TMA225 Differential Equations and Scientific Computing, part A

Problems Week 3

The Finite Element Method: Stationary Problems.

1. Let u be the solution to

$$-(au')' + cu = f \quad \text{in } (0,1), \tag{1}$$
$$u(0) = u(1) = 0, \tag{2}$$

where a, c, and f are given functions.
(a) Show that u satisfies the variational equation

$$\int_0^1 (au'v' + cuv)\,dx = \int_0^1 fv\,dx, \tag{3}$$

for all sufficiently smooth v with $v(0) = v(1) = 0$.
(b) Introduce a partition of $(0,1)$ and the corresponding space of continuous piecewise linear functions V_{h0} which are zero for $x = 0$ and $x = 1$. Formulate a finite element method based on the variational equation in (a).
(c) Let $|||u||| = \left(\int_0^1 (au'u' + cuu)\,dx\right)^{1/2}$. Verify that $|||\cdot|||$ is a norm if $a(x) > 0$ and $c(x) \geq 0$ for all $x \in (0,1)$.
(d) Prove the a priori error estimate

$$|||u - U||| \leq |||u - v|||, \tag{4}$$

for all $v \in V_{h0}$.
(e) Assume that there are constants C_a and C_c such that $\|a\|_{L_\infty(0,1)} \leq C_a$ and $\|c\|_{L_\infty(0,1)} \leq C_c$, and that $\|u''\|_{L^2(0,1)}$ is bounded. Show that $|||u - U|||$ converges to zero as the meshsize tends to zero.
2. Let u be the solution to

$$-u'' = 1 \quad \text{in } (0,1), \tag{5}$$
$$u(0) = u(1) = 0. \tag{6}$$

(a) Solve the problem analytically.
(b) Let $I = (0,1)$ be divided into a uniform mesh with $h = 1/N$. Calculate (by hand) the finite element approximation U for $N = 2, 3$.
(c) Plot your solutions in a figure. Compare your results.
3*.
(a) Show that the finite element approximations U that you have computed in Problem 2

actually are exactly equal to u *at the nodes*, by simply evaluating u and U at the nodes.
(b) Prove this result. *Hint:* Show that the error $e = u - U$ can be written

$$e(z) = \int_0^1 g_z'(x) e'(x)\, dx, \quad 0 \le z \le 1,$$

where

$$g_z(x) = \begin{cases} (1-z)x, & 0 \le x \le z, \\ z(1-x), & z \le x \le 1, \end{cases}$$

and then use the fact the $g_{x_j} \in V_{h0}$.
(c) Does the result in (b) extend to variable $a = a(x)$?

The Finite Element Method: Time Dependent Problems.

4. Consider the system of ODE:

$$\begin{aligned} M\dot{\xi}(t) + A\xi(t) &= b \quad \text{in } (0, T), && (7) \\ \xi(0) &= \xi^0. && (8) \end{aligned}$$

Assume that

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 14 \\ 4 & 8 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \xi^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \qquad (9)$$

Make a uniform partition of the time interval $(0, 1)$ into two sub-intervals and compute an approximation of $\xi(1)$ with the *backward Euler* method.
5. Show that, for the time dependent reaction-diffusion problem with Robin boundary conditions,

$$\begin{aligned} \dot{u} - (au')' + cu &= f(x,t), \quad x_{\min} < x < x_{\max}, \quad 0 < t < T, \\ a(x_{\min})u'(x_{\min}, t) &= \gamma(x_{\min})(u(x_{\min}, t) - g_D(x_{\min})) + g_N(x_{\min}), \quad 0 < t < T, \\ -a(x_{\max})u'(x_{\max}, t) &= \gamma(x_{\max})(u(x_{\max}, t) - g_D(x_{\max})) + g_N(x_{\max}), \quad 0 < t < T, \\ u(x, 0) &= u_0(x), \quad x_{\min} < x < x_{\max}, \end{aligned}$$

semi-discretization in space leads to the following system of ODE:

$$M\,\dot{\xi}(t) + (A + M_c + R)\,\xi(t) = b(t) + rv, \quad 0 < t < T. \qquad (10)$$