Supramenable groups and their actions on locally compact Hausdorff spaces

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Outline

1. Dynamical systems and crossed product C^*-algebras
2. Paradoxical sets - Tarski's theorem
3. Supramenable groups
4. Universal Γ-spaces

X = compact or locally compact Hausdorff space,
Γ = (countable) discrete group which acts on X.

- $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is **nuclear** $\iff$ $\Gamma \curvearrowright X$ *amenable* [Anantharaman-Delaroche]
- **Tracial states** on $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ $\leftrightarrow$ Γ-invariant probability measures on X
- $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is **simple** $\Leftarrow$ $\Gamma \curvearrowright X$ is *topologically free* and *minimal*. [Archbold–Spielberg]

Definition

$\Gamma \curvearrowright X$ is *topologically free* if

$$\forall t \in \Gamma \setminus \{e\}\colon \{x \in X \mid t.x \neq x\} \text{ is dense in } X.$$

Note: $C^*_{\mathrm{red}}(\mathbb{F}_n)$ is simple, but $\mathbb{F}_n \curvearrowright \{\mathrm{pt}\}$ is not topologically free (or free).

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Definition (Anantharaman-Delaroche)

An action of a countable discrete group Γ on a locally compact space X is said to be *amenable* if there is a net of continuous maps $m_i \colon X \to \mathrm{Prob}(\Gamma)$ (written $x \mapsto m_i^x$) such that

$$\|t.m_i^x - m_i^{t.x}\|_1 \to 0$$

uniformly on all compact subsets of X and for all $t \in \Gamma$.

- If Γ is amenable, then we can choose the m_i's above being constant. Hence any action of an amenable group is amenable.
- If $X = \{\mathrm{pt}\}$, then Γ acts amenably on X iff Γ is amenable.
- Any *proper* action is amenable.
- If X has an invariant probability measure, then $\Gamma \curvearrowright X$ is amenable if and only if Γ is amenable.

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Definition: $\Gamma \curvearrowright X$ is *regular* if $C_0(X) \rtimes_{\mathrm{full}} \Gamma = C_0(X) \rtimes_{\mathrm{red}} \Gamma$.

Anantharaman-Delaroche proved the following:

- $\Gamma \curvearrowright X$ amenable $\implies$ $\Gamma \curvearrowright X$ *regular*.
- $\Gamma \curvearrowright X$ amenable $\iff$ $C_0(X) \rtimes \Gamma$ is nuclear.

Jean-Louis Tu proved:

- $\Gamma \curvearrowright X$ amenable $\implies$ $C_0(X) \rtimes \Gamma$ is in the UCT class.

Example (The Roe algebra)

The Roe algebra associated with a discrete group Γ is the crossed product:

$$\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma = C(\beta\Gamma) \rtimes_{\mathrm{red}} \Gamma$$

where Γ acts on $\ell^\infty(\Gamma)$ by left translation. The left multiplication action $\Gamma \curvearrowright \Gamma$ extends (uniquely) to an action $\Gamma \curvearrowright \beta\Gamma$.

- $\Gamma \curvearrowright \beta\Gamma$ is amenable $\iff$ Γ is exact. [Ozawa]
- $\Gamma \curvearrowright \beta\Gamma$ is free for all Γ.
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Recall:

- $\Gamma \curvearrowright X$ is *regular* if $C_0(X) \rtimes_{\mathrm{full}} \Gamma = C_0(X) \rtimes_{\mathrm{red}} \Gamma$,
- $\Gamma \curvearrowright X$ amenable $\implies$ $\Gamma \curvearrowright X$ regular.
- $\Gamma \curvearrowright X$ amenable $\iff$ $C_0(X) \rtimes \Gamma$ is nuclear.

Proposition (Archbold-Spielberg, 1993)

$C_0(X) \rtimes_{\mathrm{full}} \Gamma$ is simple $\iff$ $\Gamma \curvearrowright X$ is minimal, topologically free and regular.

Corollary (Anantharaman-Delaroche + Archbold-Spielberg)

$C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is simple and nuclear $\iff$ $\Gamma \curvearrowright X$ is minimal, topologically free and amenable.

