Algebras of Multilinear Forms on Hypergroups

Bertram Schreiber
(Joint with Rupert Lasser)

Wayne State University
Detroit, MI USA

E-mail: bert@math.wayne.edu
URL:http://www.math.wayne.edu/∼bert

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Outline of the Talk

1. Background
   - Hypergroups and Convolution of Measures
   - Haagerup Tensor Product Duals
2. Algebras of Multilinear Forms
   - Definition
   - Remarks on the Proof
3. Fourier Transform
   - Characters
   - Fourier Transform

Hypergroups and Convolution of Measures

K locally compact, $\omega : K \times K \to \mathcal{P}(K)$ weak*-cont.
Let $\omega(x,y)(f) = \int_K f(z)\, d\omega(x,y)(z)$,
$x \to \tilde{x}$ a homeomorhism of K onto K.
The triple $K = (K, \omega, \tilde{\ })$ is called a *hypergroup* if:

- (associativity) For all $x, y, z \in K$,
  $\int_K \omega(x,w)(f)\, d\omega(y,z)(w) = \int_K \omega(w,z)(f)\, d\omega(x,y)(w)$.
- Every $\omega(x,y)$ has compact support.
- (involution) For all $x, y \in K$ and $E \in \mathcal{B}(K)$, $\tilde{\tilde{x}} = x$ and
  $\omega(x,y)(\tilde{E}) = \omega(\tilde{y},\tilde{x})(E)$.
- (identity) There exists (unique) $e \in K$ s.t.
  $\omega(e,x)(f) = \omega(x,e)(f) = f(x)$, $f \in C_0(K)$.
- The element e is in $\operatorname{supp}\omega(x,\tilde{y})$ if and only if $x = y$.
- (continuity) The mapping $(x,y) \to \operatorname{supp}\omega(x,y)$ is continuous w.r.t. the Michael topology.

For $\mu, \nu \in M(K)$, let

$$\int_K f \, d\mu * \nu = \int_{K \times K} \omega(x,y)(f) \, d(\mu \times \nu), \quad \tilde{\mu}(E) = \mu(\tilde{E}).$$

Then $M(K)$ becomes a Banach *-algebra.

Let $L_x f(y) = \omega(x,y)(f) = R_y f(x)$ and

$$L_\mu f(x) = \mu * f(x) = \int_K R_x f \, d\tilde{\mu}, \quad R_\mu f(x) = f * \mu(x) = \int_K L_x f \, d\tilde{\mu}$$

Then

$$\int_K f \, d\mu * \nu = \int_K \tilde{\mu} * f \, d\nu = \int_K f * \tilde{\nu} \, d\mu.$$

$K_1, \ldots, K_n$ hypergroups. $K_1 \times \cdots \times K_n$ becomes a hypergroup if we set

$$\omega((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \omega(x_1, y_1) \times \cdots \times \omega(x_n, y_n),$$

$$(x_1, \ldots, x_n)\tilde{} = (\tilde{x}_1, \ldots, \tilde{x}_n).$$

Let

$$CB(K_1, \ldots, K_n) = [C_0(K_1) \otimes_h \cdots \otimes_h C_0(K_n)]^*,$$

the space of completely bounded multilinear forms on $C_0(K_1) \times \cdots \times C_0(K_n)$.

Natural embedding $M(K_1 \times \cdots \times K_n) \subset CB(K_1, \ldots, K_n)$:

$$u_\mu(f_1 \otimes \cdots \otimes f_n) = \int_{K_1 \times \cdots \times K_n} f_1(x_1) \cdots f_n(x_n)\, d\mu(x_1, \ldots x_n)$$

Review: Haagerup Tensor Product Duals

$n = 3$ for simplicity. $X_1, X_2, X_3; Y_1, Y_2, Y_3$ operator spaces

Theorem (CSPS)

$u \in (X_1 \otimes_h X_2 \otimes_h X_3)^*$ if and only if there exists a Hilb. sp. H, for $j = 1, 2, 3$ complete isomorphisms of X_j in $B(H)$, C-subalgebras A_j of $B(H)$ containing X_j, *-rep'ns $\pi_j : A_j \to B(H)$, and $\xi, \eta \in H$ s.t. $\|u\| = \|\xi\| \, \|\eta\|$ and*

$$u(x_1, x_2, x_3) = \langle \pi_1(x_1)\pi_2(x_2)\pi_3(x_3)\xi, \eta \rangle, \quad x_j \in X_j, \; j = 1, 2, 3.$$

Replacing π_j by π_j^{**}, $j = 1, 2, 3$, we extend such a u to multilinear forms on X_j^{**}, $j = 1, 2, 3$.

