Answers to the Problems in AMBS: Third Quarter

January 30, 2003

Chapter 54

54.1 The tangent plane to the surface $x_3 = f(x_1, x_2)$ at the point $(\bar{x}_1, \bar{x}_2)$ is given by the linearized equation $x_3 = f(\bar{x}_1, \bar{x}_2) + f'(\bar{x}_1, \bar{x}_2) \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix}$, where $f'(x_1, x_2) = \nabla f(x_1, x_2)$ is the gradient vector. We choose the point $(\bar{x}_1, \bar{x}_2) = (1, 1)$.

(a) $x_3 = 2 + \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$, i.e., $x_3 = 2x_1 + 2x_2$.

(b) $x_3 = 0 + \begin{bmatrix} 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$, i.e., $x_3 = 2x_1 - 2x_2$.

(c) $x_3 = 2 + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$, i.e., $x_3 = x_1 + 2x_2 - 1$.

(d) $x_3 = 1 + \begin{bmatrix} 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$, i.e., $x_3 = 4x_1 + 4x_2 - 7$.

54.2 (a) $f(x) = \|x\|^2 x$, $f(x) - f(y) = \|x\|^2 x - \|y\|^2 y = (\|x\|^2 - \|y\|^2)x + \|y\|^2(x - y) = \big((x + y) \cdot (x - y)\big)x + \|y\|^2(x - y)$, $\|f(x) - f(y)\| \leq \big|(x+y)\cdot(x-y)\big|\|x\| + \|y\|^2\|x-y\| \leq \|x+y\|\|x-y\|\|x\| + \|y\|^2\|x-y\| = \big(\big(\|x + y\|\big)\|x\| + \|y\|^2\big)\|x - y\| \leq \big(\big(\|x\| + \|y\|\big)\|x\| + \|y\|^2\big)\|x - y\| \leq 3\|x - y\|$, i.e., $L_f \leq 3$. If, instead, we argue as in Example 55.10, then we get $\left|\dfrac{\partial f_i(x)}{\partial x_j}\right| \leq 5$, so that $L_f \leq \sqrt{3}\sqrt{3}\,5 = 15$, which a worse estimate of L_f.

(b) $\nabla f(x) = \cos(\|x\|^2) \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \end{bmatrix} = 2\cos(\|x\|^2)x$. The mean value theorem gives

$$\begin{aligned} |f(x) - f(y)| &= |\nabla f(\hat{x}) \cdot (x - y)| = 2\cos(\|\hat{x}\|^2)|\hat{x} \cdot (x - y)| \\ &\leq 2\cos(\|\hat{x}\|^2)\|\hat{x}\|\|x - y\| \leq 2\|x - y\|, \qquad L_f = 2. \end{aligned}$$

(c) Using (b) we get

$$\begin{aligned}\|f(x)-f(y)\| &= \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(\sin(\|x\|^2-\sin(\|y\|^2))^2} \\ &\le \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+4\|x-y\|^2} \le \sqrt{5}\|x-y\|.\end{aligned}$$

(d) Not Lipschitz continuous.

(e)–(f) are too difficult and should be removed.

54.3 is too difficult and should be removed.

54.4 The linearization is $\tilde{f}(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x})$.

(a) $\tilde{f}(x) = 14 + \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1-1 \\ x_2-2 \\ x_3-3 \end{bmatrix} = 14 + 2(x_1-1) + 4(x_2-2) + 2(x_3-3)$.

(b) $\tilde{f}(x) = \sin(14) + \cos(14) \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1-1 \\ x_2-2 \\ x_3-3 \end{bmatrix}$

(c) $\tilde{f}(x) = \begin{bmatrix} 14 \\ \sin(2) \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 0 & \cos(2) & 0 \end{bmatrix} \begin{bmatrix} x_1-1 \\ x_2-2 \\ x_3-3 \end{bmatrix}$

(d) $\tilde{f}(x) = \begin{bmatrix} 14 \\ \sin(2) \\ 2\cos(3) \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 0 & \cos(2) & 0 \\ 2\cos(3) & \cos(3) & -2\sin(3) \end{bmatrix} \begin{bmatrix} x_1-1 \\ x_2-2 \\ x_3-3 \end{bmatrix}$

54.5 (a) $9x_2(2x_1^3 - x_1^2x_2 + x_2^3)$

(b) 0

54.6 $P(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{1}{2}(x-\bar{x})^t f''(\bar{x})(x-\bar{x})$.

(a)

$$\begin{aligned}P(x) &= 1 + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= 1 + \frac{1}{2}(x_1+x_2+x_3) - \frac{1}{8}(x_1^2+x_2^2+x_3^2+2x_1x_2+2x_1x_3+2x_2x_3)\end{aligned}$$

(b)

$$\begin{aligned}P(x) &= 0 + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= -x_1x_3\end{aligned}$$

(c) too difficult

(d)

$$P(x) = 1 + \begin{bmatrix}0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} + \frac{1}{2}\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}-2 & 0 & 0\\0 & -2 & 0\\0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$
$$= 1 - x_1^2 - x_2^2 - x_3^2$$

(e) too difficult

54.7 The problem should be formulated: Linearize $f\,o\,s$, where $f(x) = x_1x_2x_3$, at $t = 1$, with ...

We have $g(t) = f(s(t))$, $g'(t) = f'(s(t))s'(t)$. The linearization is $\tilde{g}(t) = g(1) + g'(1)(t-1) = f(s(1)) + f'(s(1))s'(1)(t-1)$.