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X = metrizable, locally compact, totally disconnected space,
Γ = countable discrete group, and $\Gamma \curvearrowright X$.

Proposition (Archbold-Spielberg, Laca-Spielberg)

Suppose that

- $\Gamma \curvearrowright X$ *minimal and topologically free,*
- *every non-zero projection in* $C_0(X)$ *is properly infinite in* $C_0(X) \rtimes_{\mathrm{red}} \Gamma$.

Then $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ *is simple and purely infinite.*

Corollary

$C_0(X) \rtimes_{\mathrm{red}} \Gamma$ *is a Kirchberg algebra in the UCT class if and only if*

- $\Gamma \curvearrowright X$ *is minimal, topologically free, and amenable,*
- *every non-zero projection in* $C_0(X)$ *is properly infinite in* $C_0(X) \rtimes_{\mathrm{red}} \Gamma$.

Def: Kirchberg algebra = simple, nuclear, separable, *purely infinite* C^*-algebra.

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Definition

$\Gamma \curvearrowright X$ and X totally disconnected. Let $E \subseteq X$ be compact-open. Then E is (X, Γ)-*paradoxical* if $\exists$ compact-open pairwise disjoint subsets $E_1, E_2, \ldots, E_{n+m}$ of E and $t_1, t_2, \ldots, t_{n+m} \in \Gamma$ st

$$E \subseteq \bigcup_{j=1}^{n} t_j.E_j, \qquad E \subseteq \bigcup_{j=n+1}^{n+m} t_j.E_j.$$

- We say that $\Gamma \curvearrowright X$ is *purely infinite* if all non-empty compact-open subsets of X are (X, Γ)-paradoxical.

- Being purely infinite is the opposite extreme from having a non-zero invariant Radon measure on X.

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Definition (Paradoxical sets)

Let Γ be a discrete group acting on a (discrete) set X. A set $E \subseteq X$ is said to be *paradoxical* if there are pairwise disjoint subsets $V_1, V_2, \ldots, V_{n+m} \subseteq E$ and $t_1, t_2, \ldots, t_{n+m} \in \Gamma$ st

$$\bigcup_{j=1}^{n} t_j.V_j = \bigcup_{j=n+1}^{n+m} t_j.V_j = E.$$

Suppose $X = \Gamma$, consider $\Gamma \curvearrowright \Gamma$, and let $E \subseteq \Gamma$.

- An injective map $\sigma\colon E \to \Gamma$ is said to be a *piecewise translation* there is a finite set $S \subseteq \Gamma$ such that $\sigma(t)t^{-1} \in S$ for all $t \in E$.
  - $\forall A \subseteq E : \sigma(A) \sim_\Gamma A$, when $\sigma\colon E \to \Gamma$ is a piecewise translation.
  - $E \subseteq \Gamma$ is paradoxical iff there exist piecewise translations $\sigma^{\pm}\colon E \to E$ such that $\sigma^+(E) \cap \sigma^-(E) = \emptyset$.

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Theorem (Tarski)

Let Γ be a discrete group acting on a set X, and let $E \subseteq X$. Then there exists a Γ-invariant finitely additive measure μ on $P(X)$ such that $0 < \mu(E) < \infty$ if and only if E is not Γ-paradoxical.

Theorem (R.-Sierakowski)

Let Γ be a countable discrete group acting on a set X (eg. X could be Γ itself). The following are equivalent for every $E \subseteq X$:

(i) *E is Γ-paradoxical.*

(ii) *1_E is properly infinite in $\ell^\infty(X) \rtimes_{\mathrm{red}} \Gamma$.*

(iii) *There is no lower semicontinuous tracial weight τ on $\ell^\infty(X) \rtimes_{\mathrm{red}} \Gamma$ for which $0 < \tau(1_E) < \infty$.*

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Definition (Rosenblatt, 1974)

A group Γ is *supramenable* if for all $A \subseteq \Gamma$ there exists a Γ-invariant finitely additive measure on Γ such that $\mu(A) = 1$. Equivalenty, Γ is supramenable if Γ contains no paradoxical subsets.

