Algebras of Multilinear Forms on Hypergroups

Bertram Schreiber
(Joint with Rupert Lasser)

Wayne State University
Detroit, MI USA

E-mail: bert@math.wayne.edu
URL: http://www.math.wayne.edu/~bert

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Outline of the Talk

1. Background
   - Hypergroups and Convolution of Measures
   - Haagerup Tensor Product Duals

2. Algebras of Multilinear Forms
   - Definition
   - Remarks on the Proof

3. Fourier Transform
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   - Fourier Transform
Let \( \omega(x, y)(f) = \int_K f(z) \, d\omega(x, y)(z) \),
\( x \to \tilde{x} \) a homeomorphism of \( K \) onto \( K \).

The triple \( K = (K, \omega, \tilde{\cdot}) \) is called a \textit{hypergroup} if:

- (associativity) For all \( x, y, z \in K \),
  \[
  \int_K \omega(x, w)(f) \, d\omega(y, z)(w) = \int_K \omega(w, z)(f) \, d\omega(x, y)(w).
  \]
- Every \( \omega(x, y) \) has compact support.
- (involution) For all \( x, y \in K \) and \( E \in \mathcal{B}(K) \), \( \tilde{\tilde{x}} = x \) and
  \[
  \omega(x, y)(\tilde{E}) = \omega(\tilde{y}, \tilde{x})(E).
  \]
- (identity) There exists (unique) \( e \in K \) s.t.
  \[
  \omega(e, x)(f) = \omega(x, e)(f) = f(x), \quad f \in C_0(K).
  \]
- The element \( e \) is in \( \text{supp} \, \omega(x, \tilde{y}) \) if and only if \( x = y \).
- (continuity) The mapping \( (x, y) \to \text{supp} \, \omega(x, y) \) is continuous w.r.t.
  the Michael topology.
For $\mu, \nu \in M(K)$, let
\[
\int_K f \, d\mu \ast \nu = \int_{K \times K} \omega(x, y)(f) \, d(\mu \times \nu), \quad \tilde{\mu}(E) = \mu(\tilde{E}).
\]
Then $M(K)$ becomes a Banach $*$-algebra.

Let $L_x f(y) = \omega(x, y)(f) = R_y f(x)$ and
\[
L_\mu f(x) = \mu \ast f(x) = \int_K R_x f \, d\tilde{\mu}, \quad R_\mu f(x) = f \ast \mu(x) = \int_K L_x f \, d\tilde{\mu}.
\]
Then
\[
\int_K f \, d\mu \ast \nu = \int_K \tilde{\mu} \ast f \, d\nu = \int_K f \ast \tilde{\nu} \, d\mu.
\]
$K_1, \ldots, K_n$ hypergroups. $K_1 \times \cdots \times K_n$ becomes a hypergroup if we set

$$\omega((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \omega(x_1, y_1) \times \cdots \times \omega(x_n, y_n),$$

$$(x_1, \ldots, x_n) \sim = (\tilde{x}_1, \ldots, \tilde{x}_n).$$

Let

$$CB(K_1, \ldots, K_n) = [C_0(K_1) \otimes_h \cdots \otimes_h C_0(K_n)]^*,$$

the space of completely bounded multilinear forms on $C_0(K_1) \times \cdots \times C_0(K_n)$.

Natural embedding $M(K_1 \times \cdots \times K_n) \subset CB(K_1, \ldots, K_n)$:

$$u_\mu(f_1 \otimes \cdots \otimes f_n) = \int_{K_1 \times \cdots \times K_n} f_1(x_1) \cdots f_n(x_n) \, d\mu(x_1, \ldots, x_n)$$
Review: Haagerup Tensor Product Duals

$n = 3$ for simplicity. $X_1, X_2, X_3; Y_1, Y_2, Y_3$ operator spaces

**Theorem (CSPS)**

\[ u \in (X_1 \otimes_h X_2 \otimes_h X_3)^* \text{ if and only if there exists a Hilb. sp. } H, \text{ for } j = 1, 2, 3 \text{ complete isomorphisms of } X_j \text{ in } B(H), \text{ C*-subalgebras } A_j \text{ of } B(H) \text{ containing } X_j, \text{ *-rep’ns } \pi_j : A_j \rightarrow B(H), \text{ and } \xi, \eta \in H \text{ s.t.} \]

\[ \|u\| = \|\xi\| \|\eta\| \text{ and} \]

\[ u(x_1, x_2, x_3) = \langle \pi_1(x_1)\pi_2(x_2)\pi_3(x_3)\xi, \eta \rangle, \quad x_j \in X_j, \quad j = 1, 2, 3. \]

