91. Linearization and stability

Note: these problems repeat some of the problems in **90. Linearization and Newton's method**. But problems 91.4 and 91.5 are not exactly the same as 90.4 and 90.5!

91.1. Compute the Jacobi matrix $f'(x)$ (also denoted $Df(x)$). Compute the linearization of f at $\bar{x}$.

$$\text{(a)}\quad f(x) = \begin{bmatrix} \sin(x_1) + \cos(x_2) \\ \cos(x_1) + \sin(x_2) \end{bmatrix}, \quad \bar{x} = 0; \qquad \text{(b)}\quad f(x) = \begin{bmatrix} 1 \\ 1 + x_1 \\ 1 + x_1 e^{x_2} \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

91.2. Compute the gradient vector $\nabla f(x)$ (also denoted $f'(x) = Df(x)$). Compute the linearization of f at $\bar{x}$.

$$\text{(a)}\quad f(x) = e^{-x_1} \sin(x_2), \quad \bar{x} = 0; \qquad \text{(b)}\quad f(x) = \|x\|^2, \quad x \in \mathbf{R}^3, \quad \bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

91.3. Compute the tangent vector $f'(t)$. Compute the linearization of f at $\bar{t}$. Illustrate with a figure.

$$\text{(a)}\quad f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \quad \bar{t} = \pi/2; \qquad \text{(b)}\quad f(t) = \begin{bmatrix} t \\ 1 + t^2 \end{bmatrix}, \quad \bar{t} = 0.$$

91.4. (a) Write the system

$$\begin{aligned} u_1'(t) &= u_2(t)\big(1 - u_1(t)^2\big), \\ u_2'(t) &= 2 - u_1(t)u_2(t) \end{aligned}$$

in the form $u' = f(u)$. Find the stationary points, i.e., solve the equation $f(u) = 0$ by hand calculation.

(b) Compute the Jacobi matrix $DF(u)$. Linearize the system at each stationary point $\bar{u}$, i.e., write down the linearized system $v' = Df(\bar{u})v$.

(c) Solve this system analytically. Is $\bar{u}$ stable?

(d) Solve both the nonlinear system $u' = f(u)$ and the linearized systems $v' = Df(\bar{u})v$ in Matlab with your program `my_ode`. Plot the solutions $u(t)$ and $\bar{u} + v(t)$ in the same figure. Remember that we should have $u(t) \approx \bar{u} + v(t)$ as long as the perturbation $v(t)$ is small.

91.5. (a) Write the system

$$\begin{aligned} u_1'(t) &= u_1(t)\big(1 - u_2(t)\big), \\ u_2'(t) &= u_2(t)\big(1 - u_1(t)\big), \end{aligned}$$

in the form $u' = f(u)$. Find the stationary points, i.e., solve the equation $f(u) = 0$ by hand calculation.

(b) Compute the Jacobi matrix $DF(u)$. Linearize the system at each stationary point $\bar{u}$, i.e., write down the linearized system $v' = Df(\bar{u})v$.

(c)Solve this system analytically. Is $\bar{u}$ stable?

(d) Solve both the nonlinear system $u' = f(u)$ and the linearized systems $v' = Df(\bar{u})v$ in Matlab with your program `my_ode`. Plot the solutions $u(t)$ and $\bar{u} + v(t)$ in the same figure. Remember that we should have $u(t) \approx \bar{u} + v(t)$ as long as the perturbation $v(t)$ is small.

Answers and solutions

91.1.

(a)

$$f'(x) = \begin{bmatrix} \cos(x_1) & -\sin(x_2) \\ -\sin(x_1) & \cos(x_2) \end{bmatrix}, \qquad g(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(b)

$$f'(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ e^{x_2} & x_1 e^{x_2} \end{bmatrix}, \qquad g(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) = \begin{bmatrix} 1 \\ 2 \\ 1+e \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ e & e \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}.$$

91.2.