(a) $\tilde{g}(t) = 1 + \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}(t-1) = 1 + 6(t-1) = 6t - 5.$

(b) $\tilde{g}(t) = \cos(1)\sin(1) + \begin{bmatrix}\sin(1) & \cos(1) & \cos(1)\sin(1)\end{bmatrix}\begin{bmatrix}-\sin(1)\\ \cos(1)\\ 1\end{bmatrix}(t - 1) = \cos(1)\sin(1) + (-\sin^2(1) + \cos^2(1) + \cos(1)\sin(1))(t-1).$

(c) $\tilde{g}(t) = 1 + \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}(t-1) = 1.$

54.8 On the one hand: $f(x) = \int_0^\infty e^{-xy}\,dy$, $f^{(n)}(x) = \int_0^\infty (-y)^n e^{-xy}\,dy$. On the other hand: $f(x) = \int_0^\infty e^{-xy}\,dy = \left[\frac{e^{-xy}}{-x}\right]_{y=0}^\infty = x^{-1}$, $f^{(n)}(x) = (-1)^n n! x^{-(n+1)}$. We conclude that $\int_0^\infty y^n e^{-xy}\,dy = n! x^{-(n+1)}$.

54.9 This should be done by means of a computer program. But we can compute one step by hand as follows. The method of steepest descent with $x^{(0)} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}$ gives

$$x^{(1)} = x^{(0)} - \alpha\nabla u(x^{(0)}) = \begin{bmatrix}1 & 1 & 1\end{bmatrix} - \alpha\begin{bmatrix}2 & 2 & 4\end{bmatrix}$$
$$= \begin{bmatrix}1-2\alpha & 1-2\alpha & 1-4\alpha\end{bmatrix}.$$

We minimize the function $f(\alpha) = u(x^{(1)}) = (1-2\alpha)^2 + (1-2\alpha)^2 + 2(1-4\alpha)^2$ by solving the equation $f'(\alpha) = 0$ and find $\alpha = 0.3$, so that $x^{(1)} = \begin{bmatrix}0.4 & 0.4 & -0.2\end{bmatrix}$. This is closer to the minimum, which is located at the origin.

54.10 —

54.11 Should be removed.

54.12 Evaluate f on the lines $x_2 = 0$, $x_1 = 0$, and $x_1 = x_2$. This gives $f(x_1, 0) = 1$, $f(0, x_2) = -1$, $f(x_1, x_1) = 0$. So f is not well-defined at the origin. Moreover,

$$\nabla f(x) = \frac{4x_1x_2}{\|x\|^4} \begin{bmatrix} x_2 & -x_1 \end{bmatrix}.$$

Hence, on the line $x_1 = x_2$ we have $\nabla f(x_1, x_1) = x_1^{-1} \begin{bmatrix} 1 & -1 \end{bmatrix}$, which is arbitrarily large near the origin. Therefore, f is not Lipschitz continuous near the origin.

Chapter 55

55.1 —

55.2 (a) $x = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$, $0 \le t < 2\pi$.

(b) the same as (a)

(c) empty (no curve)

(d) $x = \begin{bmatrix} \sqrt{3}\cos(t) \\ \sqrt{\frac{3}{2}}\sin(t) \end{bmatrix}$, $0 \le t < 2\pi$.

55.3 Using Theorem 56.3 we check that $\partial u(1,1,1)/\partial x_3 = 9 \ne 0$. Hence, the surface can be expressed as $x_3 = g(x_1, x_2)$.

55.4 (a) $\nabla f(x) = \begin{bmatrix} nx_1^{n-1}(x_2^n + x_3^n) & nx_1^n x_2^{n-1} & nx_1^n x_3^{n-1} \end{bmatrix}$

(b) $\nabla f(x) = \dfrac{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}}{\|x\|} = \dfrac{x^t}{\|x\|}$

(c) $\nabla f(x) = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \end{bmatrix} = 2x^t$

(d) $\nabla f(x) = -\dfrac{x^t}{\|x\|^3}$

(e) $\nabla f(x) = e^{x_1x_2x_3} \begin{bmatrix} x_2x_3 & x_1x_3 & x_1x_2 \end{bmatrix}$

55.5 The equation of the tangent plane is $\nabla f(\bar{x})(x - \bar{x}) = 0$.

(a) $n \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \\ x_3 - 1 \end{bmatrix} = 0$, i.e., $2x_1 + x_2 + x_3 = 4$.

(b,c,d) The same curve, namely, $\|x\| = \sqrt{3}$. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \\ x_3 - 1 \end{bmatrix} = 0$, i.e., $x_1 + x_2 + x_3 = 3$.

(e) $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \\ x_3 - 1 \end{bmatrix} = 0$, i.e., $x_1 + x_2 + x_3 = 3$.

55.6 (a) $3x_1 + \frac{3}{2}x_2 - x_3 = 3$

(b) $2x_1 + 4x_2 + 6x_3 = 28$

(c) $2\pi x_1 - x_2 = 2\pi - 2$

55.7 $x_1 + 3\sqrt{3}x_2 = 10$, $n = \begin{bmatrix} 1 & 3\sqrt{3} \end{bmatrix}$.

55.8 Assume that f is differentiable in Q. There are three cases:

(a) There is a point $\hat{x} \in Q$ such that $f(\hat{x}) > 0$. Since $f(x) = 0$ on the boundary we conclude that f attains its maximum at an *interior* point $\bar{x} \in Q$. At this point we have $\nabla f(\bar{x}) = 0$.