Replacing $\pi_j$ by $\pi_j^{**}$, $j = 1, 2, 3$, we extend such a $u$ to multilinear forms on $X_j^{**}, \ j = 1, 2, 3.$
If $u \in (X_1 \otimes_h X_2 \otimes_h X_3)^*$ and $v \in (Y_1 \otimes_h Y_2 \otimes_h Y_3)^*$, then $\exists$!

$$u \otimes v \in [(A_1 \otimes B_1) \otimes_h (A_2 \otimes B_2) \otimes_h (A_3 \otimes B_3)]^*$$

s.t. $\|u \otimes v\| \leq \|u\| \|v\|$ and

$$u \otimes v((x_1 \otimes y_1) \otimes (x_2 \otimes y_2) \otimes (x_3 \otimes y_3)) = u(x_1 \otimes x_2 \otimes x_3) \cdot v(y_1 \otimes y_2 \otimes y_3).$$

Namely, with the obvious meaning, extend the expression

$$u \otimes v((z_1 \otimes z_2 \otimes z_3) = \langle ((\theta_1 \otimes \pi_1)(z_1)(\theta_2 \otimes \pi_2)(z_2)(\theta_3 \otimes \pi_3)(z_3)\xi \otimes \xi', \eta \otimes \eta' \rangle.$$

In particular, if $\mu, \nu \in M(K)$, then clearly

$$u_\mu \otimes u_\nu = u_{\mu \times \nu}.$$
Want to make $CB(K_1, \ldots, K_n)$ into a Banach *-algebra with an explicit formula for the product that mimics convolution of measures. Again let $n = 3$.

For $f$ locally integrable on $K$ a hypergroup, let

$$Mf(x, y) = \omega(x, y)(f), \quad \tilde{f}(x) = f(\tilde{x}), \quad f^*(x) = \overline{f(\tilde{x})}.$$ 

Then

$$Mf^*(x, y) = \overline{Mf(\tilde{y}, \tilde{x})}.$$ 

**Definition**

For $u, v \in CB(K_1, K_2, K_3)$, let

$$u \ast v(f_1 \otimes f_2 \otimes f_3) = u \otimes v(Mf_1 \otimes Mf_2 \otimes Mf_3),$$

$$u^*(f_1 \otimes f_2 \otimes f_3) = \overline{u(f_1^* \otimes f_2^* \otimes f_3^*)}.$$
Theorem

This multiplication and adjoint operation makes $CB(K_1, K_2, K_3)$ into a Banach *-algebra with isometric involution.

In particular, for $\mu, \nu \in M(K_1 \times K_2 \times K_3)$, $u_\mu \ast u_\nu = u_{\mu \ast \nu}$ and $u_{\mu^*} = u_{\mu}^*$.

Definition

For $u \in CB(K_1, K_2, K_3)$, $f \in C_0(K_1) \otimes_h C_0(K_2) \otimes_h C_0(K_3)$, and $x = (x_1, x_2, x_3) \in K_1 \times K_2 \times K_3$, define

$$u \ast f(x) = u^*(R_x f) \quad \text{and} \quad f \ast u(x) = u^*(L_x f).$$

Theorem

The functions $u \ast f$ and $f \ast u$ are in $C_b(K_1) \otimes_h C_b(K_2) \otimes_h C_b(K_3)$, and

$$u \ast v(f) = v(u^* \ast f) = u(f \ast v^*).$$
Lemma

Let \( \theta, \pi, \delta \) be representations of \( C_0(K) \) on \( H \). Then for \( f \in C_0(K) \),

\[
\left\{ \theta \otimes \left[ (\pi \otimes \delta)M \right] \right\}(Mf) = \left\{ \left[ (\theta \otimes \pi)M \right] \otimes \delta \right\}(Mf).
\]

Proof.

Define \( T_1, \ T_2 : C_b(K \times K) \to C_b(K \times K \times K) \) by

\[
T_1 h(x, y, z) = \omega(x, y)(h(\cdot, z)), \quad T_2 h(x, y, z) = \omega(y, z)(h(x, \cdot)).
\]

Then for \( f, g \in C_0(K) \),

\[
\left\{ \theta \otimes \left[ (\pi \otimes \delta)M \right] \right\}(f \otimes g) = \theta(f) \otimes (\pi \otimes \delta)(Mg) = \left[ \theta \otimes (\pi \otimes \delta) \right](T_2(f \otimes g)).
\]
Proof (Continued).

