Answers to the Problems in AM:B&S

August 29, 2001

For errors in the answers or statements of the problems, please send email to Axel Målqvist: `axel@math.chalmers.se`. Updated versions of this document will be available on: `http://www.md.chalmers.se/education/courses/2001/ala-a/Kurser/AnalysA/`.

Chapter 3

3.2 $17x = 10$

3.5 $x^2 = 3$

3.6 $170x = 45(12 - x)$

Chapter 5

5.1 1) My age, 2) Number of my children, 3) Number of contries that I have seen, 4) Number of languages that I speak, 5) Number of Vivaldi music CD that I own

5.5 In $m \times n = 0$ if and only if $m = 0$ *or* $n = 0$, *or* means *either, or* i.e. either $m = 0$ or $n = 0$. If in $p \times m = p \times n$, $p = 0$, then m and n could be any nonzero (because 0×0 is NOT defined) integer number, (for example $m = 17, n = -1$).

5.8 (a) $102 = 5 \times 18 + 12$
(b) $-4301 = -69 \times 63 + 46$
(c) $650912 = 2106 \times 309 + 158$

5.9 (a) $40 = 2^3 \times 5 \Longrightarrow \{1, 2, 4, 8, 10, 20, 40\}|40$
(b) $80 = 2^4 \times 5 \Longrightarrow \{1, 2, 4, 8, 10, 16, 20, 40, 80\}|80$

5.12 (a) $2^3 \times 3 \times 5$
(b) $2^5 \times 3$
(c) $2^4 \times 7$

(d) 3×43

5.14 $(a+b)^2 = a^2 + b^2$ is not valid. Simply take $a = b = 1$, then the left hand side is 4 while the right hand side is 2.

$ac < bc$ implies $a < b$ is an invalid implication. Just take $a = 2, b = 1, c = -1$, then we are getting: $-2 < -1$ implies $2 < 1$, i.e., we derive from a correct statement a wrong conclusion.

Finally $a + bc = (a+b)c$. Take, for example, $a = b = 1$ and $c = 0$, you get $1 = 0$.

5.15 (a) $-2 \leq x \leq 20$

(b) $8 < x < 20$

(c) $-13 < x < 25$

(d) $1 \leq x \leq 3$

Chapter 6

6.1 (a) The inductive step: $1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 =$ (the inductive assumption) $= \frac{(n-1)n(2(n-1)+1)}{6} + n^2 = \frac{2n^3-3n^2+n+6n^2}{6} = \frac{n(n+1)(2n+1)}{6}$

(b) The inductive step: $1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 + n^3 =$ (the inductive assumption) $= (\frac{(n-1)n}{2})^2 + n^3 = \frac{n^4-2n^3+n^2+4n^3}{4} = (\frac{n(n+1)}{2})^2$

6.2 Note: error in the problem statement $1/(n+1)$ should be $n/(n+1)$. The inductive step: $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} =$ (the inductive assumption) $= \frac{n-1}{(n-1)+1} + \frac{1}{n(n+1)} = \frac{(n-1)(n+1)+1}{n(n+1)} = \frac{n}{n+1}$

6.3 (a) The inductive step: $3n^2 = 3((n-1)+1)^2 = 3(n-1)^2 + 6n - 3 \geq$ (the inductive assumption) $\geq 2(n-1)+1+6n-3 = 2n+(6n-4) \geq 2n+1$

(b) The inductive step: $4^n = 4 \times 4^{n-1} \geq$ (the inductive assumption) $\geq 4(n-1)^2 = n^2 + (3n^2 - 8n + 4) = n^2 + 3(n-2)^2 + 4(n-2) \geq n^2$ (for $n \geq 2$)

6.4 Let P_n denote the size of the population year n. The modeling assumption is that $P_n = KP_{n-1}^2$, which iterated n times gives $P_n = K^{2^n-1}P_0^{2^n}$.

6.5 Let P_n denote the size of the population year n. The modeling assumption is that $P_n = K_1 P_{n-1} - K_2 P_{n-1}^2$.

6.6 Let P_n denote the size of the population year n. The modeling assumption is that $P_n = KP_{n-1} + KP_{n-2}$.

6.7 –

6.8 The inductive step: Since, by long division,

$$\begin{array}{r|llll} & & & p^n & \\ \hline p-1 & p^{n+1} & & & -1 \\ & -(& p^{n+1} & -p^n &) \\ \hline & & & p^n & -1 \end{array}$$

we get $\frac{p^{n+1}-1}{p-1} = p^n + \frac{p^n-1}{p-1} =$ (the inductive assumption) $= p^n + p^{n-1} + \ldots + 1$

Chapter 7

7.3 Proof of Commutative law for addition:
$(p,q)+(r,s) = (ps,qs)+(qr,qs) = (ps+qr,qs) = (rq+sp,sq) = (rq,sq)+(sp,sq) = (r,s)+(p,q)$
Proof of Commutative law for multiplication:
$(p,q)\times(r,s) = (pr,qs) = (rp,sq) = (r,s)\times(p,q)$
Proof of Distributive law:
$(p,q)\times((r,s)+(t,u)) = (p,q)\times(ru+st,su) = (p(ru+st),qsu) = pru+pst,qsu) = (pru,qsu)+(pst,qsu) = (pr,qs)+(pt,qu) = (p,q)\times(r,s)+(p,q)\times(t,u)$

7.4 For rational numbers $r = \frac{r_1}{r_2}$, $s = \frac{s_1}{s_2}$ and $t = \frac{t_1}{t_2}$, one has

$$r(s+t) = \frac{r_1}{r_2}\left(\frac{s_1}{s_2} + \frac{t_1}{t_2}\right) = \frac{r_1}{r_2}\,\frac{s_1t_2+s_2t_1}{s_2t_2} = \frac{r_1s_1t_2+r_1s_2t_1}{r_2s_2t_2}$$

$$= \frac{r_1s_1t_2}{r_2s_2t_2} + \frac{r_1s_2t_1}{r_2s_2t_2} = \frac{r_1s_1}{r_2s_2} + \frac{r_1t_1}{r_2t_2} = rs + rt$$

7.5 (a) $\{x \in Q : 1 \le x \le 5/3\}$
(b) $\{x \in Q : -\frac{4}{5} < x < \frac{8}{5}\}$
(c) $\{x \in Q : x < -\frac{1}{14}$ or $x > \frac{13}{14}\}$
(d) $\{x \in Q : x \le -\frac{1}{8}$ or $x \ge \frac{5}{8}\}$

7.7 Using the fact that one mile is 5280 feet, and one hour 3600 seconds, the speed of the runner is 16 miles/hour plus 8.8 feet/second, that is $16 \times 5280/3600 + 8.8 = \frac{16\cdot5280}{3600} + \frac{8.8\cdot3600}{3600} = \frac{84480+31680}{3600} = \frac{116160}{3600}$ feet/second, that is 32.26666.. feet/second.

7.8 (a) 0.428571 428571 42..
(b) 0.153846 153846 15..
(c) 0.2941176470588235 29411..

7.9 (a) 3.456
(b) 0.5975

7.10 (a) 42/99, that is 14/33

(b) 8811/9999
(c) 42905/99999

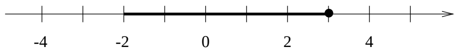
7.11 (a) Skippas!
(b) Skippas!
(c) 

Figure 1:

(d) Skippas!

7.12

7.13 $C_0(1+0.09)^n$

Chapter 9

9.1

9.2 (a) $(-10, 14]$
(b) $(10, \infty)$
(c) $\{-14, 6, 22, 30\}$

9.3 $(-\infty, 0) \cup [\frac{1}{13}, \frac{3}{13}]$

9.4 domain: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
range: $\{75, 75.01, 75.08, 75.27, 75.64, 76.25, .., 82.29\}$

9.5 $D_H = [0, \sqrt{50}]$, $R_H = [0, 50]$

9.6 $\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ..\}$

9.7 $D_f = Q = (-\infty, \infty)$. B could be any set containing $R_f = (0, 1]$

9.8 (a) $\{x \in Q : x \neq -2$ and $x \neq 4$ and $x \neq 5\}$
(b) $\{x \in Q : x \neq -2$ and $x \neq 2\}$
(c) $\{x \in Q : x \neq -1/2$ and $x \neq 8\}$

9.9 $\{0, 1, 2, 3, 4\}$

9.10 Skippas!

9.11 (a) (b), (c) skippas!