Rosenblatt proved the following:

- All abelian groups are supramenable.
- Every group of subexponential growth (in particular, of polynomial growth) is supramenable.
- Any group that contains a free semigroup with two (or more) generators is not supramenable.
- In general: bad permanence properties. Not known if Γ, Λ supramenable $\implies$ $\Gamma \times \Lambda$ supramenable.

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Theorem (Rosenblatt)

Let Γ be a solvable group. TFAE:

1. *Γ is supramenable,*
2. *Γ has sub-exponential growth,*
3. *Γ does not contain a free semigroup (of two or more generators),*
4. *Γ is essentially nilpotent.*

- A group Γ is amenable if and only if whenever it acts on a compact Hausdorff space X, then X has a Γ-invariant probability measure.

Proposition (Characterization of supramenability, I)

A group Γ is supramenable if and only if whenever it acts co-compactly on a locally compact Hausdorff space X, then X has a non-zero Γ-invariant Radon measure.

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Example (Groups that are non-supramenable)

- The $ax + b$ group (over $\mathbb{Q}$) is solvable (hence amenable), but non-supramenable.
- The Thompson group F is non-supramenable. It is not known if F is amenable or not.
- $BS(1, m) = \langle a, b \mid bab^{-1} = a^m \rangle$ (with $m \neq \pm 1$) is solvable, finitely generated, and non-supramenable (it contains a free semigroup: $\langle b^2, b^2 a \rangle$).

Supramenability and the Roe algebra

- $E \subseteq \Gamma$ is paradoxical iff 1_E is a properly infinite projection in $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$. One can use this to prove:

Proposition (Characterization of supramenability, II)

Let Γ be a discrete group. Then Γ is supramenable if and only if the Roe algebra $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ contains no properly infinite projections.

- Γ is non-amenable $\iff$ Γ is paradoxical $\iff$ $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is properly infinite.
- Γ non-supramenable $\implies$ $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is infinite.
- If Γ contains an element of infinite order, then $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is infinite.
- If Γ is locally finite $\implies$ $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is finite.

Question: $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is finite $\overset{??}{\iff}$ Γ locally finite?

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Purely infinite actions

$\mathbf{K}$ = usual compact Cantor set.
$\mathbf{K}^*$ = non-compact locally compact Cantor set.

Theorem

Let Γ be a countable exact group. Then the following are equivalent:

(1) *Γ is not supramenable.*
(2) *Γ admits a free, minimal, amenable, purely infinite action on $\mathbf{K}^*$.*

If $\Gamma \curvearrowright \mathbf{K}^*$ is as in (2), then $C_0(\mathbf{K}^*) \rtimes \Gamma$ is a stable Kirchberg algebra in the UCT class.

Note: $\Gamma \curvearrowright \mathbf{K}^*$ amenable $\not\Rightarrow$ Γ exact.

Construction of actions of groups on locally compact spaces

Begin with $\Gamma \curvearrowright \beta\Gamma$. Recall that this action is free; and amenable if Γ is exact.

Take $A \subseteq \Gamma$ and put $K_A = \bar{A} \subseteq \beta\Gamma$. Note that K_A is compact-open in $\beta\Gamma$. Put

$$X_A = \bigcup_{t \in \Gamma} t.K_A \subseteq \beta\Gamma.$$

- X_A is open (hence locally compact), $\Gamma \curvearrowright X_A$, and the action is co-compact.
- X_A admits a non-zero Γ-invariant Radon measure iff A is non-paradoxical.

$\rightsquigarrow$ If all co-compact actions of Γ on any locally compact Hausdorff space admit a non-zero invariant Radon measure, then Γ must be supramenable.

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$\rightsquigarrow$ If all co-compact actions of Γ on any locally compact Hausdorff space admit a non-zero invariant Radon measure, then Γ must be supramenable.

Construction of actions of groups on locally compact spaces, ctd.

$A \subseteq \Gamma \rightsquigarrow \Gamma \curvearrowright X_A$. The action is free and co-compact. It is amenable if Γ is exact.

Since $X_A = \bigcup_{t\in\Gamma} t.K_A$, where $K_A = \bar{A}$, and K_A is compact, it follows that X_A has a maximal proper Γ-invariant open subset, and hence a minimal (non-empty) Γ-invariant closed subset Z.