(b) There is a point $\hat{x} \in Q$ such that $f(\hat{x}) < 0$. Since $f(x) = 0$ on the boundary we conclude that f attains its minimum at an *interior* point $\bar{x} \in Q$. At this point we have $\nabla f(\bar{x}) = 0$.

(c) If $f(x) = 0$ for every $x \in Q$, then $\nabla f(x) = 0$ for every $x \in Q$.

Chapter 61

61.1 (a) $\nabla \cdot F = -1$ (b) $\nabla \times F = (\sin(1)e^2, -\cos(1) - 2\sin(1)e^2, \cos^2(1) - \sin^2(1) + 4)$ (c) $\nabla(\nabla \cdot F) = (0, -3, 0)$

61.2 $(\nabla \times \nabla) = 0$ and thus $(\nabla \times \nabla)u = 0$) or $(\nabla \times \nabla)u = \nabla \times (\nabla u) = 0$, for any sufficiently smooth scalar function u. But $\nabla \times \nabla \times u = \nabla \times (\nabla \times u)$ is not zero. Recall that the cross product is not associative.

61.4 $\nabla(rF(r)) = (x_1, x_2)r^{-1}F'(r)$.

61.5 $\nabla \cdot (\omega \times x) = (\nabla \times \omega) \cdot x - \omega \cdot (\nabla \times x) = 0$. By the Divergence Theorem $0 = \int_\Omega \nabla \cdot v = \int_\Gamma n \cdot v$. Thus the mass of the fluid inside any subdomain Ω is constant.

61.6 Note that for a rigid transformation we have $\nabla_{\tilde{x}} = T\nabla_x$ with T an orthogonal matrix. Then $\Delta_{\tilde{x}} = \nabla_{\tilde{x}} \cdot \nabla_{\tilde{x}} = T\nabla_x \cdot T\nabla_x = \nabla_x \cdot T^t T \nabla_x = \nabla_x \cdot \nabla_x = \Delta_x$.

61.8 $\text{rot}u = (u_{x_2}, -u_{x_1})$ and $\text{rot}(u_{x_2}, -u_{x_1}) = u_{x_2x_2} + u_{x_1x_1}$.

61.9 This follows by directly computing the Laplacian.

61.10 This follows by directly computing the Laplacian.

Chapter 63

63.1 (a) $2\sinh 1$

(b) $4\pi\sqrt{2}$

(c) 8

(d) $\frac{8}{27}(10\sqrt{10}-1)$

(e) Note! Incomplete statement of problem. "$0 \leq t < 2\pi$" should be added.
6

63.2 (a) $2\pi\sqrt{2}$

(b) $2\pi^2\sqrt{2}$

(c) $-\frac{\pi\sqrt{2}}{2}$

63.3 (a) $\frac{1}{2}$

(b) 1

(c) $\frac{\sqrt{2}}{6}$

63.4 (a) 0

(b) 0 (why?)

63.5 (a) 0 (clockwise and counter-clockwise)

(b) 2π (counter-clockwise), -2π (clockwise)

63.6 1

63.7 (a) 2

(b) $\frac{7}{3}$

(c) $3-\frac{4}{\pi}$

(d) $3-\frac{2}{n+1}$

63.8 1

63.9 $\{x=(x_1,x_2)\in\mathbb{R}^2 : |x-(0,\frac{1}{2})|=\frac{1}{2}\}$ See the plot below.

63.10 $\kappa=\frac{1}{R}$

63.11 –

63.12 (a) $\kappa(x_1)=\frac{2}{(1+4x_1^2)^{\frac{3}{2}}}$

(b) $\kappa(x_1)=\frac{6x_1}{(1+9x_1^4)^{\frac{3}{2}}}$ There is a change of signs in the curvature at the inflection point $x=(x_1,x_2)=(0,0)$. See the plot below.

63.13 Note! Misprint in statement of problem. The third sentence should read "Derive the equilibrium equation $y'(x)=\frac{1}{c}s(x)=\frac{1}{c}\int_0^x\sqrt{1+(y'(\xi))^2}\,d\xi$, with c a constant.".
We study the part of the hanging chain between 0 and x (see the plot below). Letting ρ denote the density (mass per unit length) of the chain, the gravitational force on this part of the chain is $\rho g s(x)$. Now letting ψ be the angle the tangent at $(x,y(x))$ makes with the horizontal axis,

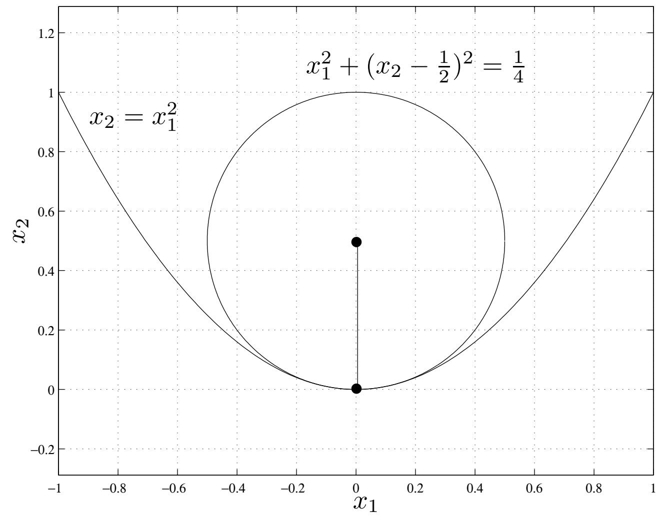

Figure 1: A plot of the circle of curvature $\{x = (x_1, x_2) \in \mathbb{R}^2 : |x - (0, \frac{1}{2})| = \frac{1}{2}\}$ of $x_2 = x_1^2$ at $x_1 = 0$. (Problem 60.9)

since the chain force is always directed along the tangent horizontal force equilibrium requires that $T(0) = T(x)\cos\psi$ and vertical force equilibrium that $\rho g s(x) = T(x)\sin\psi$. Dividing these two equilibrium equations gives $\tan\psi = y'(x) = \frac{\rho g}{T(0)} s(x) = \frac{1}{c} s(x)$, where $c = \frac{T(0)}{\rho g}$ can be thought of as a length of the chain acted upon by gravity with a force equal to the tension $T(0)$ at the lowest point of the chain.