9.12 Skippas!

9.13 Skippas!

9.14 Skippas!

9.15 Skippas!

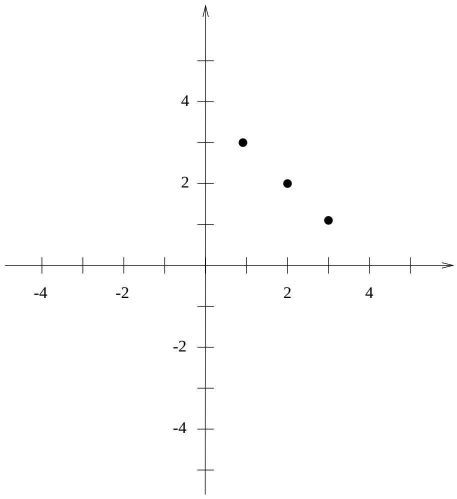


Figure 2:

Chapter 10

10.1 (a) $y = 4x - 1$

(b) $y - 2 = -\frac{1}{2}(x + 4)$

(c) $y - 7 = 0$

(d) $y = -7.4x + 27$

(e) $x = -3$

(f) $y = -3x + 5$

10.2 (a) $y = \frac{4}{5}x - \frac{46}{5}$

(b) $y = -\frac{3}{2}x + \frac{5}{2}$

(c) $x = 13$

(d) $y = 4$

(e) $y = 10x + 7$

(f) $y = -2x - 1$

10.3 $x = -\frac{b}{m}$

10.4 See the plot of the functions below.

10.5 Yes!

10.6 Yes!

10.7 (a) $(-4/7, 2/7)$

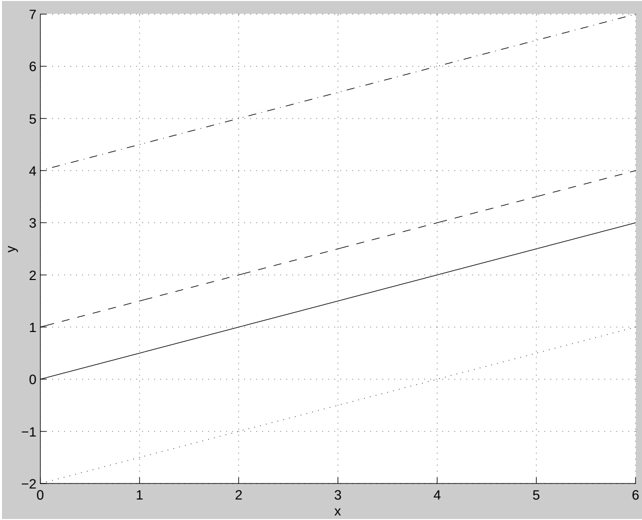


Figure 3: A plot of the functions $y = \frac{1}{2}x$ (__), $y = \frac{1}{2}x - 2$ (...), $y = \frac{1}{2}x + 4$ (_._) and $y = \frac{1}{2}x + 1$ (_ _ _). (Problem 10.4)

(b) $(35/11, 223/11)$

10.8 $y = -\frac{1}{10}(x - 3)$

10.9 (a) $y = 0.1x + 1.9$

(b) $y = -10x + 12$

10.10 $x_1 < x_2 < 0$ implies $x_2^2 - x_1^2 = (x_2 + x_1)(x_2 - x_1) < 0$, because $x_2 + x_1 < 0$ and $x_2 - x_1 > 0$, that is $x_2^2 < x_1^2$.

10.11 See the plots of the functions below.

10.12 See the plots of the functions below.

10.13 See the plots of the functions below.

10.14 See plot below.

10.15 See plot below

(a) $y = x^2 + 4x + 5 = (x + 2)^2 + 1$

(b) $y = 2x^2 - 2x - \frac{1}{2} = 2(x - \frac{1}{2})^2 - 1$

(c) $y = -\frac{1}{3}x^2 + 2x - 1 = -\frac{1}{3}(x - 3)^2 + 2$

10.16 (a) $\sum_{i=1}^{n} i^{-2}$

(b) $\sum_{i=1}^{n} (-1)^i i^{-2}$

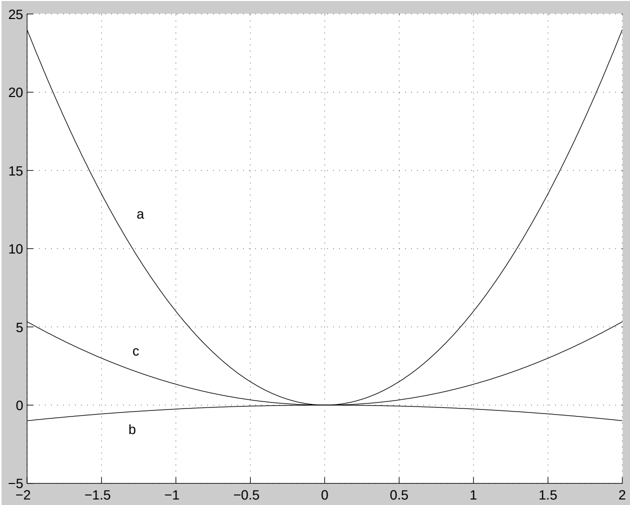


Figure 4: Plot of the functions a) $y = 6x^2$, b) $y = -\frac{1}{4}x^2$, and c) $y = \frac{4}{3}x^2$. (Problem 10.11)

(c) $\frac{1}{2} + \sum_{i=1}^{n} \frac{1}{i(i+1)}$

(d) $\sum_{i=0}^{n}(2i+1)$

(e) $\sum_{i=0}^{n-4} x^{4+i}$

(f) $\sum_{i=0}^{n} x^{2i}$

10.17 (a) $\sum_{i=-1}^{n-2}(i+2)^2$

(b) $\sum_{i=15}^{n+14}(i-14)^2$

(c) $\sum_{i=-3}^{n-4}(i+4)^2$

(d) $\sum_{i=8}^{n+7}(i-7)^2$

10.18 (a) $-4 + 6x - 8x^2 + 11x^3 - 16x^5$

(b) $48 - 72x + 6x^2 - 87x^3 + 12x^5$

(c) $-2 + 6x + 2x^2 + 6x^3 - x^4 + 4x^5$

(d) $-8x^2 + 16x^3 - 6x^4 - 2x^5 + 17x^6 + 28x^8$

(e) $-8 + 12x + 14x^3 + 4x^4 - 6x^5 - 7x^7$

(f) $4x^2 - 2x^3 + 8x^5 - 2x^6 + x^7 - 4x^9$

(g) $-8 + 12x - 2x^2 + 15x^3 + 4x^4 - 10x^5 - 7x^7$

(h) $-8 + 12x + 4x^2 + 12x^3 + 4x^4 + 2x^5 - 2x^6 - 6x^7 - 4x^9$

(i) $-16x^2 + 32x^3 - 12x^4 - 4x^5 + 42x^6 - 16x^7 + 62x^8 + 2x^9 - 17x^{10} - 28x^{12}$

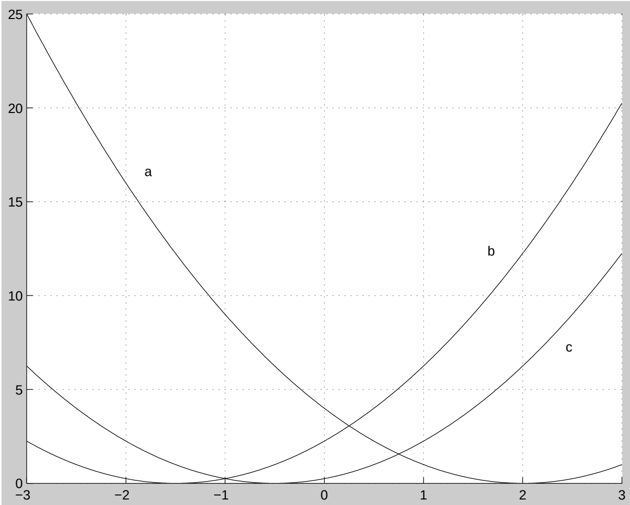


Figure 5: Plots of the functions a) $y = (x-2)^2$, b) $y = (x+1.5)^2$, and c)$y = (x+0.5)^2$. (Problem 10.12a)

10.19 (a) $x^2 + 2xa + a^2$

(b) $x^3 + 3x^2a + 3xa^2 + a^3$

(c) $x^3 - 3x^2a + 3xa^2 - a^3$

(d) $x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$

10.20 $p_1p_2 = \sum_{i=0}^{8}\sum_{j=0}^{11}\frac{i^2}{j+1}x^{i+j}$

10.21 The polynomial $p(x) = 360x - 942x^2 + 949x^3 - 480x^4 + 130x^5 - 18x^6 + x^7$ is zero for 0,1,2,3,4,5, and has the property that $p(x) \to +\infty$ when $x \to +\infty$ and $p(x) \to -\infty$ when $x \to -\infty$, see plots below.The polynomial can be factored into $p(x) = x(x-1)(x-2)(x-3)^2(x-4)(x-5)$, which explains the behavior.

10.22 (a) *Has increasing/decreasing been defined in the book?* A function f is increasing in an interval (a, b) if $a < x \leq y < b$ implies $f(x) \leq f(y)$. From $x^3 - y^3 = \frac{1}{2}(x-y)(x^2 + y^2 + (x+y)^2)$ it is seen that $x^3 - y^3$ has the same sign as $x - y$, hence x^3 is increasing.

(b) A function f is decreasing in an interval (a, b) if $a < x \leq y < b$ implies $f(x) \geq f(y)$. From $x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$ it is seen that $x^4 \leq y^4$ if $0 < x \leq y$, and $x^4 \geq y^4$ if $x \leq y < 0$.