- Z is locally compact and totally disconnected.
- Z is (typically) **not** 2nd countable.
- $\Gamma \curvearrowright Z$ is free and minimal (and amenable if Γ is exact).
- If A is paradoxical, then $\Gamma \curvearrowright Z$ is purely infinite.

▶ The desired action of Γ on $\mathbf{K}^*$ can be obtained from this provided that Z is non-compact and non-discrete.

▶ As $\Gamma \curvearrowright Z$ is purely infinite, Z cannot be discrete.

▶ We shall later give criterions on A that will ensure that Z is non-compact.

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- If A is paradoxical, then $\Gamma \curvearrowright Z$ is purely infinite.

▶ The desired action of Γ on $\mathbf{K}^*$ can be obtained from this provided that Z is non-compact and non-discrete.

▶ As $\Gamma \curvearrowright Z$ is purely infinite, Z cannot be discrete.

▶ We shall later give criterions on A that will ensure that Z is non-compact.

Construction of actions of groups on locally compact spaces, ctd.

$A \subseteq \Gamma \rightsquigarrow \Gamma \curvearrowright X_A$. The action is free and co-compact. It is amenable if Γ is exact.

Since $X_A = \bigcup_{t \in \Gamma} t.K_A$, where $K_A = \bar{A}$, and K_A is compact, it follows that X_A has a maximal proper Γ-invariant open subset, and hence a minimal (non-empty) Γ-invariant closed subset Z.

- Z is locally compact and totally disconnected.
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Definition

Let Γ and Λ be discrete groups. A map $f : \Lambda \to \Gamma$ is said to be *Lipschitz* if: $\forall$ finite $F \subseteq \Lambda$ $\exists$ finite $F' \subseteq \Gamma$ st

$$\forall s, t \in \Lambda : st^{-1} \in F \implies f(s)f(t)^{-1} \in F'.$$

f is said to be a *quasi-isometric embedding* if it is Lipschitz and satisfies: $\forall$ finite $F' \subseteq \Gamma$ $\exists$ finite $F \subseteq \Lambda$ st

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- Any group homomorphism is Lipschitz.
- Any group hom with finite kernel is a quasi-isometric embedding.

Recall:

- A piecewise translation is an injective map $\sigma : A \subseteq \Gamma \to \Gamma$ for which there is a finite $S \subseteq \Gamma$ st $\sigma(t)t^{-1} \in S$ for all $t \in A$.
- $A \subseteq \Gamma$ is paradoxical iff there are piecewise translation $\sigma^{\pm} : A \to A$ such that $\sigma^{+}(A) \cap \sigma^{-}(A) = \emptyset$.

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- If $A \subseteq \Lambda$, if $f\colon \Lambda \to \Gamma$ is injective Lipschitz, and if $\sigma\colon A \to \Lambda$ is a piecewise translation, then $\tau\colon f(A) \to \Gamma$ given by $\tau \circ f = f \circ \sigma$ is a piecewise translation:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & f(A) \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \tau} \\ \Lambda & \xrightarrow[\ f\ ]{} & \Gamma \end{array}$$

- Any injective Lipschitz function maps paradoxical sets to paradoxical sets.

Proposition (Characterization of supramenability, III)

Γ is non-supramenable $\iff$ $\exists$ injective Lipschitz map $f\colon \mathbb{F}_2 \to \Gamma$.

Proof: "$\Leftarrow$": $f(\mathbb{F}_2)$ is a paradoxical set in Γ.

"$\Rightarrow$": Construct a free binary tree in the Cayley graph of Γ from piecewise translations $\sigma^{\pm}\colon A \to \Gamma$ that witness paradoxicality of A.

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Theorem (Benjamini-Schramm, 1997)

Γ is non-amenable $\implies \exists$ injective quasi-isometric embedding $f : \mathbb{F}_2 \to \Gamma$.

(The existence of a paradoxical $A \subseteq \Gamma$ st minimal closed invariant subsets of X_A are non-compact for every non-amenable Γ follows from this theorem of Benjamini and Schramm.)

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\subseteq\ & \{\text{supramenable groups}\} \\
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It is not known if the two "$\subseteq$" are strict.