Since the arc length between 0 and x is $s(x) = \int_0^x \sqrt{1 + (y'(\xi))^2}\, d\xi$, with $y'(x) = \sinh(\frac{x}{c})$ the identity $\cosh^2(\frac{\xi}{c}) - \sinh^2(\frac{\xi}{c}) = 1$ shows that $\frac{1}{c} s(x) = \frac{1}{c}\int_0^x \cosh(\frac{\xi}{c})\, d\xi = \sinh(\frac{x}{c}) = y'(x)$, and thus, from integration, $y(x) = c\cosh(\frac{x}{c})$ is a solution.

63.14 Note! Misprint in statement of problem. The point $(1, 1, 1)$ does not lie on the given surface. Further, there seem to be many possible such curves. If we consider the point $(1, -1, -1)$, possible directions are $\alpha(1, 0, 0) + \beta(0, 1, 3)$, where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are arbitrary.

63.15 8 (For a motivation of the formula $ds^2 = \rho^2 d\theta^2 + d\rho^2$, see the plot below. The expression for the arc length then follows from the fact that $d\rho = \rho'(\theta) d\theta$).

63.16 $L\sqrt{1 + (2\pi RN)^2}$ (where L is the length of the cylinder, R is the radius of the cylinder, and N is the number of revolutions of the string per unit length (in the axial direction) of the cylinder.)

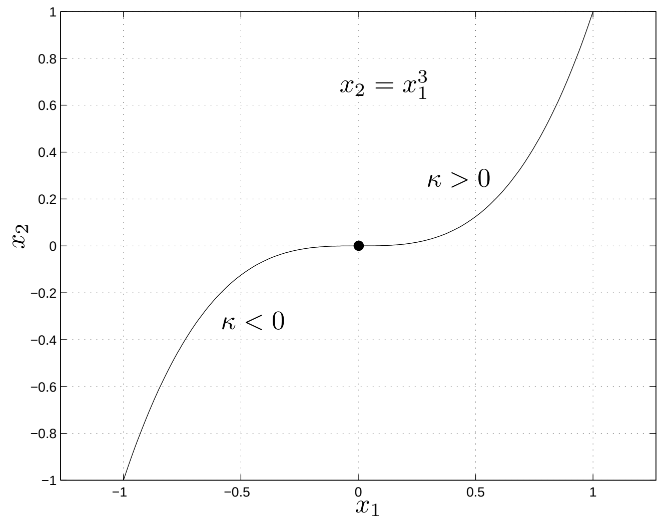

Figure 2: A plot of the function $x_2 = x_1^3$. The curvature $\kappa(x_1) = \frac{6x_1}{(1+9x_1^4)^{\frac{3}{2}}}$ changes sign at the inflection point $x = (x_1, x_2) = (0, 0)$. It is negative for $x_1 < 0$ and positive for $x_1 > 0$. (Problem 60.12 (b))

Chapter 64

64.1 (a) 1

(b) $\frac{1}{6}$

(c) $2 \ln 2$ (Use integration by parts.)

(d) $1 - \frac{1}{2!2} + \frac{1}{3!3} - \frac{1}{4!4} + ... = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!k}$ (Use the Maclaurin expansion, i.e. Taylor for $\bar{x} = 0$.)

64.2 (a) $\frac{1}{4}$

(b) $\frac{1}{4}(e^2 + 1)$

(c) $\frac{e}{2} - 1$

64.3 (a) $\int_0^{1/2} \int_{1/2}^{1-x_2} f(x_1 x_2) dx_1 dx_2$

(b) $\int_0^1 \int_0^{\sqrt{1-x_2^2}} f(x_1 x_2) dx_1 dx_2$

(c) $\int_{-1}^0 \int_0^{x_1+1} f(x_1 x_2) dx_2 dx_1$

(d) $\int_0^1 \int_{1-x_2}^1 f(x_1 x_2) dx_1 dx_2 + \int_1^2 \int_1^{x_2-1} f(x_1 x_2) dx_1 dx_2$

64.4 (a) $\frac{11}{60}$

(b) $\frac{2}{15}$

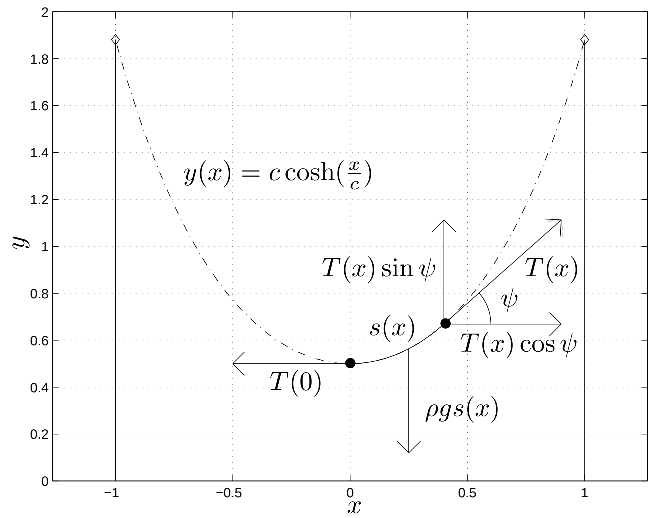

Figure 3: A plot of the hanging chain described by the function $y(x) = c\cosh(\frac{x}{c})$ (with $c = \frac{1}{2}$). Force equilibrium for the part of the chain between 0 and x requires that $T(0) = T(x)\cos\psi$ (horizontal equilibrium) and $\rho g s(x) = T(x)\sin\psi$ (vertical equilibrium). (Problem 60.13)

