97 The exponential function with a complex variable

1. The complex exponential function

AMBS 33.2. If $z = x + \mathrm{i}y \in \mathbf{C}$ with $x, y \in \mathbf{R}$, then we define

$$\exp(z) = \mathrm{e}^z = \mathrm{e}^x\big(\cos(y) + \mathrm{i}\sin(y)\big). \tag{97.1}$$

In particular, with $x = 0$, so that $z = \mathrm{i}y$ is an imaginary number, we have

$$\exp(\mathrm{i}y) = \mathrm{e}^{\mathrm{i}y} = \cos(y) + \mathrm{i}\sin(y). \tag{97.2}$$

Note that

$$\exp(-\mathrm{i}y) = \mathrm{e}^{-\mathrm{i}y} = \cos(y) + \mathrm{i}\sin(-y) = \cos(y) - \mathrm{i}\sin(y). \tag{97.3}$$

This means that $\mathrm{e}^{-\mathrm{i}y} = \overline{\mathrm{e}^{\mathrm{i}y}}$, the complex conjugate of $\mathrm{e}^{\mathrm{i}y}$. By subtracting (and adding) (97.2) and (97.3) we get

$$\sin(y) = \frac{\mathrm{e}^{\mathrm{i}y} - \mathrm{e}^{-\mathrm{i}y}}{2\mathrm{i}}, \quad \cos(y) = \frac{\mathrm{e}^{\mathrm{i}y} + \mathrm{e}^{-\mathrm{i}y}}{2}, \quad y \in \mathbf{R}. \tag{97.4}$$

Also

$$|\mathrm{e}^{\mathrm{i}y}| = \sqrt{\cos^2(y) + \sin^2(y)} = 1, \tag{97.5}$$

which means hat $\mathrm{e}^{\mathrm{i}y}$ lies on the unit circle in the complex plane.

It is easy to show that the complex exponential function satisfies the familiar identity:

$$\mathrm{e}^z\mathrm{e}^w = \mathrm{e}^{z+w}, \quad z, w \in \mathbf{C}. \tag{97.6}$$

For example, by the trigonometric identities of AMBS 32.2:

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}y}\mathrm{e}^{\mathrm{i}x} &= \big(\cos(y) + \mathrm{i}\sin(y)\big)\big(\cos(x) + \mathrm{i}\sin(x)\big) \\
&= \cos(y)\cos(x) - \sin(y)\sin(x) + \mathrm{i}\big(\sin(y)\cos(x) + \cos(y)\sin(x)\big) \\
&= \cos(y + x) + \mathrm{i}\sin(y + x) = \mathrm{e}^{\mathrm{i}(y+x)}.
\end{aligned}$$

2. The derivative of the complex exponential function

Let

$$u(t) = \exp(tz) = \mathrm{e}^{tz} = \mathrm{e}^{t(x+\mathrm{i}y)}, \quad t, x, y \in \mathbf{R}. \tag{97.7}$$

Then

$$\begin{aligned}
u'(t) &= \frac{d}{dt}\Big(\mathrm{e}^{tx}\big(\cos(ty) + \mathrm{i}\sin(ty)\big)\Big) \\
&= x\mathrm{e}^{tx}\big(\cos(ty) + \mathrm{i}\sin(ty)\big) + \mathrm{e}^{tx}\big(-y\sin(ty) + \mathrm{i}y\cos(ty)\big) \\
&= x\mathrm{e}^{tx}\big(\cos(ty) + \mathrm{i}\sin(ty)\big) + \mathrm{e}^{tx}\mathrm{i}y\big(\mathrm{i}\sin(ty) + \cos(ty)\big) \\
&= (x + \mathrm{i}y)\mathrm{e}^{tx}\big(\cos(ty) + \mathrm{i}\sin(ty)\big) \\
&= z\exp(tz) = zu(t).
\end{aligned}$$

We conclude that

$$\frac{d}{dt}\exp(tz) = z\exp(tz) \tag{97.8}$$

just as for the real-valued exponential function.

3. Polar representation of complex numbers

When a complex number is written as

(97.9) $$z = x + \mathrm{i}y, \quad x, y \in \mathbf{R},$$

we say that it is written in *Cartesian form.* Let

(97.10) $$r = |z| = \sqrt{x^2 + y^2}$$

be the absolute value of z. Then

(97.11) $$\begin{aligned} x &= r\cos(\theta), \\ y &= r\sin(\theta), \end{aligned}$$

where θ is a real number (angle), and (97.9) becomes

(97.12) $$z = r\big(\cos(\theta) + \mathrm{i}\sin(\theta)\big) = r\mathrm{e}^{\mathrm{i}\theta}.$$

The complex number is then said to be written in *polar form:*

(97.13) $$z = r\mathrm{e}^{\mathrm{i}\theta}.$$

Here

(97.14) $$r = |z|, \quad \theta = \arg(z),$$

are the absolute value and the *argument* of z. The argument is not unique: if $\theta = \arg(z)$, then

$$\theta + n2\pi, \quad n = 0, \pm1, \pm2, \dots,$$

is also an argument for z, because $\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(\theta+n2\pi)}$.