10.23 Reformulation of problem: Plot the monomials for $-2 \leq x \leq 2$. See the plot below.

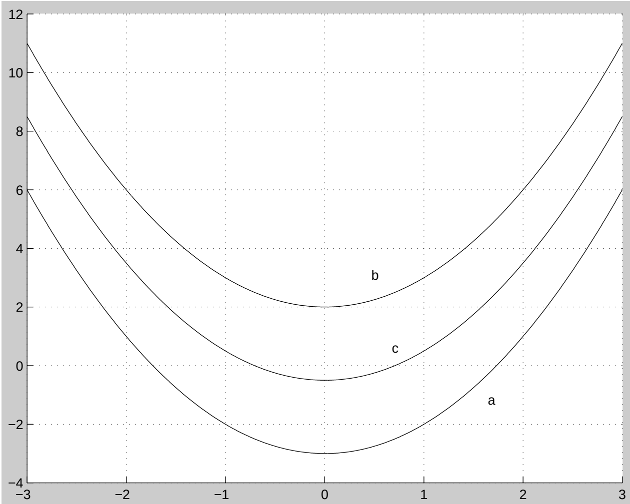


Figure 6: Plots of the functions a) $y = x^2 - 3$, b) $y = x^2 + 2$, and c) $y = x^2 - 0.5$. (Problem 10.13a)

10.24 Reformulation of problem: Plot the polynomials for in the intervals $x^* - 2, x^* + 2]$, where x^* are the symmetry or anti-symmetry point of the polynomial. The point x^* is symmetry point if for any x, $p(x^* + x) = p(x^* - x)$, correspondingly x^* is antisymmetry point if $p(x^* + x) = -p(x^* - x)$. See the plots in figure below.

10.25 See plots below of piecewise polynomials.

Chapter 11

11.1 (a) $\{x \in R : x \neq \frac{1}{3} \text{ and } x \neq 1\}$

(b) $\{x \in R : x \neq 0 \text{ and } x \neq 2 \text{ and } x \neq -\frac{1}{2}\}$

(c) $\{x \in R : x \neq 0\}$

(d) $\{x \in R : x \neq 0 \text{ and } x \neq -\frac{3}{2}\}$

(e) $\{x \in R : x \neq \frac{2}{3} \text{ and } x \neq -4\}$

(f) $\{x \in R : x \neq -2 \text{ and } x \neq -1\}$

11.2 (a) $\sum_{i=1}^{101}(i+1)x^i(x-1)^i$ (Note misprint: 100 should be 102!)

(b) $\sum_{i=1}^{13} \frac{2^i}{x-i}$

11.3 (a) Note misprint: $f(x) = ax + b$ should be $f(x) = ax$. Proof: $f(x+y) = a(x+y) = ax + ab = f(x) + f(y)$.

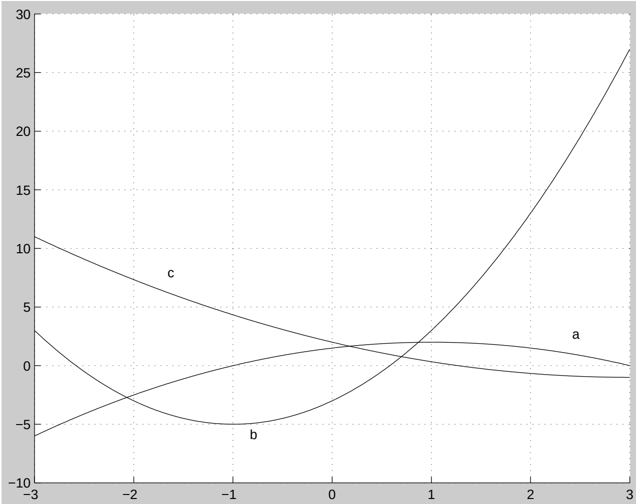


Figure 7: Plots of the functions a) $-\frac{1}{2}(x-1)^2+2$, b) $2(x+2)^2-5$ and c) $\frac{1}{3}(x-3)^2-1$ for $-3 \le x \le 3$. Note the x- and y-coordinates of the extreme points, (the points where the function has max or min value). (Problem 10.14)

(b) Proof: $g(x+y) = (x+y)^2 = x^2+2xy+y^2$ and $g(x)+g(y) = x^2+y^2$, that is $g(x+y) \neq g(x)+g(y)$ unless $x=0$ or $y=0$.

11.4 (a) $\frac{x^2+2x-3}{x-1} = x+3$

(b) $\frac{2x^2-7x-4}{2x+1} = x-4$

(c) $\frac{4x^2+2x-1}{x+6} = 4x-22+\frac{131}{x+6}$

(d) $\frac{x^3+3x^2+3x+2}{x+2} = x^2+x+1$

(e) $\frac{5x^3+6x^2-4}{2x^2+4x+1} = \frac{5}{2}x-2+\frac{\frac{11}{2}x-2}{2x^2+4x+1}$

(f) $\frac{x^4-4x^2-5x-4}{x^2+x+1} = x^2-x-4$

(g) $\frac{x^8-1}{x^3-1} = x^5+x^2+\frac{x^2-1}{x^3-1}$

(h) $\frac{x^n-1}{x-1} = x^{n-1}+x^{n-2}+\cdots+x^2+x = \sum_{i=1}^{n-1} x^i$

11.5 (a) $3(2x^2+1)-5 = 6x^2-2$

(b) $2(\frac{4}{x})^2+1 = \frac{32}{x^2}+1$

(c) $\frac{4}{3x-5}$

(d) $3(2(\frac{4}{x})^2+1)-5 = 3(\frac{32}{x^2}+1)-5 = \frac{96}{x^2}-2$

11.6 Note misprint: x/x^2 should be $1/x$. $f_1 \circ f_2 = 4(\frac{1}{x})+2 = \frac{4}{x}+2$ and $f_2 \circ f_1 = \frac{1}{4x+2}$ are *not* equal, for example for $x=1$.

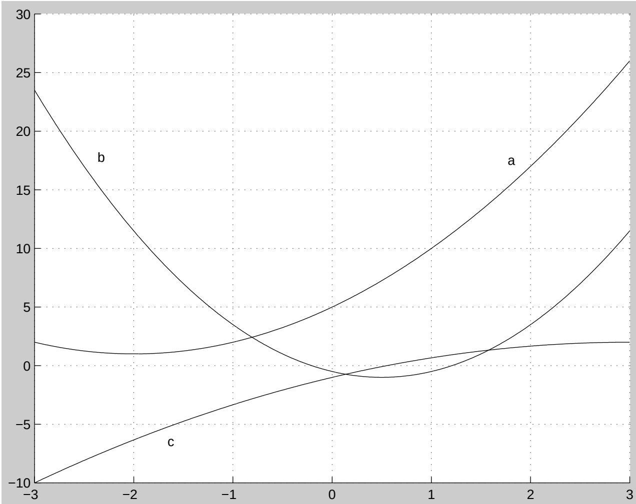


Figure 8: Plots of the functions a) $(x+2)^2+1$, b) $2(x-\frac{1}{2})^2-1$ and c) $-\frac{1}{3}(x-3)^2+2$ for $-3 \le x \le 3$. (Problem 10.15)

11.7 $f_1 \circ f_2 = a(cx+d)+b = acx+ad+b$ and $f_2 \circ f_1 = c(ax+b)+d = cax+cb+d$ are equal if and only if $ad+b = cb+d$, which is the case for example if $a=1$ and $c=1$, for any b and d, or otherwise if $d = \frac{cb-b}{a-1}$ or $b = \frac{ad-d}{c-1}$.

11.8 (a) $\{x \in R : x \neq 0 \text{ and } x \neq \frac{1}{4}\}$

(b) $\{x \in R : x \neq 1 \text{ and } x \neq \frac{1}{2} \text{ and } x \neq \frac{3}{2}\}$

Chapter 12

12.1 Since $|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 + x_2||x_1 - x_2|$, we have $|f(x_1) - f(x_2)| \le 16|x_1 - x_2|$ for $x_1, x_2 \in [-8, 8]$, and $|f(x_1) - f(x_2)| \le 800|x_1 - x_2|$ for $x_1, x_2 \in [-400, 200]$.

12.2 For $a, b \in [10, 13]$ one has $|f(a) - f(b)| = |a^2 - b^2| = |(a+b)(a-b)| = |a+b||a-b| \le 26|a-b|$

12.3 For $a, b \in [-2, 2]$ one has $|f(a) - f(b)| = |4a - 2a^2 - (4b - 2b^2)| = |4(a-b) - 2(a+b)(a-b)| = |(4 - 2a - 2b)(a-b)| = |4 - 2a - 2b||a-b| \le 12|a-b|$, because $|4 - 2a - 2b| \le 4 + 2|a| + 2|b| \le 4 + 4 + 4 = 12$, for $a, b \in [-2, 2]$.

12.4 Since $|f(x_1) - f(x_2)| = |x_1^3 - x_2^3| = |x_1 - x_2||x_1^2 + x_1x_2 + x_2^2| \le (4+4+4)|x_1 - x_2|$, we have $L = 12$.

12.5 Show that for all x_1, x_2, we have $||x_1| - |x_2|| \le |x_1 - x_2|$. Thus $|f(x_1) - f(x_2)| = ||x_1| - |x_2|| \le |x_1 - x_2|$ and $L = 1$.