If $u \in (X_1 \otimes_h X_2 \otimes_h X_3)^*$ and $v \in (Y_1 \otimes_h Y_2 \otimes_h Y_3)^*$, then $\exists!$

$$u \otimes v \in [(A_1 \otimes B_1) \otimes_h (A_2 \otimes B_2) \otimes_h (A_3 \otimes B_3)]^*$$

s.t. $\|u \otimes v\| \leq \|u\| \, \|v\|$ and

$$u \otimes v((x_1 \otimes y_1) \otimes (x_2 \otimes y_2) \otimes (x_3 \otimes y_3)) = u(x_1 \otimes x_2 \otimes x_3)\, v(y_1 \otimes y_2 \otimes y_3).$$

Namely, with the obvious meaning, extend the expression

$$u \otimes v((z_1 \otimes z_2 \otimes z_3) = \langle (\theta_1 \otimes \pi_1)(z_1)(\theta_2 \otimes \pi_2)(z_2)(\theta_3 \otimes \pi_3)(z_3)\xi \otimes \xi', \eta \otimes \eta' \rangle.$$

In particular, if $\mu, \nu \in M(K)$, then clearly

$$u_\mu \otimes u_\nu = u_{\mu \times \nu}.$$

Algebras of Multilinear Forms

Want to make $CB(K_1, \ldots, K_n)$ into a Banach *-algebra with an explicit formula for the product that mimics convolution of measures.
Again let $n = 3$.

For f locally integrable on K a hypergroup, let

$$Mf(x,y) = \omega(x,y)(f), \quad \tilde{f}(x) = f(\tilde{x}), \quad f^*(x) = \overline{f(\tilde{x})}.$$

Then

$$Mf^*(x,y) = \overline{Mf(\tilde{y},\tilde{x})}.$$

Definition

For $u, v \in CB(K_1, K_2, K_3)$, let

$$u * v(f_1 \otimes f_2 \otimes f_3) = u \otimes v(Mf_1 \otimes Mf_2 \otimes Mf_3),$$

$$u^*(f_1 \otimes f_2 \otimes f_3) = \overline{u(f_1^* \otimes f_2^* \otimes f_3^*)}.$$

Theorem

*This multiplication and adjoint operation makes $CB(K_1, K_2, K_3)$ into a Banach *-algebra with isometric involution.*

In particular, for $\mu, \nu \in M(K_1 \times K_2 \times K_3)$, $u_\mu * u_\nu = u_{\mu * \nu}$ and $u_{\mu^*} = u_\mu^*$.

Definition

For $u \in CB(K_1, K_2, K_3)$, $f \in C_0(K_1) \otimes_h C_0(K_2) \otimes_h C_0(K_3)$, and $x = (x_1, x_2, x_3) \in K_1 \times K_2 \times K_3$, define

$$u * f(x) = u^*(R_x f) \quad \text{and} \quad f * u(x) = u^*(L_x f).$$

Theorem

The functions $u * f$ and $f * u$ are in $C_b(K_1) \otimes_h C_b(K_2) \otimes_h C_b(K_3)$, and

$$u * v(f) = v(u^* * f) = u(f * v^*).$$

Remarks on the Proof

Lemma

Let Let θ, π, δ be representations of $C_0(K)$ on H. Then for $f \in C_0(K)$,

$$\{\theta \otimes [(\pi \otimes \delta)M]\}(Mf) = \{[(\theta \otimes \pi)M] \otimes \delta\}(Mf).$$

Proof.

Define $T_1,\ T_2 : C_b(K \times K) \to C_b(K \times K \times K)$ by

$$T_1 h(x,y,z) = \omega(x,y)(h(\cdot,z)), \quad T_2 h(x,y,z) = \omega(y,z)(h(x,\cdot)).$$

Then for $f, g \in C_0(K)$,

$$\begin{aligned}\{\theta \otimes [(\pi \otimes \delta)M]\}(f \otimes g) &= \theta(f) \otimes (\pi \otimes \delta)(Mg) \\ &= [\theta \otimes (\pi \otimes \delta)](T_2(f \otimes g)).\end{aligned}$$

□

Proof (Continued).

Since all our tensor products of rep'ns are weak*-to-σ-weak continuous on $C_0(K)^{**}$, for all $h \in C_b(K \times K)$,

$$\{\theta \otimes [(\pi \otimes \delta)M]\}(h) = [\theta \otimes (\pi \otimes \delta)](T_2 h).$$

A similar argument shows that

$$\{[\theta \otimes \pi)M] \otimes \delta\}(h) = [(\theta \otimes \pi) \otimes \delta](T_1 h).$$

Our lemma now follows from the identity $T_1(Mf) = T_2(Mf)$, which is a restatement of the associativity property of K. □

Corollary

The multiplication $*$ is associative.