(a)

$$\begin{aligned} \nabla f(x) &= \begin{bmatrix} -e^{-x_1}\sin(x_2), & e^{-x_1}\cos(x_2) \end{bmatrix}, \\ g(x) &= f(\bar{x}) + f'(\bar{x})(x - \bar{x}) = 0 + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2. \end{aligned}$$

(b)

$$\begin{aligned} \nabla f(x) &= \begin{bmatrix} 2x_1 & 2x_3 & 2x_3 \end{bmatrix}, \\ g(x) &= f(\bar{x}) + f'(\bar{x})(x - \bar{x}) = 3 + \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \\ x_3 - 1 \end{bmatrix} = -3 + 2x_1 + 2x_2 + 2x_3. \end{aligned}$$

91.3.

(a)

$$\begin{aligned} f'(t) &= \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}, \\ g(t) &= f(\bar{t}) + f'(\bar{t})(t - \bar{t}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} (t - \pi/2). \end{aligned}$$

(b)

$$\begin{aligned} f'(t) &= \begin{bmatrix} 1 \\ 2t \end{bmatrix}, \\ g(t) &= f(\bar{t}) + f'(\bar{t})(t - \bar{t}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t = \begin{bmatrix} t \\ 1 \end{bmatrix}. \end{aligned}$$

91.4. (a) The stationary points are given by the equation $f(u) = 0$, i.e.,

$$f(u) = \begin{bmatrix} u_2(1 - u_1^2) \\ 2 - u_1 u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find two solutions $\bar{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\bar{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

(b,c) The Jacobian is

$$Df(u) = \begin{bmatrix} -2u_1u_2 & 1-u_1^2 \\ -u_2 & -u_1 \end{bmatrix}.$$

Let $u(t) \approx \bar{u} + v(t)$. Linearization $f(\bar{u}+v) \approx f(\bar{u}) + Df(\bar{u})v = Df(\bar{u})v$ at $\bar{u}$ gives the following equation for the perturbation $v(t)$

$$v' = (u - \bar{u})' = u' = f(u) = f(\bar{u}+v) = Df(\bar{u})v.$$

Linearization at $\bar{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ leads to the linearized system

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}.$$

Its solution is

$$\begin{aligned} v(t) &= c_1 e^{\lambda_1 t} g_1 + c_2 e^{\lambda_1 t} g_2 \\ &= c_1 e^{-4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

$\bar{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is *stable.*

Linearization at $\bar{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ gives

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}.$$

Its solution is

$$\begin{aligned} v(t) &= c_1 e^{\lambda_1 t} g_1 + c_2 e^{\lambda_1 t} g_2 \\ &= c_1 e^{-4t} \begin{bmatrix} 5 \\ -2 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

The stationary point $\bar{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ is *unstable.*

91.5. (a) The stationary points are given by the equation $f(u) = 0$, i.e.,

$$f(u) = \begin{bmatrix} u_1(1-u_2) \\ u_2(1-u_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find two solutions $\bar{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\bar{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b,c) The Jacobian is

$$Df(u) = \begin{bmatrix} 1-u_2 & -u_1 \\ -u_2 & 1-u_1 \end{bmatrix}.$$

Let $u(t) \approx \bar{u} + v(t)$. Linearization $f(\bar{u}+v) \approx f(\bar{u}) + Df(\bar{u})v = Df(\bar{u})v$ at $\bar{u}$ leads to the linearized equation for the perturbation $v(t)$:

$$v' = (u - \bar{u})' = u' = f(u) = f(\bar{u}+v) = Df(\bar{u})v.$$

Linearization at $\bar{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}.$$

Its solution is

$$\begin{aligned} v(t) &= c_1 e^{\lambda_1 t} g_1 + c_2 e^{\lambda_1 t} g_2 \\ &= c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \end{aligned}$$

The stationary point $\bar{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is *unstable.*

Linearization at $\bar{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ gives

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}.$$

Its solution is

$$\begin{aligned} v(t) &= c_1 e^{\lambda_1 t} g_1 + c_2 e^{\lambda_1 t} g_2 \\ &= c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

The stationary point $\bar{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is *unstable.*

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