

## 91 Linearization

Note: these problems repeat the problems in *Lecture 1* from week 1. But problems 91.4 and 91.5 are not exactly the same as 90.4 and 90.5!

**91.1.** Compute the Jacobi matrix  $f'(x)$  (also denoted  $Df(x)$ ). Compute the linearization of  $f$  at  $\bar{x}$ .

$$(a) \quad f(x) = \begin{bmatrix} \sin(x_1) + \cos(x_2) \\ \cos(x_1) + \sin(x_2) \end{bmatrix}, \quad \bar{x} = 0; \quad (b) \quad f(x) = \begin{bmatrix} 1 \\ 1 + x_1 \\ 1 + x_1 e^{x_2} \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

**91.2.** Compute the gradient vector  $\nabla f(x)$  (also denoted  $f'(x) = Df(x)$ ). Compute the linearization of  $f$  at  $\bar{x}$ .

$$(a) \quad f(x) = e^{-x_1} \sin(x_2), \quad \bar{x} = 0; \quad (b) \quad f(x) = \|x\|^2, \quad x \in \mathbf{R}^3, \quad \bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

**91.3.** Compute the tangent vector  $f'(t)$ . Compute the linearization of  $f$  at  $\bar{t}$ . Illustrate with a figure.

$$(a) \quad f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \quad \bar{t} = \pi/2; \quad (b) \quad f(t) = \begin{bmatrix} t \\ 1 + t^2 \end{bmatrix}, \quad \bar{t} = 0.$$

**91.4.** (a) Write the system

$$\begin{aligned} u_1'(t) &= u_2(t)(1 - u_1(t)^2), \\ u_2'(t) &= 2 - u_1(t)u_2(t) \end{aligned}$$

in the form  $u' = f(u)$ . Find the stationary points, i.e., solve the equation  $f(u) = 0$  by hand calculation.

(b) Compute the Jacobi matrix  $DF(u)$ . Linearize the system at each stationary point  $\bar{u}$ , i.e., write down the linearized system  $v' = Df(\bar{u})v$ . Solve this system.

(c) Perform the first step of Newton's method for the equation  $f(u) = 0$  with initial vector  $u^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

**91.5.** (a) Write the system

$$\begin{aligned} u_1'(t) &= u_1(t)(1 - u_2(t)), \\ u_2'(t) &= u_2(t)(1 - u_1(t)), \end{aligned}$$

in the form  $u' = f(u)$ . Find the stationary points, i.e., solve the equation  $f(u) = 0$  by hand calculation.

(b) Compute the Jacobi matrix  $DF(u)$ . Linearize the system at each stationary point  $\bar{u}$ , i.e., write down the linearized system  $v' = Df(\bar{u})v$ . Solve this system.

(c) Perform the first step of Newton's method for the equation  $f(u) = 0$  with initial vector  $u^{(0)} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ .

## Answers and solutions

**91.1.**

(a)

$$f'(x) = \begin{bmatrix} \cos(x_1) & -\sin(x_2) \\ -\sin(x_1) & \cos(x_2) \end{bmatrix}, \quad g(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(b)

$$f'(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ e^{x_2} & x_1 e^{x_2} \end{bmatrix}, \quad g(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) = \begin{bmatrix} 1 \\ 2 \\ 1 + e \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ e & e \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}.$$

**91.2.**

(a)

$$\begin{aligned} \nabla f(x) &= [-e^{-x_1} \sin(x_2), \quad e^{-x_1} \cos(x_2)], \\ g(x) &= f(\bar{x}) + f'(\bar{x})(x - \bar{x}) = 0 + [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2. \end{aligned}$$

(b)

$$\begin{aligned} \nabla f(x) &= [2x_1 \quad 2x_3 \quad 2x_3], \\ g(x) &= f(\bar{x}) + f'(\bar{x})(x - \bar{x}) = 3 + [2 \quad 2 \quad 2] \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \\ x_3 - 1 \end{bmatrix} = -3 + 2x_1 + 2x_2 + 2x_3. \end{aligned}$$