(c) $\frac{7}{2}$

(d) 1

64.5 (a) $e - 1$

(b) $(1 - e^{-2})(2 - 5e^{-1})$

(c) $\frac{3}{4}$

(d) $\frac{\pi}{4}$ (Use integration by parts to obtain: $\int \sqrt{a - x^2}dx = \frac{1}{2}x\sqrt{a - x^2} - \frac{\sqrt{a}}{2}\arccos(\frac{x}{\sqrt{a}})$)

64.6 The volume of the metal removed is $\frac{4\pi}{3}(a^3 - (a^2 - b^2)^{3/2})$. (Use polar coordinates.)

64.7 $\frac{2\pi}{5}a_1 a_2$ (Use elliptic coordinates.)

64.8 $e^2 - 1$

64.9 $\frac{9\pi}{4}$

64.10 The area within the cardoid is $\frac{3\pi}{2}$. (Use polar coordinates.)

64.11 (a) $\frac{e^2-3}{2}$

(b) $e^2(2e - 1)$

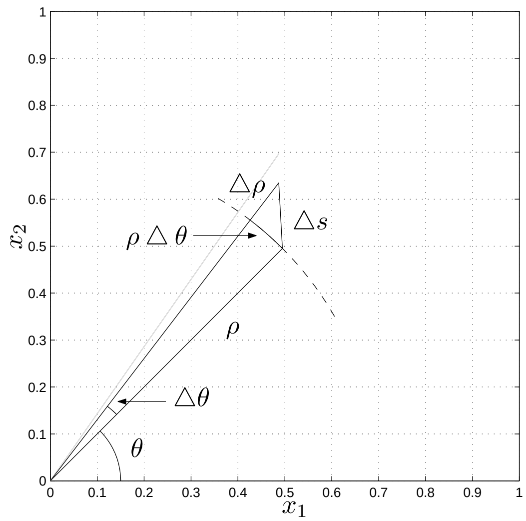

Figure 4: A plot of the relation between small changes $\triangle\rho$ and $\triangle\theta$ in ρ and θ, and the corresponding change in position $\triangle s$. From the figure and old Pythagoras we get $(\triangle s)^2 \approx \rho^2(\triangle\theta)^2 + (\triangle\rho)^2$, which in the infinitesimal limit turns into $ds^2 = \rho^2 d\theta^2 + d\rho^2$. (Problem 60.15)

(c) $\frac{1}{2}(1,1)$

64.12 (a) $2(1-\frac{2}{e})$

(b) $\frac{1}{15}(25\sqrt{5}-4\sqrt{2}-31)$

(c) $2ln2-\frac{5}{2}$ (Division of polynomials.)

64.13 (a) $\frac{4\pi}{3}$ (Complete the squares and take $x_1 = r\cos(\theta)+1$ and $x_2 = r\sin(\theta)+1$.)

(b) $\frac{1}{9^4 2}$ (Complete the squares and take $x_1 = r\cos(\theta)+\frac{1}{3}$ and $x_2 = \sqrt{3}r\sin(\theta)$.)

(c) $-\pi$

Chapter 65

65.2 $\frac{40\pi}{3}(26\sqrt{26}-125)$

65.6 (a) $\sqrt{6}$ (Since $x = s(y) = My$ is a linear transformation, the area scale from the parameter domain to the surface S is constant, equal to $|\frac{\partial s}{\partial y_1}\times\frac{\partial s}{\partial y_2}| = |(1,0,1)\times(0,1,2)| = \sqrt{6}$, where $\frac{\partial s}{\partial y_1} = (1,0,1)$ and $\frac{\partial s}{\partial y_2} = (0,1,2)$ are the columns of M. Therefore, the area of S is $\sqrt{6}$

times the area of Q, and since $Q = \{y = (y_1, y_2) \in \mathbb{R}^2 : 0 \le y_1 \le 1, 0 \le y_2 \le 1\}$ is the unit square in $\mathbb{R}^2$ with area $(1-0)(1-0) = 1$, the area of S is equal to $\sqrt{6}$. But we can equivalently view S as the parallelogram spanned by the two vectors $v = (1-0)\frac{\partial s}{\partial y_1} = (1,0,1)$ and $w = (1-0)\frac{\partial s}{\partial y_2} = (0,1,2)$ with area equal to $|v \times w|$, which obviously yields the same result. See the plot below.)

