of operators on the Banach space of continuous functions on the first uncountable ordinal

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Joint work with
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For a compact Hausdorff space $K$, consider the Banach space

$$C(K) = \{ f : K \to \mathbb{C} : f \text{ is continuous} \}.$$ 

**Fact.** $C(K)$ separable $\iff$ $K$ metrizable.

**Classification.** Let $K$ be a compact metric space. Then:

(i) $K$ has $n \in \mathbb{N}$ elements $\iff$ $C(K) \cong \ell^n$;

(ii) (Milutin) $K$ is uncountable $\iff$ $C(K) \cong C[0,1]$;

(iii) (Bessaga and Pełczyński) $K$ is countably infinite $\iff$

$$C(K) \cong C[0,\omega^\alpha] \text{ for a unique countable ordinal } \alpha.$$ 

Here, for an ordinal $\sigma$,

$$[0,\sigma] = \{ \alpha \text{ ordinal} : \alpha \leq \sigma \}$$

is equipped with the order topology, which is determined by the basis

$$[0,\beta), \quad (\alpha, \beta), \quad (\alpha, \sigma] \quad (0 \leq \alpha < \beta \leq \sigma).$$
Let $\omega_1$ be the first uncountable ordinal, so that $C[0, \omega_1]$ is the “next” $C(K)$-space after the separable ones $C[0, \omega^{\omega^a}]$ for countable $\alpha$.

**Fact.** Suppose that $f : [0, \omega_1] \rightarrow \mathbb{C}$ is continuous at $\omega_1$. Then $f$ is eventually constant.

**Proof.** For each $\varepsilon > 0$, there exists $\alpha < \omega_1$ such that $|f(\beta) - f(\omega_1)| < \varepsilon$ for each $\beta \in [\alpha, \omega_1]$.

For $n \in \mathbb{N}$, choose $\alpha_n$ corresponding to $\varepsilon = \frac{1}{n}$, and let

$$\alpha = \sup\{\alpha_n : n \in \mathbb{N}\} < \omega_1.$$  

Then, for each $\beta \in [\alpha, \omega_1]$,

$$|f(\beta) - f(\omega_1)| < \frac{1}{n} \quad (n \in \mathbb{N}),$$

so $f(\beta) = f(\omega_1)$. \qed
Introducing our main character: the Loy–Willis ideal

**Theorem** (B. E. Johnson 1967). Let $X$ be a Banach space with $X \cong X \oplus X$. Then each derivation from the Banach algebra $\mathcal{B}(X)$ of (bounded) operators on $X$ into a Banach $\mathcal{B}(X)$-bimodule is automatically continuous.

**Question:** what happens when $X \not\cong X \oplus X$?

**Theorem** (Semadeni 1960). $C[0, \omega_1] \not\cong C[0, \omega_1] \oplus C[0, \omega_1]$.

**Theorem** (Loy and Willis 1989). Each derivation from the Banach algebra $\mathcal{B}(C[0, \omega_1])$ into a Banach $\mathcal{B}(C[0, \omega_1])$-bimodule is automatically continuous.

**Starting point:** $\mathcal{B}(C[0, \omega_1])$ contains a maximal ideal $\mathcal{M}$ of codimension one.

**Key step:** $\mathcal{M}$ has a bounded right approximate identity.

We call $\mathcal{M}$ the Loy–Willis ideal.

It is defined using a representation of operators on $C[0, \omega_1]$ as scalar-valued $[0, \omega_1] \times [0, \omega_1]$-matrices; an operator belongs to $\mathcal{M}$ if and only if the final column of its matrix is continuous.
Uniqueness of the Loy–Willis ideal

**Theorem** (Kania and L 2012). An operator \( T \) on \( C[0, \omega_1] \) belongs to the Loy–Willis ideal if and only if the identity operator on \( C[0, \omega_1] \) does not factor through \( T \):

\[
\mathcal{M} = \{ T \in \mathcal{B}(C[0, \omega_1]) : \forall R, S \in \mathcal{B}(C[0, \omega_1]) : I \neq STR \}.
\]

**Corollary.** The Loy–Willis ideal is the unique maximal ideal of \( \mathcal{B}(C[0, \omega_1]) \).

**Proof.** Suppose that \( T \notin \mathcal{M} \). Then \( I = STR \) for some \( R, S \in \mathcal{B}(C[0, \omega_1]) \), so \( I \) belongs to the ideal generated by \( T \). \( \square \)

**Remark.** Many Banach spaces \( X \) share with \( C[0, \omega_1] \) the property that

\[
\mathcal{M}_X := \{ T \in \mathcal{B}(X) : \forall R, S \in \mathcal{B}(X) : I \neq STR \}
\]

is the unique maximal ideal of \( \mathcal{B}(X) \).