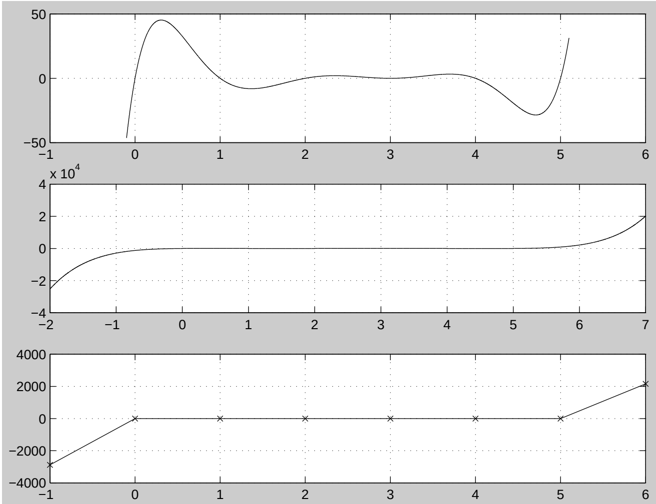


Figure 9: Three plots of the polynomial $360x - 942x^2 + 949x^3 - 480x^4 + 130x^5 - 18x^6 + x^7$, the top figure shows with matlab notation $x = -0.1 : 0.001 : 5.1$, the middle $x = -2 : 0.001 : 7$, and bottom $x = -1 : 6$. The matlab notation `x=x0:dx:x1`, means that x are the values starting with `x0` and increasing with interval `dx` until `x1` is reached. (Problem 10.21)

12.6 Realize (by plotting $f(x)$) that, given $|x_1 - x_2|$, $|f(x_1) - f(x_2)|$ attains its greatest value near $x_1 \approx x_2 \approx \pm 2$. Take $x_1 = 2$ and $x_2 = 2 - \epsilon$, where ϵ is a small number. Then show that $|f(x_1) - f(x_2)| \approx 32|x_1 - x_2|$.

12.7 For $a, b \in [1, 2]$ one has $|f(a) - f(b)| = |\frac{1}{a^2} - \frac{1}{b^2}| = |\frac{b^2 - a^2}{a^2 b^2}| = |\frac{(b+a)(b-a)}{a^2 b^2}| = \frac{|a+b|}{a^2 b^2}|a - b| \leq 4|a - b|$, because $|a + b| \leq 4$ and $a^2 b^2 \geq 1$.

12.8 Show that $|f(x_1) - f(x_2)| \leq \frac{|x_1 + x_2|}{(1+x_1^2)(1+x_2^2)}|x_1 - x_2|$. For $x_1, x_2 \in [-2, 2]$ then show the Lipschitz continuity with $L = 4$. It is, however, possible to do better and get $L = 3\sqrt{3}/8$, which is the maximum value attained by $\frac{|x_1 + x_2|}{(1+x_1^2)(1+x_2^2)}$, at $x_1 = x_2 = \pm 1/\sqrt{3}$. See the plot of this function below.

12.9 (a) $L = 100$

(b) $L = 10000$

(c) $L = 1000000$

12.10 For $x \neq y$ the Lipschitz inequality may be written $|f(x) - f(y)|/|x - y| \leq L$. Let $x = 1/n, y = 1/2n, n = 1, 2, 3, \ldots$ and observe that $|f(x) - f(y)|/|x - y| = 2 * n^2$, which is greater than any L for $n > \sqrt{L/2}$.

12.12 (a) For $x \neq y$ one can write the Lipschitz inequality as $|f(x) - f(y)|/|x -$

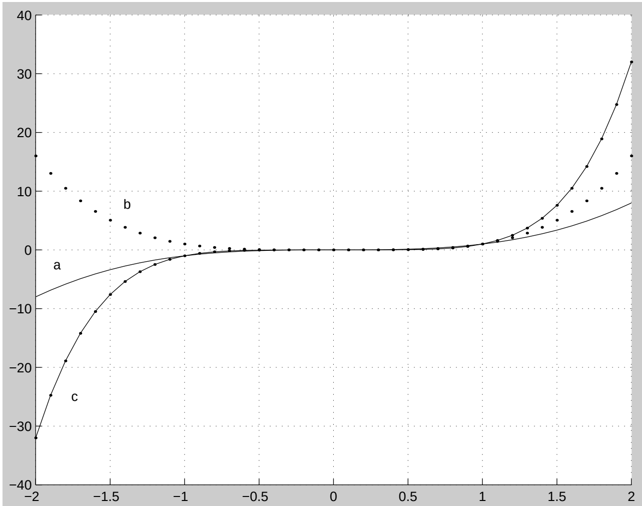


Figure 10: Plots of a) x^3, b) x^4 and c) x^5. Note that x^3 and x^5 are odd while x^4 is even. (Problem 10.23)

$y| \leq L$. With $x = 0$ and $y = -1/N$ we have $f(x) - f(y)|/|x - y| = |0 - 1|/|0 - (-1/N)| = N$, which is larger than any L for $N > L$.

(b) Yes!

12.13 If the Lipschitz constant L is extremly large then the function is close to discontinuous from a practical point of view.

12.14 Note misprint: $f_2 - f_2$ should be $f_1 - f_2$ and the Lipschitz constant of cf_1 should be $|c|f_1$. We have $|(f_1(x) - f_2(x)) - (f_1(y) - f_2(y))| \leq |f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| \leq (L_1 + L_2)|x - y|$ and $|cf_1(x) - cf_1(y)| \leq |c|\,|f_1(x) - f_1(y)| \leq |c|L|x - y|$.

12.15 Note first that a Lipschitz constant for $f(x) = x^n$ on $[-c, c]$ is nc^{n-1}, see Problem 12.14. Then using Theorem 12.1 we readily obtain the desired result.

12.17 Observe that $1/f_2$ is Lipschitz mith Lipschitz constant $1/m^2$, since $|1/f_2(x) - 1/f_2(y)| \leq |f_2(x) - f_2(y)|/m^2$. Now the Theorem follows from Theorem 12.4.

12.18 (a) Lipschitz with $L = 133$ using the formula in 12.14.

(b) Lipschitz with $L = 16/9$.

(c) Not Lipschitz continuous by Theorem 12.3, because not bounded on the given interval.

(d) Lipschitz with $L = 32$, use Theorem 12.6.

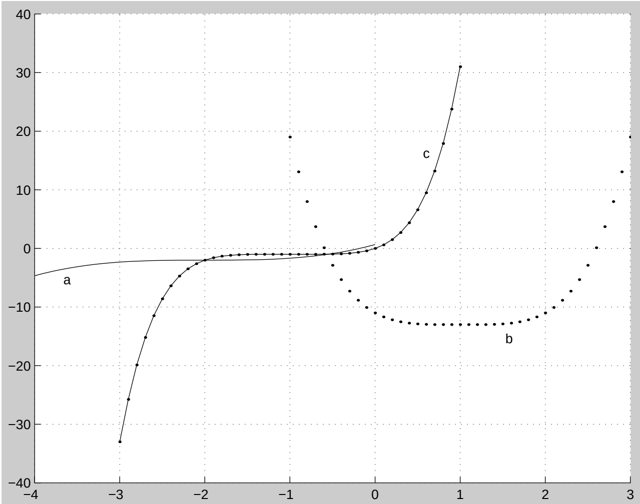


Figure 11: a) is plotted in interval [-4,0], b) is plotted in [-1,3], and c) in [-3,1]. (Problem 10.23)

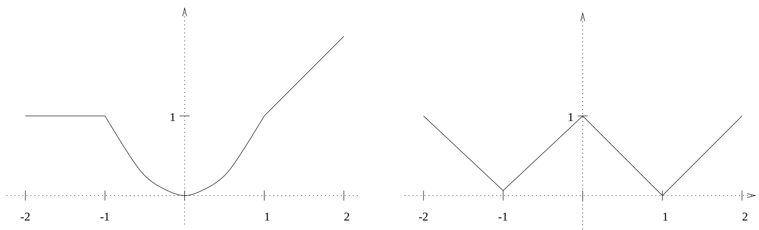


Figure 12: To the left problem a) and to the right b). (Problem 10.25)

12.19 Follows from Theorem 12.5 because $c_1x + c_2(1-x) \geq \min(c_1, c_2) > 0$ for $x \in [0,1]$.

Chapter 13

13.1

13.2

13.3 (a) $\{3^i\}_{i=0}^{\infty}$

(b) $\{4^i\}_{i=2}^{\infty}$

(c) $\{(-1)^i\}_{i=2}^{\infty}$

(d) $\{1+3i\}_{i=1}^{\infty}$

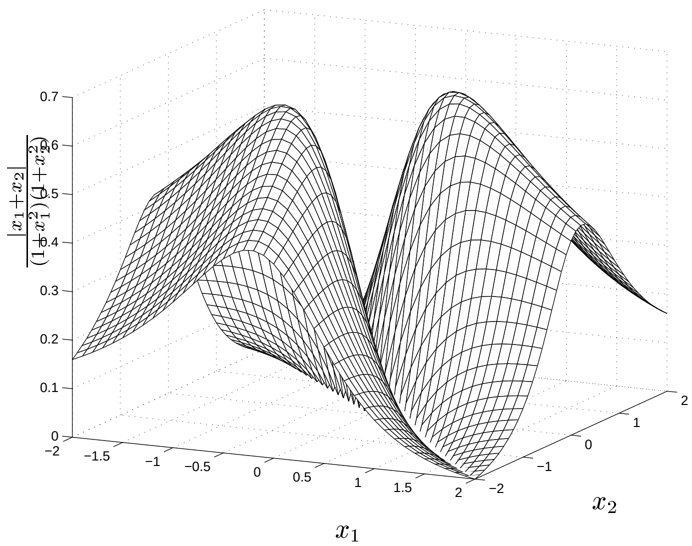


Figure 13: A plot of the function $\frac{|x_1+x_2|}{(1+x_1^2)(1+x_2^2)}$ in $[-2,2] \times [-2,2]$. (Problem 12.8)

(e) $\{3i-1\}_{i=1}^{\infty}$

(f) $\{5^{-i}\}_{i=-3}^{\infty}$

13.4 (a) $|\frac{8}{3n+1} - 0| = \frac{8}{3n+1} \leq \epsilon$ if $3n+1 \geq \frac{8}{\epsilon}$,
that is if $n \geq N$ where $N = \frac{8/\epsilon - 1}{3}$.