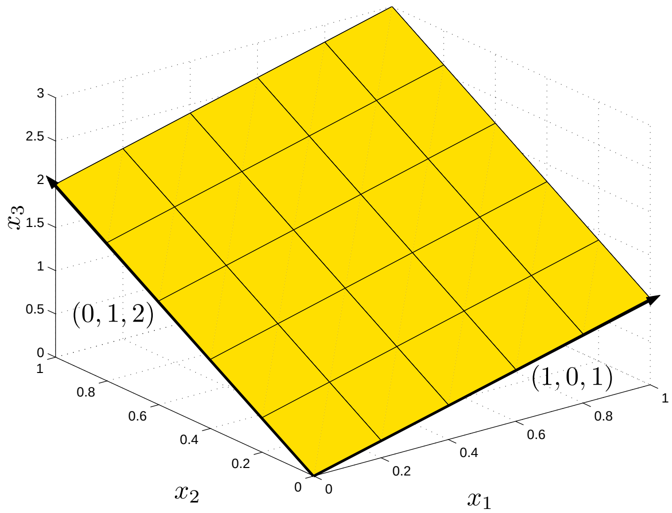

Figure 5: A plot of the surface S, a parallelogram spanned by the vectors $(1,0,1)$ and $(0,1,2)$. (Problem 62.6)

(b) $\frac{3\sqrt{6}}{2}$

65.10 Note! Parameter domain is not specified.
$\frac{4}{\sqrt{3}}$ (for $0 \le y_1 \le 1, 0 \le y_2 \le \pi$)

65.11 Note! n denotes the outward unit normal to S.
$\frac{1}{2}$

65.15 $4\pi^2 ab$

Chapter 66

66.1 Hint: Use the fact that $E = \frac{1}{2}m(r,\theta,\phi)v^2 = \frac{1}{2}m(r,\theta,\phi)(\omega r)^2$

66.2

66.3 (1) 1

(2) $\frac{(e^2-1)^2}{4}$

(3) 4π

66.4 (a) $\frac{1}{24}$
(b) $\frac{10\pi}{24}$

66.5 (1) $2\pi\log(2)+\pi^2$ Hint: Use polar coordinates for x_1 and x_2
(2) $\frac{m}{2}$
(3) $\frac{7m}{12}$

66.6 (1) $\frac{8\pi}{e}$
(2) 2
(3) $\frac{n}{2}$

66.7 (1) 0
(2) π
(3) $\frac{4\pi}{5}$

66.8

66.9 $\pi^{n/2}$ Hint: Use the fact that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-||x||^2}dx_1dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x_1^2+x_2^2)}dx_1dx_2 =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x_1^2}\cdot e^{-x_2^2}dx_1dx_2 = \int_{-\infty}^{\infty} e^{-x_1^2}dx_1 \cdot \int_{-\infty}^{\infty} e^{-x_2^2}dx_2$$

66.10 $\frac{7m}{6}$

66.11 Hint: Use the fact that $\{(x,y):\ x\in E_y,\ 0<y<\infty\} = \{(x,y):\ |f(x)| > y,\ 0<y<\infty\} = \{(x,y):\ 0<y<|f(x)|,\ x\in R^n\}$

Chapter 67

67.1 (64.4) Gauss' theorem applied to vw yields

$$\int_\Omega \frac{\partial vw}{\partial x_i}dx = \int_\Gamma vwn_i ds.$$

On the other hand the product rule for differentiation yields

$$\int_\Omega \frac{\partial vw}{\partial x_i}dx = \int_\Omega \frac{\partial v}{\partial x_i}wdx + \int_\Omega v\frac{\partial w}{\partial x_i}dx.$$

The result follows by equating the right hand sides.
(64.5) Follows immediately by the definition.
(64.6) Componentwise it reads, using (64.4),

$$\int_\Omega \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}dx = \int_\Gamma v\frac{\partial w}{\partial x_i}n_i ds - \int_\Omega v\frac{\partial^2 w}{\partial x_i^2}dx.$$

(64.8) Follows by interchanging the roles of v and w in the right hand side of (64.6) and equating the two right hand sides.

67.3 (a) $\int_\Gamma u \cdot nds = 0$ by Gauss' theorem.

(b) Nothing.

67.4

67.5 For the mapping $\overline{x}_i = x_i + a_i x_i$, $i = 1, 2$, the Jacobian J becomes $1 + a_1 + a_2 + a_1 a_2$. For area preserving maps $J = 1$. For small deformations the term $a_1 a_2$ can be neglected from which we conclude that $\text{div} u = a_1 + a_2$ is almost zero.

67.6 (a) Here the divergence theorem applies and since $\text{div}\, u = 0$ we conclude that $\int_\Gamma u \cdot nds = 0$.

(b) $\int_\Gamma u \cdot nds = 2\pi$. Here the divergence theorem does not apply since $\text{div}\, u$ is not defined in the origin. Instead compute the curve integral by using polar coordinates.

67.7 Γ and $\overline{\Gamma}$ with outward normals n and $-\overline{n}$, respectively, close a domain Ω in the plane to which the divergence theorem applies. We get

$$0 = \int_\Omega \text{div}\, u dx = \int_\Gamma u \cdot nds + \int_{\overline{\Gamma}} u \cdot (-\overline{n}) ds.$$

Consequently

$$\int_\Gamma u \cdot nds = \int_{\overline{\Gamma}} u \cdot \overline{n} ds.$$

67.8 (a) 0. Hint: Close the curve by adding the line segments from $(0, 0)$ to $(1, 0)$ and from $(0, 0$ to $(0, 1)$, respectively.

(b) $\log 5$. Hint: Close the curve by adding the line segment from $(1, 0)$ to $(1, 2)$.

67.9 $\nabla \times u = 0$, thus there exists a potential φ such that $u = \nabla \varphi$. An integration gives $\varphi(x) = x_1 \exp(x_1 x_2) + C$, where C is an arbitrary constant.