**Examples:** \( X = \ell_p \) and \( L_p[0, 1] \) for \( 1 \leq p \leq \infty \); \( c_0 \); \( C[0, 1] \); \( \ell_\infty / c_0 \); \( C[0, \omega^\omega] \); the Lorentz sequence spaces; and many more!
A generalization

**Theorem** (Kania, Koszmider and L). Let \( T \in \mathcal{B}(C_0[0, \omega_1]) \). Then TFAE:

(a) \( T \in \mathcal{M} \);
(b) \( T \) is a Semadeni operator, in the sense that \( T^{**} \) maps the subspace

\[ \left\{ \Lambda \in C_0[0, \omega_1]^{**} : \langle \lambda_n, \Lambda \rangle \to 0 \text{ as } n \to \infty \right\} \]

into the canonical copy of \( C_0[0, \omega_1] \) in its bidual;
(c) there is a closed, unbounded subset \( D \) of \([0, \omega_1)\) such that

\[ (Tf)(\alpha) = 0 \quad (f \in C_0[0, \omega_1], \alpha \in D); \]

(d) \( T \) factors through the Banach space \( C_0(L_0) \), where \( L_0 = \bigcup_{\sigma < \omega_1} [0, \sigma] \);
(e) the range of \( T \) is contained in a Hilbert-generated subspace of \( C_0[0, \omega_1] \); that is, there exist a Hilbert space \( H \) and an operator \( U : H \to C_0[0, \omega_1] \) such that \( T(C_0[0, \omega_1]) \subseteq \overline{U(H)} \);
(f) the range of \( T \) is contained in a weakly compactly generated subspace of \( C_0[0, \omega_1] \); that is, there exist a reflexive Banach space \( X \) and an operator \( V : X \to C_0[0, \omega_1] \) such that \( T(C_0[0, \omega_1]) \subseteq \overline{V(X)} \);
(g) \( T \) does not fix a copy of \( C_0[0, \omega_1] \);
(h) the identity operator on \( C_0[0, \omega_1] \) does not factor through \( T \).
Recall: $\mathcal{M}$ has a bounded right approximate identity (Loy and Willis 1989).

**Question:** does $\mathcal{M}$ also have a bounded left approximate identity?

**Answer:** yes!

**Theorem** (Kania, Koszmider and L). $\mathcal{M}$ contains a net $(Q_j)$ of projections with $\|Q_j\| \leq 2$ such that

$$\forall T \in \mathcal{M} \exists j_0 \forall j \geq j_0 : Q_j T = T.$$

**Corollary** (using Dixon 1973). $\mathcal{M}$ has a bounded two-sided approximate identity.
Other consequences: traces and $K$-theory

Since $\mathcal{M}$ has codimension one, we have a character $\varphi$ on $\mathcal{B}(C_0[0, \omega_1])$:

$$
\mathcal{B}(C_0[0, \omega_1]) \xrightarrow{- - - - - - - - - - - -} \mathbb{C}
$$

$\mathcal{B}(C_0[0, \omega_1]) / \mathcal{M} \xrightarrow{\sim} \mathbb{C}$

**Theorem** (Kania, Koszmider and L). Let $\tau : \mathcal{B}(C_0[0, \omega_1]) \to \mathbb{C}$ be linear. Then: 
$\tau$ is a trace, in the sense that $\tau(ST) = \tau(TS)$ for all $S, T \in \mathcal{B}(C_0[0, \omega_1])$, if and only if $\tau = \tau(I)\varphi$.