13.5 $|r^n - 0| \leq \epsilon$ if $(\frac{1}{2})^n \leq \epsilon$, that is if $2^n \geq \frac{1}{\epsilon}$.

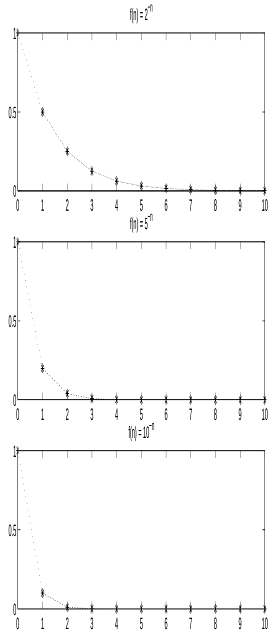
13.6

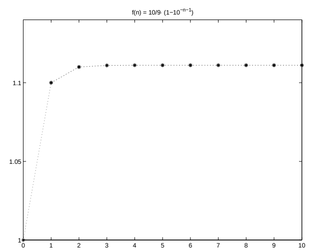
13.7 (a) Choose any $M > 0$. Now we have to show that there exists an N such
that $-4n + 1 < -M$ for all $n > N$. We see that this is true for $N = (M+1)/4$.

(b) If $\lim_{n\to\infty} n^3 = \infty$ then surely
$\lim_{n\to\infty} n^3 + n^2 = \infty$, since $n^3 < n^3 + n^2$
for $n \geq 1$. So it is sufficient to, for any $M > 0$, find an N such that $n^3 > M$, for $n > N$. This is true for $N = M^{1/3}$.

13.8 *Correction*: Should be $r > 1$, not $|r| \geq 2$.

For any $M > 0$, we want to show that there exists an N such that $r^n > M$ for all


f(n) = 2^-n
f(n) = 5^-n
f(n) = 10^-n
0 0.5 1
0 1 2 3 4 5 6 7 8 9 10


f(n) = 10/9 (1-10^-n-1)
1 1.05 1.1
0 1 2 3 4 5 6 7 8 9 10

$n > N$. But $r^n > M \Leftrightarrow n \ln r > \ln M \Leftrightarrow n > \ln(\frac{M}{r})$,
so let $N = \ln(\frac{M}{r})$.

13.9 (a) $\frac{1}{1-(-.5)} = 2/3$

(b) $3\frac{1}{1-\frac{1}{4}} = 4$

(c) $\frac{5^(-2)}{1-\frac{1}{5}} = \frac{1}{20}$.

13.10 (a) $\frac{1}{1-r^2}$

(b) $\frac{1}{1-(-r)} = \frac{1}{1+r}$.

13.11 All are equal to $\{\frac{2}{3}, \frac{4}{5}, \frac{8}{3}, \frac{16}{5}, \dots\}$
except (e) which is $\{\frac{4}{5}, \frac{8}{3}, \frac{16}{5}, \dots\}$.

13.12 (a) $\left\{\frac{2+(n+5)^2}{9^{n+5}}\right\}_{n=-4}^{\infty}$

(b) $\left\{\frac{2+(n-2)^2}{9^{n-2}}\right\}_{n=3}^{\infty}$

(c) $\left\{\frac{2+(n-1)^2}{9^{n-1}}\right\}_{n=2}^{\infty}$

13.13 STRYKES, triangelolikheten bör behandlas redan i kap
6, rational numbers.
The task is to prove the *triangle inequality*: $|a+b| \leq |a| + |b|$ for all a, b.
If $a \geq 0$, $b \geq 0$, then $|a+b| = a+b = |a|+|b|$.
If $a \leq 0$, $b \geq 0$, $a+b \geq 0$, then $|a+b| = a+b \leq -a+b = |a|+|b|$.
If $a \leq 0$, $b \geq 0$, $a+b \leq 0$, then $|a+b| = -a-b \leq -a+b = |a|+|b|$.
The remaining cases are proved in a similar way.
Proof of (13.7): $|a-b| = |a-c+c+b| \leq |a-c| + |c+b|$.

13.14 The triangle inequality gives $|(a_n - b_n) - (A-B)| = |(a_n - A) - (b_n - B)| \leq |a_n - A| + |b_n - B|$, where the right side can be made as small as desired by taking n sufficiently large. Another proof can be found in
Section 13.5.

13.15 If n is sufficiently large, then
$|a_n - A| \leq \frac{1}{2}|A|$ and $|b_n - B| \leq \frac{1}{2}|B|$, so that
$|a_n| = |a_n - A + A| \leq |a_n - A| + |A| \leq \frac{3}{2}|A|$, and
$|B| = |B - b_n + b_n| \leq |B - b_n| + |b_n| \leq \frac{1}{2}|B| + |b_n|$, and
$|b_n| \geq \frac{1}{2}|B|$, and $\frac{1}{|b_n|} \leq \frac{2}{|B|}$. For large
n we thus get

$\frac{a_n}{b_n} - \frac{A}{B} = \frac{a_n}{b_n} - \frac{a_n}{B} + \frac{a_n}{B} - \frac{A}{B} = \frac{a_n}{b_n B}(B - b_n) + \frac{1}{B}(a_n - A)$
$\leq \frac{|a_n|}{|b_n||B|}|B - b_n| + \frac{1}{|B|}|a_n - A| \leq 3\frac{|A|}{|B|^2}|b_n - B| + \frac{1}{|B|}|a_n - A|$

where the right side can be made as small as desired

by taking n sufficiently large.

Another proof can be found in Section 13.5.

13.16 (a) 1

(b) divergent to $+\infty$, because $a_n = n^2(4 - 6n^{-1}) \geq n^2$ for $n \geq 2$

(c) 0, because $|a_n - 0| = n^{-2}$

(d) 1/3

(e) divergent, because $a_n = \frac{(-1)^n}{7+n^{-2}}$ flips (approximately) between $\frac{1}{7}$ and $-\frac{1}{7}$ when n is large

(f) 2

(g) -4 (all a_n equal -4)

(h) $-5/8$

(i) divergent to $+\infty$, because $a_n = n\frac{2+n^{-2}+n^{-3}}{6-5n^{-2}} \geq 2n$

(j) -1

13.19 $|0.99\cdots99_n - 1| = (0.1)^n = 0.00\cdots01_n \leq \epsilon$

for $n \geq N$, if N is

the index of the first non-zero decimal in ϵ.

Note also that, using the geometric sum,

$$0.99\cdots99_n = 0.9\sum_{k=0}^{n-1}(0.1)^k = 0.9\frac{1-(0.1)^n}{1-0.1} = 1-(0.1)^n.$$

Chapter 14

14.2 (a) See (b).

(b) Assume $\sqrt{p} = a/b$ where largest common divisor of a and b is 1. Then $b^2p = a^2$, and since p is prime $a = p\alpha$ for some α, and thus $b^2p = p^2\alpha^2$ or $b^2 = p\alpha^2$ and thus $b = p\beta$. This is a contradiction, since p divides both a and b. Make sure you understand all details.

14.3 $4^{1/3}, 3^{1/4}, 4^{1/4}$, etc.

14.9 Assumption give $|b| = b < b - a < c$ and $|a| = -a = (b-a) - b < c - b < c$

Chapter 15

15.2 (b) Have that $|xy - x_i y_i| = |(x-x_i)y + x_i(y-y_i)| \le |(x-x_i)y| + |x_i(y-y_i)| = |y||x-x_i| + |x_i||y-y_i| \le |y|2^{-i} + (|x|+0.1)2^{-i}$, where we used the fact $|x_i| = |(x_i - x) + x| \le |x_i - x| + |x| \le 2^{-i} + |x| \le 0.1 + |x|$ for $i \ge 4$.

15.4 $x = 0.373737...$ and $y = \sqrt{2} = 1.414213..$ give $x_1 y_1 = 0.3 \times 1.4 = 0.42$, $x_2 y_2 = 0.37 \times 1.41 = 0.5217$, $x_3 y_3 = 0.373 \times 1.414 = 0.527422$, etc.

15.5 No, because if the limit x would be less than 1 then $d = (1-x)/2$ is positive, and $\frac{i}{i+1} = \frac{i+1-1}{i+1} = 1 - \frac{1}{i+1} \ge 1 - d = x + d$ for $\frac{1}{i+1} \le d$, that is for $i \ge \frac{1}{d} - 1$, which contradict the assumption that $\{\frac{i}{i+1}\}$ converges to x.