67.10 Note! Misprint in statement of problem. Reads: $v = -\frac{1}{2\pi} \log(x - \overline{x})$
Should read: $v = -\frac{1}{2\pi} \log(\|x - \overline{x}\|)$
Let $\Omega_\epsilon = \{\|x\| < R : \|x - \overline{x}\| > \epsilon\}$ On Ω_ϵ we have $\Delta v = 0$, c.f. exercise 58.9. Further, since w and $\nabla w \cdot n$ vanish for $\|x\| > R$, for some R large enough, (64.16) reduces to

$$\int_{\Omega_\epsilon} v f dx = -\int_{\partial\Omega_\epsilon} v \nabla w \cdot nds + \int_{\partial\Omega_\epsilon} w \nabla v \cdot nds.$$

In fact, since $\|x - \overline{x}\| = \epsilon$ and $n = -\frac{x - \overline{x}}{\|x - \overline{x}\|}$ on $\partial\Omega_\epsilon$, we get

$$\int_{\Omega_\epsilon} v f dx = \frac{1}{2\pi} \int_{\partial\Omega_\epsilon} (\log \epsilon) \nabla w \cdot nds + \frac{1}{2\pi\epsilon} \int_{\partial\Omega_\epsilon} w ds.$$

Using polar coordinates in the curve integrals, a passage to the limit, letting $\epsilon \to 0$ in the integrals, gives

$$\int_{\Omega_\epsilon} vf dx \to \int_{\Omega} vf dx,$$

$$\frac{1}{2\pi}\int_{\partial\Omega_\epsilon} (\log \epsilon)\nabla w \cdot n ds \to 0$$

and

$$\frac{1}{2\pi\epsilon}\int_{\partial\Omega_\epsilon} w ds \to w(\overline{x}).$$

Hence,

$$w(\overline{x}) = \int_{\Omega} vf dx.$$

67.11 Analogous arguments as in 64.10. Let $\tilde{\Omega}_\epsilon = \{x \in \Omega_\epsilon : x_2 > 0\}$. (64.16) takes the form

$$\int_{\tilde{\Omega}_\epsilon} vf dx = -\frac{1}{2\pi}\int_{\partial\tilde{\Omega}_\epsilon} \log\epsilon \nabla w \cdot n ds + \frac{1}{2\pi\epsilon^2}\int_{\partial\tilde{\Omega}_\epsilon} w \cdot n ds + \int_{\{x_2=0\}} wg \cdot n ds.$$

By noting that

$$\frac{1}{2\pi\epsilon^2}\int_{\partial\Omega_\epsilon} w \cdot n ds \to \frac{1}{2}w(\overline{x}),$$

since the curve integral only takes half a round for points $\overline{x} = (\overline{x}_1, 0)$ the result now follows.

67.12 Follows immediately from (64.16).

Chapter 68

68.3 $j = 0$, 1 and 2 gives the values 4π, 2π and 0, respectively. Compute the integrand and use polar coordinates. For the case $j = 2$ one can also compute the divergence wihich is zero and apply the divergence theorem. Compare with excercise 64.6.

68.4 Dismissed.

68.5 (1) 4π
(2) 0
(3) 4π
(4) 0.

68.6 A direct computation gives $\operatorname{div} F = (\alpha + 2)\|x\|^{\alpha-1}$.

68.7 An application of the divergence theorem gives

$$I = \int\int\int_V \text{div}\mathbf{F} dx_1 dx_2 dx_3.$$

To minimize the volume integral corresponds to solving the inequality $\text{div}\mathbf{F} \leq 0$, from which the set V of integration is determined.

$$\text{div}\mathbf{F} = x_1^2 - 2x_1 + x_2^2 - 4x_2 + 4x_3^2 + 8x_3 + 5 \leq 0.$$

A completion of the squares gives:

$$\frac{(x_1-1)^2}{4} + \frac{(x_2-2)^2}{4} + (x_3+1)^2 \leq 1,$$

which is recognized as an ellipsoid centered at (1,2,-1) with half axes 2, 2 och 1, respectively. An introduction of ellipsoidal coordinates

$$\begin{cases} x = 1 + 2r\sin\theta\cos\varphi, \\ y = 2 + 2r\sin\theta\sin\varphi, \\ z = -1 + r\cos\theta, \end{cases}$$

with the Jacobian $J = 4r^2\sin\theta$ finally yields

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^1 (4r^2-4)4r^2\sin\theta\, drd\theta d\varphi = -128\pi/15.$$

68.8 Close the surface with a lid $\tilde{\Gamma}$ from the bottom, where $\tilde{\Gamma} : x_1^2 + x_2^2 \leq 4,\ \ x_3 = 0$. By choosing downward normal $n = (0,0,-1)$ from $\tilde{\Gamma}$ we can apply the Gauss theorem.

$$\int_\Gamma F\cdot n ds = \int_\Omega \text{div}\, F dx - \int_{\tilde{\Gamma}} F\cdot n ds,$$

where Ω is the solid upper half sphere with radius 2. The surface integral vanishes on $\tilde{\Gamma}$ since $F\cdot n = (f_1, f_2, 0)\cdot(0,0,-1) = 0$ here. Moreover taking the divergence of F gives $\text{div}\, F = x_1$. A standard use of spherical coordinates gives

$$\int_\Gamma F\cdot n ds = 0,$$

by periodicity.

68.9 k. Compare with exercise 65.5 (4) for instance. Each point x_i inside the closed surface Γ contributes with the value 1, since F is normalized with $1/4\pi$.

