I. The Kadison-Singer Non-Problem
II. New Characterisations of WEP

Vern I. Paulsen

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So.... how would you like to hear my ideas for solving Connes' Embedding problem instead?!

II. New Characterizations of WEP

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Based on joint work with: Doug Farenick, Ali S. Kavruk, Ivan Todorov and Mark Tomforde.

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Let $\mathcal{A}$ be a C-algebra. Then $\mathcal{A}$ has Lance's WEP if and only if $C^*(\mathbb{F}_\infty) \otimes_{min} \mathcal{A} = C^*(\mathbb{F}_\infty) \otimes_{max} \mathcal{A}$.*

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Lance's WEP

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Definition (Lance)

A C-algebra* $\mathcal{A}$ *has WEP if and only if for every faithful representation* $\pi : \mathcal{A} \to B(\mathcal{H})$ *there exists a UCP map* $\phi : B(\mathcal{H}) \to \pi(\mathcal{A})''$ *such that* $\phi(\pi(a)) = \pi(a)$ *for every* $a \in \mathcal{A}$.

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Every C*-algebra has at least one faithful representation π for which such a ϕ exists(the reduced atomic).

So given a concrete C*-subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ the existence of a UCP $\phi : B(\mathcal{H}) \to \mathcal{A}''$ with $\phi(a) = a$ for all $a \in \mathcal{A}$ does not ensure that $\mathcal{A}$ has WEP.

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We'll give characterizations in terms of:

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- Relative Riesz interpolation

Numerical Radius versus Norm

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For many theorems about contraction operators, there are analogues for operators satisfying $w(X) \leq 1$.

Proposition

Let $X \in B(\mathcal{H})$*. Then* $\|X\| \le 1$ *if and only if the operator matrix*

$$\begin{pmatrix} I_{\mathcal{H}} & X \\ X^* & I_{\mathcal{H}} \end{pmatrix} \ge 0,$$

i.e., defines a positive operator on $\mathcal{H} \oplus \mathcal{H}$*.*

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Theorem (Ando)

Let $X \in B(\mathcal{H})$. *Then* $w(X) \le 1/2$ *if and only if there exists* $A, B \in B(\mathcal{H})$ *with* $A + B = I_{\mathcal{H}}$ *such that*

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Note: A bit like an $\ell^\infty - \ell^1$ duality.

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A key element of Bunce's proof was that, if S denotes the unilateral shift, then:
$w(X) \leq 1/2$ iff $I \otimes I + X \otimes S + X^* \otimes S^* \geq 0$,
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It is also the case that:

$w(X) \leq 1/2$ iff $I \otimes I + X \otimes U + X^* \otimes U^* \geq 0$

for every unitary U on every Hilbert space.

A Joint Numerical Radius

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Let $X_1, ..., X_n \in B(\mathcal{H})$ *then*

$$w(X_1, ..., X_n) = \sup\{w(X_1 \otimes U_1 + \cdots + X_n \otimes U_n)\}$$

where $U_1, ..., U_n \in B(\mathcal{K})$ *are arbitrary unitaries on an arbitrary Hilbert space and the tensor product is spatial.*

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Equivalently:
$w(X_1, ..., X_n) \leq 1/2$ iff
$I \otimes I + X_1 \otimes U_1 + X_1^* \otimes U_1^* + \cdots + X_n \otimes U_n + X_n^* \otimes U_n^* \geq 0$
for every tuple of unitaries on every Hilbert space.

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$$\text{such that} \begin{pmatrix} A_1 & X_1 & 0 & \cdots & 0 \\ X_1^* & A_2 & X_2 & \ddots & \\ 0 & X_2^* & A_3 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & X_n^* & A_{n+1} \end{pmatrix} \geq 0,$$

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3. *there exist* $B_1, ..., B_{n+1} \in B(\mathcal{H})$ *with* $B_1 + \cdots + B_n = I_{\mathcal{H}}$ *such that*
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- $\mathcal{A}$ *has WEP*

Theorem (FKPT)

Let $X_1, X_2 \in B(\mathcal{H})$. Then $w(X_1, X_2) < 1/2$ if and only if there exist $B, C \in B(\mathcal{H})$ positive and invertible with

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Let $\mathcal{A} \subseteq B(\mathcal{H})$ is a C-sublgebra. Then the B and C can be chosen from $M_k(\mathcal{A})$ for all $X_1, X_2 \in M_k(\mathcal{A})$ and all k if and only if $\mathcal{A}$ has WEP.*

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In general, we cannot even pick the A's and B's of previous theorem and the B's and C's from this theorem to belong to the von Neumann algebra containing $\mathcal{A}$, as the following result shows.

Theorem (FKP)

Let $\mathcal{A} = \mathcal{A}'' \subseteq B(\mathcal{H})$. *Then* $\mathcal{A}$ *is injective if and only if* $\forall\, j$ *whenever* $X_1, X_2 \in M_j(\mathcal{A})$ *with* $w(X_1, X_2) \leq 1/2$ *we may choose* $A_1, A_2, A_3 \in M_j(\mathcal{A})$.