15.8 (a) $\{x \in R : -2\sqrt{2} \le x \le 5\sqrt{2}\}$

(b) $\{x \in R : x < 2\sqrt{2} - 2/3 \text{ or } x > 2\sqrt{2} + 2/3\}$

15.11 (a) The sequence is $\{\frac{1}{i^2}\}$ (ok to shift the index since we are only concerned with the *limit*). For $|\frac{1}{i^2} - \frac{1}{j^2}| = |\frac{j^2 - i^2}{i^2 j^2}| = \frac{|j^2 - i^2|}{i^2 j^2} \le \frac{j^2 + i^2}{i^2 j^2} = \frac{1}{i^2} + \frac{1}{j^2} \le \epsilon$ if $i, j \ge N$ and $N = \frac{1}{\sqrt{2\epsilon}}$.

15.12 Assume that i^2 is a Cauchy sequence. Choose $\epsilon > 0$ and N, and take $j = N$ and $i = j + 1$. Compute $|i^2 - j^2|$ and derive a contradiction.

15.13 (b) 1/3

15.15 Let $\bar{c}$ denote the smallest of all c:s. Choose an $\epsilon > 0$. Then $\bar{c} - \epsilon \le x_i \le \bar{c}$ for all $i > N(\epsilon)$. So $\bar{c}$ is the limit by the formal definition.

15.18 $\sqrt{2} = 1.414...$ gives $f(1.4) = \frac{1.4}{1.4+2} = 0.4117647....$, $f(1.41) = \frac{1.41}{1.41+2} = 1.41348973...$, $f(1.414) = 0.4141769185...$, etc. (Hmm, looks familiar, like $\sqrt{2} - 1$. Could it be that $f(\sqrt{2}) = \frac{\sqrt{2}}{\sqrt{2}+2} = \sqrt{2} - 1$? Check!).

15.19 6

15.23 (a) $(-2, 4]$

(b) $(-3, -1) \cup (-1, 2]$

(c) $[-2, -2] \cup [0, \infty)$

(d) $(-\infty, 0) \cup (1, \infty)$

15.24 $[2, 3)$

15.28 For $a, b > \delta$ one has $|f(a) - f(b)| = |\sqrt{a} - \sqrt{b}| = |\frac{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})}{\sqrt{a}+\sqrt{b}}| = |\frac{a-b}{\sqrt{a}+\sqrt{b}}| = \frac{|a-b|}{\sqrt{a}+\sqrt{b}} \le L|a-b|$ where $L = \frac{1}{2\sqrt{\delta}}$

Chapter 16

Problems

16.1 Use the Bisection Algorithm to find a solution, accurate to within 10^{-2}, to the equation $x + 0.5 + 2\cos \pi x = 0$ on the interval $[0.5, 1.5]$.

16.2 Use the Bisection Algorithm to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}.

16.3 Find a bound for the number of iterations needed to approximate a solution to the equation $x^3 + x - 4 = 0$ on the interval $[1, 4]$ to an accuracy of 10^{-3}.

16.4 A trough of water of length $L = 10$ feet has a cross section in the shape of a semicircle with radius $r = 1$ foot. When filled with water to within a distance h of the top, the volume $V = 12.4 ft^3$ of the water is given by the formula

$$12.4 = 10[0.5\pi - \arcsin h - h(1 - h^2)^{1/2}].$$

Determine the depth of the water to within 0.01 feet.

16.5 Suppose $f(x)$ has only simple roots in (a, b). If $f(a)f(b) < 0$, show that there are an odd number of roots of $f(x) = 0$ in (a, b). If $f(a)f(b) > 0$, show that there are an even number (possibly zero) of roots of $f(x) = 0$ in (a, b).

16.6 Show that the Bisection method converges linearly, that is, $\lim_{n\to\infty} \frac{r_{n+1}-r}{r_n-r}$ is constant.

16.7 Find all the roots of the function $f(x) = \cos x - \cos 3x$.

16.8 Find the root or roots of $\ln[(1 + x)/(1 - x^2)] = 0$.

16.9 Find where the graphs of $y = 3x$ and $y = e^x$ intersect by finding roots of $e^x - 3x = 0$ correct to four decimal digits.

16.10 Consider the bisection method, determine how many steps are required to guarantee an approximation of a root to six decimal places (rounded).

16.11 By graphical methods, locate approximations to all roots of the nonlinear equation $\ln(1 + x) + \tan(2x) = 0$.

16.12 Equation $xe^x - 2 = 0$ has a simple root r in $[0, 1]$. Use the bisection method to estimate r within seven decimal digits.

16.13 Use the bisection method to find, as accurately as you can, all real roots for each equation.

(a) $x^3 - x^2 - x - 1 = 0$

(b) $x^2 = e^{-x^2}$

(c) $\ln|x| = \sin x$

16.14 A certain technical problem requires solution of the equation

$$21.13 - \frac{3480}{T} - 5.08 \log T = 0$$

for a temprature T. Technical information indicates that the temperature should lie between 400° and 500°. Use the bisection method to estimate the desired temperature to nearest degree.

16.15 Use the bisection method with some calculus to find the minimum value of $f(x) = \sin x/x$ on interval $[\pi, 2\pi]$.

Answers

16.1 $r_7 = 0.711$

16.2 $\sqrt{3} \approx r_{14} = 1.7320$

16.3 $r_{12} = 1.3787$

16.4 $h \approx r_{13} = 0.1617$ so the dept is $r - h \approx 1 - 0.1617 = 0.838$ feet

16.7 $\{0, \pm\pi/2, \pm\pi, \pm 3\pi/2, \pm 2\pi \dots\}$

16.8 $x = 0$

16.9 0.61906, 1.51213

16.10 20 steps

16.11 $\{0,\ \frac{\pi}{4} + \varepsilon,\ \frac{3\pi}{4} + \varepsilon,\ \frac{5\pi}{4} + \varepsilon, \dots\}$, where ε starts at approximatelt 1/2 and decreases.

16.12 $r = 0.8526055$

16.14 475°

Chapter 18

18.2 Yes $L = 1,\ \theta = 1/3$

18.3 $(b),\ (c)$

18.4 No

18.6 No

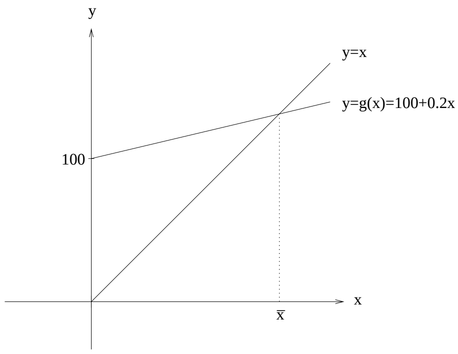


Figure 14: Problem 19.1

Chapter 19

19.1 Break even if sales x equals expenses $100 + 0.2x$, that is if $x = g(x)$ where $g(x) = 100 + 0.2x$.

19.2 (a) For example $f(x) = \frac{x^3-1}{x+2} - x = 0$ or $f(x) = x^3 - 1 - x(x+2) = x^3 - x^2 - 2x - 1 = 0$

(b) For example $f(x) = x^5 - x^3 + 4 - x = 0$ or $f(x) = \frac{x^5-x^3}{x-4} - 1 = 0$.

19.3 (a) For example $x = g(x) = x - 0.1(7x^5 - 4x^3 + 2)$ or $x = g(x) = (\frac{7}{4}x^5 + \frac{1}{2})^{1/3}$

(b) For example $x = g(x) = x + 0.3(\frac{2}{x} - x^3)$ or $x = g(x) = \frac{2}{x^3}$ or $x = g(x) = (3x + \frac{2}{x^3})/4$.

19.4 Skippas

```
functionx = fixedpoint(g,x0,max_iter,tol)
iter = 0;
xold = x0;
x = x_old;
x_new = eval(g);
19.5  whileiter < max_iter&abs(x_new - x_old) > tol
  x = x_old;
  x_new = eval(g);
  iter = iter + 1;
end
x = x_new;
```

19.6 Rewrite the equation $x(0.02 + 2x)^2 = 1.8\,10^{-5}$ as $100\,x(2 + 2\,100\,x)^2 = 18$, and rescale by introducing $y = 100x$ to obtain $y(2 + 2y)^2 = 18$. Writing this as $y = g(y) = \frac{18}{(2+2y)^2}$, the fixed point iterations $y_{j+1} = g(y_j)$ does not converge. Try instead $y = g(y) = (y + \frac{18}{(2+2y)^2})/2$ for which the iteration

$y_{j+1} = g(y_j)$ with $y_0 = 2$ gives the sequence

2.00000000000000
1.06944444444444
1.06010396178350
1.06020866771593
1.06020713377205
1.06020715618506
1.06020715585756
1.06020715586235

that is, $x = y/100 \approx 0.010602071559$.

19.7 Rewrite the equation $x(0.037+2x)^2 = 1.57\,10^{-9}$ with $y = 1000x$ as $y(37+2y)^2 = 1.57$. Write this as $y = g(y) = \frac{1.57}{(37+2y)^2}$ and compute $y_{j+1} = g(y_j)$ with $y_0 = 1$ to obtain

1.00000000000000
0.00114669453395
0.00114668034327
0.00114668034503

that is, $x = y/1000 \approx 0.0000011466803$.