Some calculations become easier if we use the polar form. For example, if $z = r\mathrm{e}^{\mathrm{i}\theta}$, $w = \rho\mathrm{e}^{\mathrm{i}\omega}$, then

(97.15) $$\frac{z}{w} = \frac{r\mathrm{e}^{\mathrm{i}\theta}}{\rho\mathrm{e}^{\mathrm{i}\omega}} = r\mathrm{e}^{\mathrm{i}\theta}(\rho\mathrm{e}^{\mathrm{i}\omega})^{-1} = \frac{r}{\rho}\mathrm{e}^{\mathrm{i}(\theta-\omega)}, \quad z^m = (r\mathrm{e}^{\mathrm{i}\theta})^m = r^m\mathrm{e}^{\imath m\theta}.$$

It becomes easy to solve equations of the form (binomial equation):

(97.16) $$z^n = w,$$

if we write both z and b in polar form.

Example: Solve the binomial equation

(97.17) $$z^4 = -4.$$

In polar form, with $z = r\mathrm{e}^{\mathrm{i}\theta}$, $-4 = 4\mathrm{e}^{\mathrm{i}\pi}$, this becomes

(97.18) $$r^4\mathrm{e}^{\mathrm{i}4\theta} = 4\mathrm{e}^{\mathrm{i}\pi}.$$

We identify the absolute values and the arguments,

$$r^4 = 4, \quad 4\theta = \pi + n2\pi, \quad n = 0, \pm1, \pm2, \dots,$$

so that

$$r = \sqrt{2}, \quad \theta = \frac{\pi}{4} + n\frac{\pi}{2}, \quad z = \sqrt{2}\mathrm{e}^{\mathrm{i}(\frac{\pi}{4}+n\frac{\pi}{2})}.$$

For $n = 0, 1, 2, 3$ we get four roots:

$$\begin{aligned}
z_1 &= \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} = \sqrt{2}(\cos(\tfrac{\pi}{4}) + \mathrm{i}\sin(\tfrac{\pi}{4})) = \sqrt{2}(\tfrac{1}{\sqrt{2}} + \mathrm{i}\tfrac{1}{\sqrt{2}}) = 1 + \mathrm{i}, \\
z_2 &= \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}} = \sqrt{2}(\cos(\tfrac{3\pi}{4}) + \mathrm{i}\sin(\tfrac{3\pi}{4})) = \sqrt{2}(-\tfrac{1}{\sqrt{2}} + \mathrm{i}\tfrac{1}{\sqrt{2}}) = -1 + \mathrm{i}, \\
z_3 &= \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{5\pi}{4}} = \sqrt{2}(\cos(\tfrac{5\pi}{4}) + \mathrm{i}\sin(\tfrac{5\pi}{4})) = \sqrt{2}(-\tfrac{1}{\sqrt{2}} - \mathrm{i}\tfrac{1}{\sqrt{2}}) = -1 - \mathrm{i}, \\
z_4 &= \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{7\pi}{4}} = \sqrt{2}(\cos(\tfrac{7\pi}{4}) + \mathrm{i}\sin(\tfrac{7\pi}{4})) = \sqrt{2}(\tfrac{1}{\sqrt{2}} - \mathrm{i}\tfrac{1}{\sqrt{2}}) = 1 - \mathrm{i}.
\end{aligned}$$

For other values of n these roots are repeated.

4. Problems

97.1. Show that $e^z \neq 0$.

97.2. Compute $|z|$ and $\arg(z)$ and write the following in polar form.

$$(a)\ z = 2 - 2\mathrm{i} \qquad (b)\ z = 3i \qquad (c)\ z = 1 + \mathrm{i}\sqrt{3}$$

97.3. Write the following in Cartesian form.

$$(a)\ z = 5\mathrm{e}^{\mathrm{i}\pi} \qquad (b)\ z = \mathrm{e}^{-\mathrm{i}\pi} \qquad (c)\ z = 2\mathrm{e}^{\mathrm{i}3\pi/4}$$

97.4. Solve the binomial equations.

$$(a)\ z^3 = 1 \qquad (b)\ z^2 = \mathrm{i}$$

97.5. Compute

$$\frac{3i}{1 + \mathrm{i}\sqrt{3}}$$

5. Answers and solutions

97.1. $|\mathrm{e}^z| = \mathrm{e}^x > 0$.

97.2.

(a) $|z| = 2\sqrt{2}$, $\arg(z) = \frac{7\pi}{4} + n2\pi$, $z = 2\sqrt{2}\mathrm{e}^{\mathrm{i}7\pi/4} = 2\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}$.

(b) $|z| = 3$, $\arg(z) = \frac{\pi}{2} + n2\pi$, $z = 3\mathrm{e}^{\mathrm{i}\pi/2}$.

(c) $|z| = 2$, $\arg(z) = \frac{\pi}{3} + n2\pi$, $z = 2\mathrm{e}^{\mathrm{i}\pi/3}$.

97.3.

(a) $z = -5$

(b) $z = -1$

(c) $z = -\sqrt{2} + \mathrm{i}\sqrt{2}$

97.4.

(a) $z = \mathrm{e}^{\mathrm{i}n2\pi/3}$, $z_1 = 1$, $z_2 = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$, $z_3 = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$.

(b) $z = \mathrm{e}^{\mathrm{i}(\frac{\pi}{4} + n\pi)}$, $z_1 = \frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}$, $z_2 = -\frac{1}{\sqrt{2}} - \mathrm{i}\frac{1}{\sqrt{2}}$.

97.5.

$$\frac{3\mathrm{e}^{\mathrm{i}\pi/2}}{2\mathrm{e}^{\mathrm{i}\pi/3}} = \frac{3}{2}\mathrm{e}^{\mathrm{i}\pi/6} = \frac{3}{2}\Big(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\Big)$$

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