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Another view: For $X_1, X_2 \in M_j(\mathcal{A})$, define

$$w_{\mathcal{A}}(X_1, X_2) = \frac{1}{2} \inf\{\|A_1 + A_2 + A_3\| : (2) \text{ is positive } \}$$

with $A_1, A_2, A_3 \in M_j(\mathcal{A})$.

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with $A_1, A_2, A_3 \in M_j(\mathcal{A})$.
Then $\mathcal{A}$ is WEP iff $w_{\mathcal{A}}(X_1, X_2) = w(X_1, X_2)\ \forall\, X_1, X_2$.

An Application

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Theorem (Bhattacharya-Farenick)

Let α be an amenable action of a discrete group G on a unital C-algebra $\mathcal{A}$. Then $\mathcal{A}$ has WEP iff $\mathcal{A} \times_{\alpha} G$ has WEP.*

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Recall that $X_1, X_2, Y_1, Y_2 \in B(\mathcal{H})_h$ with $X_i << Y_j$, then there need not be an element H that interpolates, i.e., $X_i << H << Y_j$.

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For example,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} << \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & 3 \end{pmatrix},$$

but there is no matrix that interpolates.

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Theorem (Kavruk)

Every C-subalgebra* $\mathcal{A} \subseteq B(\mathcal{H})$ *has the complete relative (2,2)-Riesz interpolation property.*

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Corollary (Kavruk)

Let $\mathcal{A} = \mathcal{A}'' \subseteq B(\mathcal{H})$. Then $\mathcal{A}$ is injective iff $\mathcal{A}$ has the complete relative (2,3)-Riesz interpolation property.

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In fact, Connes' Embedding Conjecture is true if and only if $C^*(G) \otimes_{min} C^*(H) = C^*(G) \otimes_{max} C^*(H)$ for any pair of these groups.

Assume that $\mathcal{U} \subseteq G$ is a finite generating set for G and let

$$S_{\mathcal{U}} = span\{1, g, g^* : g \in \mathcal{U}\} \subseteq C^*(G),$$

denote the finite dimensional operator system.

Proposition (FKPT)

If G characterizes WEP and $C^*_e(S_{\mathcal{U}}) = C^*(G)$, then $\mathcal{A}$ has WEP iff $S_{\mathcal{U}} \otimes_{min} \mathcal{A} = S_{\mathcal{U}} \otimes_{max} \mathcal{A}$.

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This applies to the “canonical” generating sets for each of the groups from the list that characterize WEP.

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This reduces Connes' to determining if $\mathcal{U} \otimes_{min} \mathcal{V} = \mathcal{U} \otimes_c \mathcal{V}$ for any pair of these operator systems of groups.

Quotients of Operator Systems

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Construction: $(x_{i,j} + J) >> 0$ in $M_n(\mathcal{S}/J)$ iff $\exists y_{i,j} \in J$ with $(x_{i,j} + y_{i,j}) >> 0$ in $M_n(\mathcal{S})$.

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Surprise: Even though operator systems are normed spaces, this quotient gives a very different norm than the usual quotient norm!

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Let $\mathcal{T}_n = span\{E_{i,j} : |i-j| \leq 1\} \subseteq M_{n+1}$, *and let* J *denote the subspace of diagonal matrices of trace 0,*

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But in the operator system quotient $\|E_{I,i+1} + J\| = \|g_i/n\| = 1/n$.
Results on joint numerical radii follow from using this realization of S_n as an operator system quotient and applying tensor theory.
Focus on Kavruk's work instead.

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By the results on groups that characterize WEP:

Proposition

As long as $(p,q) \neq (2,2)$, then $S(p,q) \otimes_{min} \mathcal{A} = S(p,q) \otimes_{max} \mathcal{A}$ iff $\mathcal{A}$ has WEP.

Now consider the following diagram:

$$\begin{array}{ccc} \ell^\infty_{p+q} \otimes_{min} \mathcal{A} & \xrightarrow{\cong} & \ell^\infty_{p+q} \otimes_{max} \mathcal{A} \\ \phi\otimes\mathrm{id} \downarrow & & \downarrow \phi\otimes\mathrm{id} \\ S(p,q) \otimes_{min} \mathcal{A} & \xrightarrow[\mathrm{id}]{} & S(p,q) \otimes_{max} \mathcal{A}\,. \end{array}$$

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Thus, $\mathcal{A}$ has WEP iff the left-hand vertical arrow is a quotient map and we have a "positives lift to positives" characterization of WEP.

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In fact, we have multiple, “inequivalent” ways of realizing S_n as a quotient and each of these leads to a “different” characterization of WEP. Currently, the only way that we have to show that these “different” conditions on a C*-algebra are all the same is by showing that each characterizes WEP.

THANKS!