Since all our tensor products of rep’ns are weak*-to-σ-weak continuous on $C_0(K)^{**}$, for all $h \in C_b(K \times K)$,

$$\{\theta \otimes [(\pi \otimes \delta)M]\}(h) = [\theta \otimes (\pi \otimes \delta)](T_2 h).$$

A similar argument shows that

$$\{[\theta \otimes \pi)M] \otimes \delta\}(h) = [(\theta \otimes \pi) \otimes \delta](T_1 h).$$

Our lemma now follows from the identity $T_1(Mf) = T_2(Mf)$, which is a restatement of the associativity property of $K$.

Corollary

*The multiplication $\ast$ is associative.*
Since $M$ is isometric from $C_0(K)$ to $C_b(K \times K)$, the image is an operator space, and it follows that $u \ast v \in CB(K_1, K_2, K_3)$ and $\|u \ast v\| \leq \|u\| \|v\|$.

Since $x \mapsto \tilde{x}$ is an involutive homeomorphism, it is easy to see that for every $k$, the extensions to $k \times k$ matrices satisfy $\|u_k^*(F)\| = \|u_k(\tilde{F})\|$ for every $F \in M_k(C_0(K_1)) \otimes M_k(C_0(K_2)) \otimes M_k(C_0(K_3))$. (Here $\tilde{F}$ is defined by replacing every matrix entry $f_{ij}$ by $f_{ij}^*$. Hence $u^* \in CB(K_1, K_2, K_3)$ and $\|u\| = \|u^*\|$.

**Lemma**

$$(u \ast v)^* = v^* \ast u^*.$$
Proof.

\[(u \ast v)^* (f_1 \otimes f_2 \otimes f_3) = u \ast v(f_1^* \otimes f_2^* \otimes f_3^*) = u \otimes v(Mf_1^* \otimes Mf_2^* \otimes Mf_3^*).\]

and

\[v^* \ast u^*(f_1 \otimes f_2 \otimes f_3) = v^* \otimes u^*(Mf_1 \otimes Mf_2 \otimes Mf_3).\]

Now

\[v^* \otimes u^*((f_1 \otimes g_1) \otimes (f_2 \otimes g_2) \otimes (f_3 \otimes g_3)) = v^*(f_1 \otimes f_2 \otimes f_3) u^*(g_1 \otimes g_2 \otimes g_3) = u(g_1^* \otimes g_2^* \otimes g_3^*) v(f_1^* \otimes f_2^* \otimes f_3^*) = u \otimes v((g_1^* \otimes f_1^*) \otimes (g_2^* \otimes f_2^*) \otimes (g_3^* \otimes f_3^*)).\]

Passing to limits and using the formula for \(Mf^*\), we obtain our lemma. \(\square\)
For $K$ a commutative hypergroup, let

$$\mathcal{X}(K) = \{ \alpha \in C_b(K) : \alpha \neq 0, \, \omega(x, y)(\alpha) = \alpha(x)\alpha(y), \, x, y \in K \},$$

$$\hat{K} = \{ \alpha \in \mathcal{X}(K) : \alpha(\tilde{x}) = \overline{\alpha(x)}, \, x \in K \}.$$

Elements of $\mathcal{X}(K)$ are called \textit{characters} of $K$, and those of $\hat{K}$ are \textit{symmetric characters}.

It can be shown that $\hat{K}$ is the Gelfand space of the Banach algebra $L^1(K)$, defined in terms of a Haar measure on $K$ and with multiplication defined by restricting convolution of measures to this space. The Gelfand topology on $\mathcal{X}(K)$ is the compact-open topology.
For $K_1, \ldots, K_n$ commutative and $u \in CB(K_1, \ldots, K_n)$, define the Fourier transform of $u$ by

$$\hat{u}(\alpha_1, \ldots, \alpha_n) = u(\overline{\alpha_1} \otimes \ldots \otimes \overline{\alpha_n}), \quad \alpha_j \in \mathcal{X}(K_j), \ j = 1, \ldots, n.$$ 

Then this transform satisfies the usual properties:

**Theorem**

The mapping $u \rightarrow \hat{u}$ is a norm-decreasing, injective homomorphism from the algebra $CB(K_1, \ldots, K_n)$ to $C_b(\mathcal{X}(K_1) \times \cdots \times \mathcal{X}(K_n))$ which satisfies $\hat{u}^* = \overline{\hat{u}}$ on $\hat{K}_1 \times \cdots \times \hat{K}_n$.

**Proof.**

The computations are straightforward. The mapping is injective because there are sufficiently many symmetric characters to separate points of each $K$, so their span is a weak*–dense in each $C_0(K_j)^{**}$, and $u$ is separately weak* continuous.
Concluding Remarks

A more abstract version of these arguments appears in the paper E.G. Effros and Z-J Ruan, Operator space tensor products and Hopf convolution algebras, *J. Operator Theory*, **50** (2003), 131-156.
Thank you.