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Outline

1. Dynamical systems and crossed product C^*-algebras
2. Paradoxical sets - Tarski's theorem
3. Supramenable groups
4. Universal Γ-spaces

We consider locally compact Γ-spaces with a dense orbit:

Proposition

Let Γ be a countable group acting on a locally compact Hausdorff space X. Consider the following properties:

- (i) $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ *is prime,*
- (ii) $\Gamma \curvearrowright X$ *is topologically transitive:* $\forall \emptyset \neq U, V \subseteq X$ *open* $\exists g \in \Gamma : g.U \cap V \neq \emptyset$,
- (iii) $\exists$ *dense* Γ*-orbit in* X,
- (iv) $\Gamma.x$ *is dense in* X *for all* x *in a dense* G_δ *subset of* X.

Then

$$(i) \overset{(*)}{\Longleftrightarrow} (ii) \overset{(**)}{\Longleftrightarrow} (iii) \overset{(**)}{\Longleftrightarrow} (iv).$$

(∗) *holds if* $C_0(X)$ *separates ideals in* $C_0(X) \rtimes_{\mathrm{red}} \Gamma$*. This happens if* Γ *is exact and* Γ *acts residually topologically freely on* X *(Sierakowski).*

(∗∗) *holds if* X *is second countable.*

- The Γ-space $\beta\Gamma$ is universal for all compact Γ-spaces with a dense orbit: If X is a compact Γ-space with a dense orbit, then there is a continuous surjective Γ-map $\beta\Gamma \to X$.

- Every *minimal* closed invariant subset of $\beta\Gamma$ is universal for all minimal compact Γ-spaces.

Thm (Ellis, Gutman–Li): Any two minimal closed invariant subsets of $\beta\Gamma$ are isomorphic.

They show that any such Γ-space X is *coalescent*, i.e., any surjective continuous Γ-map $X \to X$ is injective.

Questions:

- Do the locally compact Γ-spaces X_A, $A \subseteq \Gamma$, have a similar universal property for *locally compact* Γ-spaces?

- Are any two minimal closed invariant subsets of X_A isomorphic?

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Universal property of the Γ-space X_A

Definition

Let Γ be a group and let A be a non-empty subset of Γ. Let $\mathscr{C}_\Gamma(A)$ be the class of locally compact Hausdorff free Γ-spaces X with a base point $x_0 \in X$ such that

(a) $\Gamma.x_0$ is dense in X,

(b) $K := \overline{A.x_0}$ is a compact subset of X,

(c) $\bigcup_{g\in\Gamma} g.K^{o} = X$, where K^{o} denote the interior of K,

(d) $\{g \in \Gamma : g.x_0 \in K\}$ is A-bounded.

- $(X_A, 1)$ is *universal* for $\mathscr{C}_\Gamma(A)$: If $(X, x_0) \in \mathscr{C}_\Gamma(A)$, then $\exists! \, f : X_A \to X$ continuous proper surjective Γ-map st $f(1) = x_0$.
- If X is a locally cpt Hausdorff free co-compact Γ-space with dense orbit $\Gamma.x_0$, then $(X, x_0) \in \mathscr{C}_\Gamma(A)$ for some $A \subseteq \Gamma$.

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- Any minimal closed invariant subset Z of X_A is totally disconnected. What more do we know about such Z?
  - Is Z always non-compact if $X_A \neq \beta\Gamma$?
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All questions above have negative answer!

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Absorbing sets

- A subset $A \subseteq \Gamma$ is *absorbing* if $\bigcap_{g \in F} gA \neq \emptyset$ for all finite $F \subseteq \Gamma$.
  - Equivalently, A is absorbing iff $\forall$ finite $F \subseteq \Gamma$ $\exists g \in \Gamma : Fg \subseteq A$.
- **Def:** $A \approx B$ if A is B-bounded and B is A-bounded, i.e., $A \subseteq FB$ and $B \subseteq F'A$ for some finite $F, F' \subseteq \Gamma$.

Proposition

Let $A \subseteq \Gamma$. Then some minimal closed invariant subset Z of X_A is compact iff $A \approx B$ for some absorbing set $B \subseteq \Gamma$.