19.8 Rewrite equation $1 = \frac{4^2 R}{(3+R)^2}$ with $x = R$ as $x = g(x) = x - 1.5(\frac{16x}{(3+x)^2} - 1)$ and compute iterates $x_{j+1} = g(x_j)$ with $x_1 = 2$. This gives the sequence of iterates

2.00000000000000
1.27225415228543
1.09935039113477
1.02928661189918
1.00771903960222
1.00195760369188
1.00049119586678
1.00012291204137
1.00003073509157
1.00000768421569
1.00000192108160
1.00000048027213
1.00000012006814
1.00000003001704
1.00000000750426
1.00000000187607
1.00000000046902
1.00000000011725
1.00000000002931

converging to 1.

Iterating with $g(x) = x + 20(\frac{16x}{(3+x)^2} - 1)$ and $x_0 = 8$ gives a sequence of iterates

8.00000000000000
8.98369412815009
9.00182407284279
8.99979747927866
9.00002250420120
8.99999749955665
9.00000027782733
8.99999996913030
9.00000000342996
8.99999999961889
9.00000000004234

converging to 9. (Equation can also be solved analytically for R)

19.9 Rewrite the equation $(2+\frac{50}{V^2})(V-0.011) = 3\times 15$ with $x = V$ as $x = g(x) = 0.011 + 45/(2+50/x^2)$ and compute iterates $x_{j+1} = g(x_j)$ with $x_0 = 20$ to obtain

20.00000000000000
21.32406304182749
21.33843041035536
21.33992672744253
21.34008240256070
21.34009859707175
21.34010028172915
21.34010045697786
21.34010047520834
21.34010047710479
21.34010047730207
21.34010047732259

19.10 Proof by induction: True for $n = 3$. Assume true for n. Then

$$\begin{aligned} x_{n+1} &= \tfrac{1}{4}x_n + \tfrac{1}{4} = \tfrac{1}{4}(\tfrac{1}{4^n}x_0 + \textstyle\sum_{i=1}^{n}\tfrac{1}{4^i}) + \tfrac{1}{4} \\ &= \tfrac{1}{4^{n+1}x_0} + \textstyle\sum_{i=2}^{n+1}\tfrac{1}{4^i} + \tfrac{1}{4} = \tfrac{1}{4^{n+1}x_0} + \textstyle\sum_{i=1}^{n+1}\tfrac{1}{4^i}, \end{aligned}$$

showing that formula valid for $n+1$.

19.11 (a) $x_n = 2^n x_0 + \frac{1}{4}\sum_{i=1}^{n} 2^i$

(b) For any given $M > 0$ we have that $x_n > M$ if n is large enough, because $\sum_{i=0}^{n} 2^i = (1-2^{n+1})(1-2) = 2^{n+1} - 1$.

19.12 (a) $x_n = (\frac{3}{4})^n + \sum_{i=1}^{n}\frac{3^{i-1}}{4^i}$

(b) First we point out that $lim_{n\to}\inf(\frac{3}{4})^n \to 0$. Then we look at the sum $\sum_{i=1}^{n}\frac{3^{i-1}}{4^i} = \frac{1}{3}\sum_{i=1}^{n}(\frac{3}{4})^i = \frac{1}{3}\frac{1-(\frac{3}{4})^{(n+1)}}{1-\frac{3}{4}} = \frac{1}{3}\frac{1}{1-\frac{3}{4}} = \frac{4}{3}$

19.13 $x_n = m^n x_0 + b\frac{1-m^n}{1-m}$

19.14

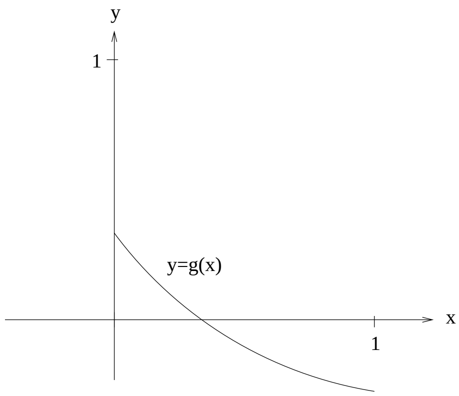


Figure 15: Problem 19.16.

We show that $g(x) = mx + b$ is a contraction mapping: $|g(x) - g(y)| = |mx + b - (my + b)| = |m||x - y| = L|x - y|$. $L < 1$ so g is a contraction mapping and the fixpoint iteration has a unique solution by Theorem 19.1. The solution is $\bar{x} = b/(1 - m)$.

19.14 Draw for example the function $g(x) = 2x - 3$ for which $g(x) \in [0, 1]$, when $x \in [1.5, 2]$.

19.15 Need to know the specific fixpoint functions used in the 19.3 problem to solve this problem.

19.16 If $g'(x)$ is bounded in the interval then $g(x)$ is Lipschitz continious in the interval. $g'(x) = \frac{2x}{(1+x^2)^2} \Rightarrow L = \max_{x \in [a,b]} |\frac{2x}{(1+x^2)^2}| \leq 1 \Rightarrow g : [a, b] \to [a, b]$, that is g is a contraction mapping. By theorem 19.2 we now have that if the starting point in the fixed point iteration $x_0 \in [a, b]$ then the sequence given by the iteration converges to a $\bar{x} \in [a, b]$.

19.17 Using $|x_{k+1} - x_k| \leq L^k |x_1 - x_0|$ we compute $(|x_{k+1} - x_k|/|x_1 - x_0|)^{1/k}$ for $k = 1, 2, 3, 4$ with data from the table. The result is 0.875 for $k = 1, 2, 3, 4$, hence $L = 0.875$.

19.18 If $g'(x)$ is bounded in the interval then $g(x)$ is Lipschitz continious in the interval. $g'(x) = \frac{4x^3(10-x)^2+2(10-x)x^4}{(10-x)^4} \Rightarrow$ $L = \max_{x \in [-1,1]} |\frac{4x^3(10-x)^2+2(10-x)x^4}{(10-x)^4}| = |\frac{4}{(10-1)^2} + \frac{2}{(10-1)^3}| = \frac{38}{729} < 0.053 < 1 \Rightarrow g$ is a contraction mapping.By theorem 19.2 we now have that if the starting point in the fixed point iteration $x_0 \in [-1, 1]$ then the sequence given by the iteration converges to a $\bar{x} \in [-1, 1]$. g is not a contraction mapping in [-9.9,9.9]. $L = \max_{x \in [-9.9,9.9]} | \frac{4x^3(10-x)^2+2(10-x)x^4}{(10-x)^4}| = |\frac{4 \cdot 9.9^3}{(10-9.9)^2} + \frac{2 \cdot 9.9^4}{(10-9.9)^3}| < 2 \cdot 10^7$, which is larger than 1.

19.19 Using the method from Problem 19.17 we get the estimates for the Lipschitz constant to be 0.6954, 0.6152, 0.5867, 0.5683 for $k = 1, 2, 3, 4$, respectively. Alternatively we can compute $|x_{k+1} - x_k|/|x_k - x_{k-1}|$ which gives 0.6954 , 0.5443 , 0.5334 , 0.5165 for $k = 1, 2, 3, 4$, respectively. Both these computations show that the convergence is not linear.

19.20 (a) If $g'(x)$ is bounded in the interval then $g(x)$ is Lipschitz continious in the interval. $g'(x) = 2x^2$. $L = \max_{x\in[-1/2,1/2]} |g'(x)| = |2 \cdot (\frac{1}{2})^2| = 0.5$.

(b)

(c) $x_i = g(x_{i-1}) = g(\bar{x} + x_{i-1} - \bar{x}) = \frac{2}{3}(\bar{x} + (x_{i-1} - \bar{x}))^3$. Using the fact that $\bar{x} = 0$ we get: $|x_i - \bar{x}| = |\frac{2}{3}(x_{i-1} - \bar{x})^3|$

19.21 Use amongst other things that $x_{i-1} \approx \sqrt{2}$.

19.22 (a) $x^2 + x - 6 = 0 \rightarrow x(x+1) = 6 \rightarrow x = \frac{6}{(x+1)}$. The error is estimated by $|x_i - \bar{x}| = |g(x_{i-1}) - g(\bar{x})| = \frac{6}{x_{i-1}+1} - \frac{6}{\bar{x}+1} \leq |\frac{6(\bar{x}-x_{i-1})}{(\bar{x}+1)^2}| \leq \frac{2}{3}|\bar{x} - x_{i-1}|$, when the sequense of the fixed point iteration has converged and $x_{i-1} \leq \bar{x}$.