Since M is isometric from $C_0(K)$ to $C_b(K \times K)$, the image is an operator space, and it follows that $u * v \in CB(K_1, K_2, K_3)$ and $\|u * v\| \leq \|u\| \, \|v\|$.

Since $x \to \tilde{x}$ is an involutive homeomorphism, it is easy to see that for every k, the extensions to $k \times k$ matrices satisfy $\|u_k^*(F)\| = \|u_k(\tilde{F})\|$ for every $F \in M_k(C_0(K_1)) \otimes M_k(C_0(K_2)) \otimes M_k(C_0(K_3))$. (Here $\tilde{F}$ is defined by replacing every matrix entry f_{ij} by f_{ij}^*.) Hence $u^* \in CB(K_1, K_2, K_3)$ and $\|u\| = \|u^*\|$.

Lemma

$(u * v)^* = v^* * u^*$.

Proof.

$$\begin{aligned}(u*v)^*(f_1\otimes f_2\otimes f_3) &= \overline{u*v(f_1^*\otimes f_2^*\otimes f_3^*)}\\ &= \overline{u\otimes v(Mf_1^*\otimes Mf_2^*\otimes Mf_3^*)}.\end{aligned}$$

and

$$v^* * u^*(f_1\otimes f_2\otimes f_3) = v^*\otimes u^*(Mf_1\otimes Mf_2\otimes Mf_3).$$

Now

$$\begin{aligned}&v^*\otimes u^*((f_1\otimes g_1)\otimes(f_2\otimes g_2)\otimes(f_3\otimes g_3))\\ &\quad= v^*(f_1\otimes f_2\otimes f_3)\,u^*(g_1\otimes g_2\otimes g_3)\\ &\quad= \overline{u(g_1^*\otimes g_2^*\otimes g_3^*)}\;\overline{v(f_1^*\otimes f_2^*\otimes f_3^*)}\\ &\quad= \overline{u\otimes v((g_1^*\otimes f_1^*)\otimes(g_2^*\otimes f_2^*)\otimes(g_3^*\otimes f_3^*))}.\end{aligned}$$

Passing to limits and using the formula for Mf^*, we obtain our lemma. $\square$

Fourier Transforms

For K a commutative hypergroup, let

$$\mathcal{X}(K) = \{\alpha \in C_b(K) : \alpha \neq 0,\ \omega(x,y)(\alpha) = \alpha(x)\alpha(y),\ x, y \in K\},$$

$$\widehat{K} = \{\alpha \in \mathcal{X}(K) : \alpha(\tilde{x}) = \overline{\alpha(x)},\ x \in K\}.$$

Elements of $\mathcal{X}(K)$ are called *characters* of K, and those of $\widehat{K}$ are *symmetric characters*.

It can be shown that $\widehat{K}$ is the Gelfand space of the Banach algebra $L^!(K)$, defined in terms of a Haar measure on K and with multiplication defined by restricting convolution of measures to this space. The Gelfand topology on $\mathcal{X}(K)$ is the compact-open topology.

For $K_1, \ldots, K_n$ commutative and $u \in CB(K_1, \ldots, K_n)$, define the *Fourier transform* of u by

$$\hat{u}(\alpha_1, \ldots, \alpha_n) = u(\overline{\alpha}_1 \otimes \ldots \otimes \overline{\alpha}_n), \quad \alpha_j \in \mathcal{X}(K_j),\ j = 1, \ldots, n.$$

Then this transform satisfies the usual properties:

Theorem

The mapping $u \to \hat{u}$ is a norm-decreasing, injective homomorphism from the algebra $CB(K_1, \ldots, K_n)$ to $C_b(\mathcal{X}(K_1) \times \cdots \times \mathcal{X}(K_n))$ which satisfies $\hat{u^*} = \overline{\hat{u}}$ on $\widehat{K}_1 \times \cdots \times \widehat{K}_n$.

Proof.

The computations are straight forward. The mapping is injective because there are sufficiently many symmetric characters to separate points of each K, so their span is a weak*-dense in each $C_0(K_j)^{**}$, and u is separately weak* continuous. □

Concluding Remarks

A more abstract version of these arguments appears in the paper E.G. Effros and Z-J Ruan, Operator space tensor products and Hopf convolution algebras, *J. Operator Theory*, **50** (2003), 131-156.

Thank you.