Chapter 69

69.2 We have $\nabla\cdot(u\times a) = a\cdot\nabla\times u - u\cdot\nabla\times a$, but a is constant $\Rightarrow \nabla\times a = 0 \Rightarrow$

$$\int_\Omega \nabla\cdot(u\times a)\ dx = \int_\Omega a\cdot(\nabla\times u)\ dx = a\cdot\int_\Omega \nabla\times u\ dx. \qquad (1)$$

Also $(u\times a)\cdot n = (n\times u)\cdot a \Rightarrow$

$$\int_\Gamma (u\times a)\cdot n\ ds = \int_\Gamma (n\times u)\cdot a\ ds = a\cdot\int_\Gamma (n\times u)\ ds \qquad (2)$$

Now (1), (2) and the divergence theorem gives

$$a\cdot(\int_\Omega \nabla\times u\ dx - \int_\Gamma (n\times u)\ ds) = 0.$$

And this holds for all constant vectors $a \Rightarrow$

$$\int_\Omega \nabla\times u\ dx = \int_\Gamma (n\times u)\ ds.$$

69.3 $(u_1, u_2) = (-v_2, v_1) \Rightarrow \nabla\cdot u = \nabla\times v$ and $u\cdot n = n\times v$.

69.5 14π. Hint: $u = \frac{(-x_2,x_1,x_3)}{(x_1^2+x_2^2)\|x\|}$ conservative.

69.6 We have $\nabla\times(va) = v\nabla\times a + (\nabla v)\times a = (\nabla v)\times a$ (a constant $\Rightarrow \nabla\times a = 0$), and $(\nabla v)\times a\cdot n = n\times(\nabla v)\cdot a \Rightarrow \int_S(\nabla\times(va))\cdot n\ ds = a\cdot\int_S n\times(\nabla v)\ ds$. Also $\int_\Gamma (va)\cdot ds = a\cdot\int_\Gamma v\ ds$, and Stoke's theorem gives that $a\cdot(\int_S n\times(\nabla v)\ ds - \int_\Gamma v\ ds) = 0$, but this is true for all constant vectors $a \Rightarrow \int_S n\times(\nabla v)\ ds = \int_\Gamma v\ ds$.

69.7 $\nabla\times(x_2, 2x_3, 3x_1) = -(2,3,1)$, $s(x_1,x_2) = (x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \Rightarrow s'_{,1} = (1, 0, -x_1/\sqrt{1-x_1^2-x_2^2})$, $s'_{,2} = (0,1,-x_2/\sqrt{1-x_1^2-x_2^2}) \Rightarrow s'_{,1}\times s'_{,2} = (x_1/\sqrt{1-x_1^2-x_2^2}), x_2/\sqrt{1-x_1^2-x_2^2}), 1) \Rightarrow$

$$|s'_{,1}\times s'_{,2}| = \frac{1}{1-x_1^2-x_2^2}$$

$$n = \frac{s'_{,1}\times s'_{,2}}{|s'_{,1}\times s'_{,2}|} = (\frac{x_1}{1-x_1^2-x_2^2)^{3/2}}, \frac{x_2}{1-x_1^2-x_2^2)^{3/2}}, \frac{1}{1-x_1^2-x_2^2})$$

$\Rightarrow (\nabla\times(x_2, 2x_3, 3x_1))\cdot n = \frac{-2x_1-3x_2-\sqrt{1-x_1^2-x_2^2}}{(1-x_1^2-x_2^2)^{3/2}}$

69.8 We have $\frac{1}{2}\int_\Gamma u\cdot\ \ ds = \frac{1}{2}\int_\Omega(\nabla\times u)\cdot n\ ds$ by Stoke's theorem. Further $\nabla\times u = \nabla\times(-x_2, x_1, 0) = (0,0,2)$ and $n = (0,0,1) \Rightarrow (\nabla\times u)\cdot n = 2 \Rightarrow \frac{1}{2}\int_\Omega 2\ ds = \int_\Omega\ \ ds = A(\Omega)$.

Chapter 70

70.1 Using the formula $\nabla \times u = (\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2})$ we find that

(a) $\nabla \times u = (0 - 0, 0 - 0, 0 - 0) = (0, 0, 0)$ so that $u = \nabla\phi$ for some potential ϕ. Integrating $\nabla\phi = (x_1, x_2, x_3)$, we obtain $\phi(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + C = \frac{1}{2}\|x\|^2 + C$, where C is an arbitrary constant.

(b) $\nabla \times u = (1 - 0, 1 - 0, 1 - 0) = (1, 1, 1) \neq (0, 0, 0)$ so that u is not a potential field!

(c) $\nabla \times u = (0 - 0, -1 - (-1), 2x_2 - 2x_2) = (0, 0, 0)$ so that $u = \nabla\phi$ for some ϕ. Integrating we find that $\phi(x) = x_1 x_2^2 - x_1 x_3 + x_3^3 + C$.

70.2 Hint: Verify by direct differentiation (under the integral sign) that $\nabla \times \psi = u$. At some point, use the fact that $\nabla \cdot u = 0$.

70.3 We find for $u(x) = \frac{(-x_2, x_1, 0)}{||x||^2}$ that $\nabla \times u = (\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}) = (0, 0, 0)$ for $(x_1, x_2) \neq (0, 0)$, while for $s(t) = (r\cos(t), r\sin(t), 0)$ we have $\int_\Gamma u \cdot ds = \int_0^{2\pi} \frac{1}{r^2}(-r\sin(t), r\cos(t), 0) \cdot (-r\sin(t), r\cos(t), 0)\, dt = \int_0^{2\pi} dt = 2\pi$, just as in the two dimensional counter example.