Example

$A = \mathbb{N} \subseteq \mathbb{Z} = \Gamma$ is absorbing, but $\mathbb{N} \not\approx \mathbb{Z}$, so $X_{\mathbb{N}}$ is non-compact (and $X_{\mathbb{N}} \neq \beta\mathbb{Z}$).
$X_{\mathbb{N}}$ has at least one minimal closed invariant subset which is compact.

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- Let $A \subseteq \Gamma$. The corner $1_A(\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma)1_A$ has a character $\iff \nexists A', A'' \subseteq A$ st $A' \cap A'' = \emptyset$ and $A' \approx A'' \approx A$.

- There are several infinite sets $A \subseteq \Gamma$ satisfying the above conditions! Eg. $\Gamma = \mathbb{Z}$ and $A = \{k^2 : k \in \mathbb{N}\}$.

- A subset $A \subseteq \Gamma$ is *infinitely divisible* if there exists a sequence $\{A_n\}_{n=1}^\infty$ of pairwise disjoint subsets of A st $A_n \approx A$ for all n (i.e., A is A_n-bounded for all n)

- Equivalently, A is infinitely divisible if for each n there exist pairwise disjoint subsets $A_1, \dots, A_n$ of A st $A_k \approx A$ for $k = 1, 2, \dots, n$.

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If $A \subseteq \Gamma$ is absorbing and not infinitely divisible, then X_A contains minimal closed invariant subsets Z_1 and Z_2 where Z_1 is discrete (i.e., isomophic to Γ) and Z_2 is compact. In particular, Z_1 and Z_2 are not isomorphic.

Example: $\Gamma = \mathbb{Z}$ and $A = -\mathbb{N} \cup \{k^2 : k \in \mathbb{N}\}$.

Note: $(\ell^\infty(\mathbb{Z}) \rtimes \mathbb{Z})/(c_0(\mathbb{Z}) \rtimes \mathbb{Z}) \cong \mathcal{A}_1 \oplus \mathcal{A}_2$ (its center is isomorphic to $\mathbb{C} \oplus \mathbb{C}$).

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Theorem: Every countable infinite group contains a subset which is infinitely divisible and not equivalent to an absorbing set.

- In every countable infinite group G there is $A \subseteq \Gamma$ st every minimal closed invariant subset of X_A is non-compact and non-discrete.

Theorem: Every countable infinite group Γ admits a free minimal action on the non-compact locally compact Cantor set. If Γ is exact, then the action can be chosen to be amenable.

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Remarks on possible applications to general actions

Suppose that X is a locally compact Hausdorff space Γ is a countable group acting freely and co-compactly on X. Then $(X, x_0) \in \mathscr{C}_\Gamma(A)$ for some $A \subseteq \Gamma$.

Hence $\exists f : X_A \to X$ continuous proper surjective Γ-map.

- If A is non-paradoxical, then there is a non-zero invariant Radon measure on X.
- If A is equivalent to an absorbing set, then X contains a minimal closed invariant subset, which is compact.
- If A is not infinitely divisible, then X contains a minimal closed invariant subset, which is discrete.
- If A is infinitely divisible and not equivalent to an absorbing set, then all minimal closed invariant subsets of X are non-compact and non-discrete.

Theorem

Let Γ be a countable group.

(i) *If Γ contains an infinite exact subgroup, then Γ admits a free minimal amenable action on* $\mathbf{K}^*$.

(ii) *If Γ contains an element of infinite order, or if Γ contains an infinite amenable subgroup of infinite index, then Γ admits a free minimal amenable action on* $\mathbf{K}^*$ *such that* $\mathbf{K}^*$ *admits an invariant non-zero Radon measure.*

(iii) *If Γ contains an exact non-supramenable subgroup, then Γ admits a free minimal amenable purely infinite action on* $\mathbf{K}^*$.

Questions:

- Do all countable infinite groups admit a free minimal amenable action on $\mathbf{K}^*$ such that $\mathbf{K}^*$ admits an invariant non-zero Radon measure?
- Do all countable non-supramenable groups admit a free minimal amenable purely infinite action on $\mathbf{K}^*$?

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