**91.3.**

(a)

$$\begin{aligned} f'(t) &= \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}, \\ g(t) &= f(\bar{t}) + f'(\bar{t})(t - \bar{t}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} (t - \pi/2). \end{aligned}$$

(b)

$$\begin{aligned} f'(t) &= \begin{bmatrix} 1 \\ 2t \end{bmatrix}, \\ g(t) &= f(\bar{t}) + f'(\bar{t})(t - \bar{t}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t = \begin{bmatrix} t \\ 1 \end{bmatrix}. \end{aligned}$$

**91.4.** (a) The stationary points are given by the equation  $f(u) = 0$ , i.e.,

$$f(u) = \begin{bmatrix} u_2(1 - u_1^2) \\ 2 - u_1 u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find two solutions  $\bar{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\bar{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ .

(b) The Jacobian is

$$Df(u) = \begin{bmatrix} -2u_1 u_2 & 1 - u_1^2 \\ -u_2 & -u_1 \end{bmatrix}.$$

Let  $u(t) = \bar{u} + v(t)$ . Linearization  $f(\bar{u} + v) \approx f(\bar{u}) + Df(\bar{u})v = Df(\bar{u})v$  at  $\bar{u}$  gives the following equation for the perturbation  $v(t)$

$$v' = (u - \bar{u})' = u' = f(u) = f(\bar{u} + v) = Df(\bar{u})v.$$

Linearization at  $\bar{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  leads to the linearized system

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}.$$

Its solution is

$$\begin{aligned} v(t) &= c_1 e^{\lambda_1 t} g_1 + c_2 e^{\lambda_2 t} g_2 \\ &= c_1 e^{-4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Linearization at  $\bar{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$  gives

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}.$$

Its solution is

$$\begin{aligned} v(t) &= c_1 e^{\lambda_1 t} g_1 + c_2 e^{\lambda_2 t} g_2 \\ &= c_1 e^{-4t} \begin{bmatrix} 5 \\ -2 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

(c) The first step of Newton's method:

evaluate  $A = Df(1, 1) = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix}$  and  $b = -f(1, 1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

solve  $Ah = b$ ,  $\begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\begin{cases} -2h_1 = 0, \\ -h_1 - h_2 = -1, \end{cases} \quad h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

update  $u^{(1)} = u^{(0)} + h = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \bar{u}$

bingo!

**91.5.** (a) The stationary points are given by the equation  $f(u) = 0$ , i.e.,

$$f(u) = \begin{bmatrix} u_1(1 - u_2) \\ u_2(1 - u_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find two solutions  $\bar{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  and  $\bar{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

(b) The Jacobian is

$$Df(u) = \begin{bmatrix} 1 - u_2 & -u_1 \\ -u_2 & 1 - u_1 \end{bmatrix}.$$

Let  $u(t) = \bar{u} + v(t)$ . Linearization  $f(\bar{u} + v) \approx f(\bar{u}) + Df(\bar{u})v = Df(\bar{u})v$  at  $\bar{u}$  leads to the linearized equation for the perturbation  $v(t)$ :

$$v' = (u - \bar{u})' = u' = f(u) = f(\bar{u} + v) = Df(\bar{u})v.$$

Linearization at  $\bar{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  gives

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}.$$

Its solution is

$$\begin{aligned} v(t) &= c_1 e^{\lambda_1 t} g_1 + c_2 e^{\lambda_1 t} g_2 \\ &= c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \end{aligned}$$

Linearization at  $\bar{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  gives

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}.$$

Its solution is

$$\begin{aligned} v(t) &= c_1 e^{\lambda_1 t} g_1 + c_2 e^{\lambda_1 t} g_2 \\ &= c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

(c) The first step of Newton's method:

evaluate  $A = Df(2, 2) = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$  and  $b = -f(2, 2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

solve  $Ah = b$ ,  $\begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ,

$$\begin{cases} -h_1 - 2h_2 = 2, \\ -2h_1 - h_2 = 2, \end{cases} \quad h = \begin{bmatrix} -2/3 \\ -2/3 \end{bmatrix}$$

update  $u^{(1)} = u^{(0)} + h = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4/3 \end{bmatrix}$

getting closer to  $\bar{u}$ !