

92 Analytical computation of integrals

92.1. You must be familiar with the following elementary primitive functions. Prove them or find them in the book. Here $F'(x) = f(x)$.

$$(92.1) \quad \begin{array}{ll} f(x) & F(x) \\ x^a & \frac{x^{a+1}}{a+1} \quad (a \neq -1) \end{array}$$

$$(92.2) \quad \begin{array}{ll} x^{-1} & \log(|x|) \end{array}$$

$$(92.3) \quad \begin{array}{ll} \frac{h'(x)}{h(x)} & \log(|h(x)|) \end{array}$$

$$(92.4) \quad \begin{array}{ll} e^x & e^x \end{array}$$

$$(92.5) \quad \begin{array}{ll} a^x & \frac{a^x}{\log(a)} \quad (a \neq 1) \end{array}$$

$$(92.6) \quad \begin{array}{ll} \sin(x) & -\cos(x) \end{array}$$

$$(92.7) \quad \begin{array}{ll} \cos(x) & \sin(x) \end{array}$$

$$(92.8) \quad \begin{array}{ll} \frac{1}{\cos^2(x)} & \tan(x) \end{array}$$

$$(92.9) \quad \begin{array}{ll} \frac{1}{\sin^2(x)} & -\cot(x) \end{array}$$

$$(92.10) \quad \begin{array}{ll} \frac{1}{\sqrt{a^2 - x^2}} & \arcsin\left(\frac{x}{a}\right) \end{array}$$

$$(92.11) \quad \begin{array}{ll} \frac{1}{\sqrt{x^2 + a}} & \log(|x + \sqrt{x^2 + a}|) \end{array}$$

$$(92.12) \quad \begin{array}{ll} \frac{1}{a^2 + x^2} & \frac{1}{a} \arctan\left(\frac{x}{a}\right) \end{array}$$

92.2. Compute (hint: integrate by parts)

$$(a) \int_1^x \log(y) dy \quad (b) \int_1^x y \log(y) dy \quad (c) \int_1^x y^2 \log(y) dy$$

92.3. Compute (hint: integrate by parts)

$$(a) \int_0^x t \sin t dt \quad (b) \int_0^x t^2 \cos t dt \quad (c) \int_0^x t^2 e^t dt \quad (d) \int_0^x e^t \cos t dt$$

92.4. Compute $F(x) = \int_0^x f(t) dt$ with $f(t) = t$ for $t \in [0, 1]$, $f(t) = -1$ for $t \in [1, 2]$. Draw the graphs of f and F . Compute and plot F with MATLAB.

92.5. Compute the following. For which values of t are the integrals defined? (hint for (c): partial fractions)

$$(a) \int_0^t \frac{1}{x+3} dx \quad (b) \int_0^t \frac{1}{x-5} dx \quad (c) \int_0^t \frac{1}{x^2 - 2x - 15} dx$$

92.6. Compute (hint: change of variable)

- (a) $\int_0^2 x e^{-x^2} dx$
- (b) $\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x} dx$ (hint: $t = \sin x$)
- (c) $\int_1^2 x \sqrt{x-1} dx$ (hint: $t = \sqrt{x-1}$)
- (d) $\int_0^y \cos^3 x dx$ (hint: $t = \sin x$)

92.7. A function $f : [-a, a] \rightarrow \mathbf{R}$ is called *even* if $f(-x) = f(x)$ for all $x \in [-a, a]$, and *odd* if $f(-x) = -f(x)$ for all $x \in [-a, a]$. (a) Find examples of even and odd functions. Sketch their graphs. (b) Show that

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad \text{if } f \text{ is even,} \quad \int_{-a}^a f(x) dx = 0 \quad \text{if } f \text{ is odd.}$$

92.8. Use the result in Problem 92.7 to compute

- (a) $\int_{-\pi}^{\pi} |x| \cos x dx$ (b) $\int_{-\pi}^{\pi} \sin^2 x dx$
- (c) $\int_{-\pi}^{\pi} x \sin^2 x dx$ (d) $\int_{-\pi}^{\pi} \arctan(x + 3x^3) dx$

92.9. (Advanced.) (Hyperbolic functions, AMBS 32.5) Show that

- (a) $\operatorname{arsinh}(y) = \log(y + \sqrt{y^2 + 1})$ for $-\infty < y < \infty$
- (b) $\operatorname{arcosh}(y) = \log(y + \sqrt{y^2 - 1})$ for $y \geq 1$
- (c) $\cosh^2(x) - \sinh^2(x) = 1$
- (d) $D \operatorname{arsinh}(y) = \frac{1}{\sqrt{y^2 + 1}}$
- (e) $D \operatorname{arcosh}(y) = \frac{1}{\sqrt{y^2 - 1}}$ for $y > 1$
- (f) $\int_0^x \frac{dy}{\sqrt{y^2 + 1}} = \log(x + \sqrt{x^2 + 1})$
- (g) $\int_1^x \frac{dy}{\sqrt{y^2 - 1}} = \log(x + \sqrt{x^2 - 1})$ for $x > 1$

Answers and solutions

92.2.