With any ring $\mathcal{A}$, one can associate an abelian group $K_0(\mathcal{A})$, which reflects the structure of the idempotent matrices over $\mathcal{A}$. Since $C_0[0, \omega_1]^m$ does not embed in $C_0[0, \omega_1]^n$ for $m < n$, the element $[I]_0 \in K_0(\mathcal{B}(C_0[0, \omega_1]))$ corresponding to the identity operator has infinite order. Now:

**Theorem** (Kania, Koszmider and L). $K_0(\mathcal{B}(C_0[0, \omega_1])) = \mathbb{Z}[I]_0$.

**Remark.** Work of Edelstein and Mityagin (1970) implies that

$$K_1(\mathcal{B}(C_0[0, \omega_1])) = \{0\}.$$
The second-largest proper ideal of $\mathcal{B}(C_0[0, \omega_1])$

**Theorem** (Kania and L). Let $T \in \mathcal{B}(C_0[0, \omega_1])$. Then TFAE:

(a) $T$ fixes a copy of $C_0(L_0)$, where $L_0 = \bigsqcup_{\sigma < \omega_1} [0, \sigma]$;
(b) the identity operator on $C_0(L_0)$ factors through $T$;
(c) the Szlenk index of $T$ is uncountable.

**Corollary.** The set

$$\mathcal{I}_{C_0(L_0)}(C_0[0, \omega_1]) = \{ T \in \mathcal{B}(C_0[0, \omega_1]) : T \text{ does not fix a copy of } C_0(L_0) \}$$

$$= \{ T \in \mathcal{B}(C_0[0, \omega_1]) : \text{the identity operator on } C_0(L_0) \text{ does not factor through } T \}$$

$$= \{ T \in \mathcal{B}(C_0[0, \omega_1]) : \text{the Szlenk index of } T \text{ is countable} \}$$

is the second-largest proper closed ideal of $\mathcal{B}(C_0[0, \omega_1])$; that is, for each proper ideal $\mathcal{I}$ of $\mathcal{B}(C_0[0, \omega_1])$, either $\mathcal{I} = \mathcal{M}$ or $\mathcal{I} \subseteq \mathcal{I}_{C_0(L_0)}(C_0[0, \omega_1])$. 
Partial structure of the lattice of closed ideals of $\mathcal{B} = \mathcal{B}(C_0[0, \omega_1])$

$\mathcal{B} \supseteq \mathcal{M} = \mathcal{G}C_0(L_0) \supseteq \mathcal{I}_0(L_0)$

$K_\alpha = [0, \omega^{\omega_\alpha}], \alpha < \omega_1$

$\mathcal{G}C(K_{\alpha+1}) \supseteq \mathcal{G}C(K_\alpha) \supseteq \mathcal{G}C(K_1) \supseteq \mathcal{G}C_0(\omega_1)$

$\{0\} \subseteq \mathcal{K} \subseteq \mathcal{G}C_0$
Conventions

(i) We suppress \( C_0[0, \omega_1] \) everywhere, thus writing \( \mathcal{K} \) instead of \( \mathcal{K}(C_0[0, \omega_1]) \) for the ideal of compact operators on \( C_0[0, \omega_1] \), etc.;

(ii) \( \mathcal{I} \hookrightarrow \mathcal{J} \) means that the ideal \( \mathcal{I} \) is properly contained in the ideal \( \mathcal{J} \);

(iii) a double-headed arrow indicates that there are no closed ideals between \( \mathcal{I} \) and \( \mathcal{J} \);

(iv) \( \mathcal{G}_{C(K)} \) denotes the set of operators that factor through \( C(K) \), and \( \overline{\mathcal{G}}_{C(K)} \) its closure;

(v) \( \mathcal{X} \) denotes the set of operators with separable range.
A view behind the scenes: the topological dichotomy

**Theorem** (Kania, Koszmider and L). Let $K$ be a weakly* compact subset of $C_0[0,\omega_1]^*$. Then:

- either $K$ is homeomorphic to a weakly compact subset of a Hilbert space ($K$ is uniformly Eberlein compact);
- or $K$ contains a homeomorphic copy of $[0,\omega_1]$ of the form

$$\{\rho + s\delta_\alpha : \alpha \in D\} \cup \{\rho\},$$

where $\rho \in C_0[0,\omega_1]^*$, $s \in \mathbb{C} \setminus \{0\}$, $\delta_\alpha$ is the point evaluation at $\alpha$, and $D$ is a closed and unbounded subset of $[0,\omega_1)$.

**Note:** $[0,\omega_1]$ is not uniformly Eberlein compact.
Some references (in chronological order)


- T. Kania, P. Koszmider and N. J. Laustsen, $K$-theory for the Banach algebra of bounded operators on the Banach space $C[0, \omega_1]$; arXiv:1303.2606.