- (a) $\int_1^x \log(y) dy = \int_1^x 1 \log(y) dy = [y \log(y)]_1^x - \int_1^x y \frac{1}{y} dy = x \log(x) - x + 1.$
- (b) $\frac{x^2}{2} \log(x) - \frac{x^2}{4} + \frac{1}{4}.$
- (c) $\frac{x^3}{3} \log(x) - \frac{x^3}{9} + \frac{1}{9}.$

92.3.

- (a) $-x \cos x + \sin x$.
 (b) $(x^2 - 2) \sin x + 2x \cos x$.
 (c) $(x^2 - 2x + 2)e^x - 2$.
 (d) $\frac{1}{2}(e^x \cos x + e^x \sin x - 1)$.

92.4.

$$F(x) = \begin{cases} \int_0^x t \, dt = \left[\frac{t^2}{2} \right]_0^x = \frac{x^2}{2} & \text{for } x \in [0, 1] \\ \int_0^1 t \, dt + \int_1^x (-1) \, dt = \left[\frac{t^2}{2} \right]_0^1 + [-t]_1^x = \frac{1}{2} + 1 - x & \text{for } x \in [1, 2] \end{cases}$$

MATLAB function file:

```
function y=funk1(x,u)
if x<1
y=x;
else
y=-1;
end
```

MATLAB command:

```
>> [x,U]=my_ode('funk1',[0 2],0,1e-2); plot(x,U)
```

92.5.

(a)

$$\int_0^t \frac{1}{x+3} \, dx = \left[\log(x+3) \right]_0^t = \log(t+3) - \log(3) = \log\left(\frac{t+3}{3}\right), \quad \text{for } t > -3$$

(b)

$$\begin{aligned} \int_0^t \frac{1}{x-5} \, dx &= - \int_0^t \frac{1}{5-x} \, dx = \left\{ \begin{array}{l} z = 5-x \\ dz = -dx \end{array} \right\} = \int_5^{5-t} \frac{1}{z} \, dz \\ &= \left[\log(z) \right]_5^{5-t} = \log(5-t) - \log(5) = \log\left(\frac{5-t}{5}\right), \quad \text{for } t < 5 \end{aligned}$$

(c) We split the function into *partial fractions*:

$$\frac{1}{x^2 - 2x - 15} = \frac{1}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3} = \frac{A(x+3) + B(x-5)}{(x-5)(x+3)}$$

Identification of the coefficients gives $A = 1/8$, $B = -1/8$, so that

$$\frac{1}{x^2 - 2x - 15} = \frac{1}{(x-5)(x+3)} = \frac{1/8}{x-5} - \frac{1/8}{x+3}$$

Hence, using (a) and (b), we get

$$\begin{aligned} \int_0^t \frac{1}{x^2 - 2x - 15} \, dx &= \frac{1}{8} \int_0^t \frac{1}{x-5} \, dx - \frac{1}{8} \int_0^t \frac{1}{x+3} \, dx \\ &= \frac{1}{8} \log\left(\frac{5-t}{5}\right) - \frac{1}{8} \log\left(\frac{t+3}{3}\right) = \frac{1}{8} \log\left(\frac{3(5-t)}{5(t+3)}\right), \quad \text{for } -3 < t < 5 \end{aligned}$$

92.6.

(a) $\int_0^2 x e^{-x^2} dx = \left\{ t = x^2, dt = 2x dx \right\} = \frac{1}{2} \int_0^4 e^{-t} dt = \frac{1}{2}(1 - e^{-4}).$

(b) $\pi/4.$

(c) $\frac{2}{5} + \frac{2}{3}.$

(d) $\sin y - \frac{1}{3} \sin^3 y.$

92.7.

(a) even: $1, x^2, \cos x$, odd: $x, x^3, \sin x$

92.8.

(a) $2 \int_0^\pi x \cos x dx = -4.$ (b) $\pi.$ (c) $0.$ (d) $0.$

92.9.

(a) Hint: in the equation $\frac{1}{2}(e^x - e^{-x}) = y$ we set $z = e^x$ and find $z^2 - 2yz - 1 = 0.$