(b) $x^2 + x - 6 = 0$ adding x^2 on each side gives: $2x^2 + x = x^2 + 6 \rightarrow x = \frac{x^2+6}{2x+1}$. The error is estimated using $\bar{x} = 2$ and $x_i = g(\bar{x} + (x_{i-1} - \bar{x})) = \frac{\bar{x}+(x_{i-1}-\bar{x})^2+6}{2(\bar{x}+(x_{i-1}-\bar{x}))+1} = \frac{4x_{i-1}+2+(x_{i-1}-\bar{x})^2}{4+1}$. $|x_i - \bar{x}| = |2 + \frac{(x_{i-1}-\bar{x})^2}{5} - 2| = |\frac{(x_{i-1}-\bar{x})^2}{5}|$

19.23 The equation for the line through the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ is given by $y = f(x_{i-1}) + \frac{x-x_{i-1}}{x_i-x_{i-1}}(f(x_i) - f(x_{i-1})$ which for $y = 0$ has the solution $x_{i+1} = x_{i-1} - f(x_i)(x_i - x_{i-1})/(f(x_i) - f(x_{i-1}))$. Convergence factor?

Chapter 20

20.1 Proof (example, analytical) of $\lambda(\mu a) = (\lambda\mu)a$: By definition

$$\lambda(\mu a) = \lambda(\mu(a_1, a_2)) = \lambda(\mu a_1, \mu a_2) = (\lambda(\mu a_1), \lambda(\mu a_2)).$$

and

$$(\lambda\mu)a = (\lambda\mu)(a_1, a_2) = ((\lambda\mu)a_1, (\lambda\mu)a_2).$$

The desired identity thus follows from the associative law for real number multiplication.

20.2 $f(x) = x + 2(P_a(x)x - x) = 2P_a(x) - x = 2\frac{a\cdot x}{|a|^2}a - x$. The corresponding matrix is

$$\begin{pmatrix} 2\frac{a_1^2}{|a|^2} - 1 & 2\frac{a_1 a_2}{|a|^2} \\ 2\frac{a_1 a_2}{|a|^2} & 2\frac{a_2^2}{|a|^2} - 1 \end{pmatrix}$$

20.3 (a) $\sqrt{13}$ (b) $\sqrt{17}$ (c) $\sqrt{52}$ (d) $\sqrt{8}$ (e) $(3,2)/\sqrt{13}$ (f) $(1,4)/\sqrt{17}$

20.4 $|\frac{a}{|a|}| = |(\frac{a_1}{|a|}, \frac{a_2}{|a|})| = \sqrt{(\frac{a_1}{|a|})^2 + (\frac{a_2}{|a|})^2} = \sqrt{\frac{a_1^2}{|a|^2} + \frac{a_2^2}{|a|^2}} = \sqrt{\frac{a_1^2+a_2^2}{|a|^2}} = \sqrt{1} = 1$

20.5 (b) $a \cdot b = |a||b|\cos(\theta) \leq |a||b|$. (a) $|a+b|^2 = (a_1+b_1)^2 + (a_2+b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2a_1b_1 + 2a_2b_2 = |a|^2 + |b|^2 + 2a \cdot b \leq |a|^2 + |b|^2 + 2|a||b| = (|a|+|b|)^2$

20.6 (a) 7 (b) 5 (c) 0

20.7 (a), (c) and (e) makes sense.

20.8 $\theta = \arccos(\frac{10}{\sqrt{2}\sqrt{58}})$

20.9 All $a = (a_1, a_2)$ such that $2a_1 + a_2 = 2$. A line in the a_1, a_2 plane, with normal $(2,1)$ passing through, for example, the point $(0,2)$.

20.10 (a) $P_b(a) = \frac{b \cdot a}{|b|^2} b = \frac{5}{5}(1,2) = (1,2)$ (b) $P_b(a) = \frac{0}{5}(1,2) = (0,0)$ (c) $P_b(a) = \frac{6}{5}(1,2)$ (d) $P_b(a) = \frac{3\sqrt{2}}{5}(1,2)$

20.11 $b = c + d$ where $d = b - c$ and (a) $c = P_a(b) = \frac{a \cdot b}{|a|^2} a = \frac{13}{5}(1,2)$ (b) $c = \frac{-1}{5}(-2,1)$ (c) $c = \frac{16}{8}(2,2)$ (d) $c = \frac{8\sqrt{2}}{4}(\sqrt{2}, \sqrt{2})$

20.12 $|c|^2 = |a-b|^2 = (a_1-b_1)^2 + (a_2-b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2(a_1b_1 + a_2b{-}2) = |a|^2 + |b|^2 - 2a \cdot b = |a|^2 + |b|^2 - 2|a||b|\cos(\phi)$

20.13 See previous problem.

20.14 (a) $Ax = (5,11)^\top$ and $A^\top x = (7,10)^\top$ (b) $Ax = (3,7)^\top$ and $A^\top x = (4,6)^\top$

20.15 (a) $\begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$ (b) $\begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$ (c) $\begin{pmatrix} 26 & 30 \\ 38 & 44 \end{pmatrix}$ (d) $\begin{pmatrix} 17 & 23 \\ 39 & 53 \end{pmatrix}$ (e) $\begin{pmatrix} 19 & 43 \\ 22 & 50 \end{pmatrix}$ (f) $\begin{pmatrix} 19 & 43 \\ 22 & 50 \end{pmatrix}$ (g) $\begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$ (h) $\begin{pmatrix} -4 & 3 \\ 3.5 & -2.5 \end{pmatrix}$ (i) $\begin{pmatrix} 12.5 & -5.5 \\ -10.75 & 4.75 \end{pmatrix}$ (j) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

20.16 The matrix element in row i and column j of $(AB)^\top$ (which is the same as in row j and column i of AB) is the scalar product of row j of A and column i of B. The matrix element in row i and column j of $B^\top A^\top$ is the scalar product of row i of $B^\top$ and column j of $A^\top$, that is, of column i of B and row j of A. That is, the matrices $(AB)^\top$ and $B^\top A^\top$ have the same elements and are therefore equal.

20.17 (a) A is symmetric. (b) A is invertible with inverse B.

20.18 The 2×2-matrix P corresponding to the projection $P_a(b)$ is

$$\frac{1}{|a|^2}\begin{pmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{pmatrix}$$

Obviously, $P^\top = P$. Computing, one finds that

$$PP = \frac{1}{|a|^4}\begin{pmatrix} a_1^4 + a_1^2a_2^2 & a_1^3a_2 + a_1a_2^3 \\ a_1^3a_2 + a_1a_2^3 & a_1^2a_2^2 + a_2^4 \end{pmatrix} = \frac{1}{|a|^4}\begin{pmatrix} a_1^2|a|^2 & a_1a_2|a|^2 \\ a_1a_2|a|^2 & a_2^2|a|^2 \end{pmatrix} = P$$

20.19 $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ rotates x the angle θ counter clock-wise.

20.20 See the section about Reflection above!

20.21 (a) -4 (b) 0 (c) 10

Chapter 21

21.2 Only the rightmost one.

21.4 $(-7, 2, 1)$

21.5 2

21.6 $\sqrt{72}/2$

21.7 (a) $\arccos(\frac{5}{\sqrt{11}\sqrt{3}})$ (b) $\frac{5}{\sqrt{11}\sqrt{3}}(1,1,1)$ (c) $(1,0,-1)/\sqrt{2}$ (or $(-1,0,1)/\sqrt{2}$)

21.8 $(-1, 0, 1)$

21.9 (a) true, (b) true, (c) true

21.12

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

21.14

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2}$$

21.17 $(-1, -3, 6)$, exception.

21.18 Intersection of the two planes: $\lambda(2, -1, -1)$, intersection of two planes with the $x_1 - x_2$ plane: $(0, 0, 0)$, of course.

21.20 $r + \lambda b(a - 2P_na)$, $(\lambda \geq 0)$, where (see figure) $b = a - 2P_na$, $P_na = \frac{a \cdot n}{|n|^2}n$.

Chapter 22

22.2 (a) $\frac{1}{2}$ (b) $\frac{31}{50}$ (c) $\frac{2xy}{x^2+y^2}$ if $z = x + iy$.

22.3 (a) $23 + i2$ (b) $\frac{-7-i22}{13}$ (c) $\frac{34-i22}{40}$

22.5 (a) $\sqrt{2}(\cos(45^o), \sin(45^o))$
(b) $(\cos(90^o), \sin(90^o))$
(c) $\frac{\sqrt{13}}{\sqrt{41}}(\cos(\theta - \phi), \sin(\theta - \phi))$, where $\theta = \text{Arg}(2 + 3i)$, $\phi = \text{Arg}(5 + 4i)$.

22.6 (a) $z_1 = (\cos(45^o), \sin(45^o))$, $z_2 = (\cos(135^o), \sin(135^o))$
(b) $z_i = (\cos(i * 45^o), \sin(i * 45^o))$, $i = 1, 2, .., 8$.
(c) $z_1 = -\frac{1}{2}+\sqrt{r}(\cos(\theta/2), \sin(\theta/2))$ and $z_2 = -\frac{1}{2}+\sqrt{r}(\cos(\theta/2+180^o), \sin(\theta/2+180^o))$, where $r = |-3/4 - i| = 5/4$ and $\theta = \text{Arg}(-3/4 - i)$.
(d) hint: first solve for $w = z^2$ to find that

$$w_1 = \frac{3}{2}(1 + 2i) + \sqrt{r}(\cos(\theta/2), \sin(\theta/2))$$

and

$$w_2 = \frac{3}{2}(1 + 2i) + \sqrt{r}(\cos(\theta/2 + 180^o), \sin(\theta/2 + 180^o)),$$

where $r = |27/4 - 15i|$ and $\theta = \text{Arg}(27/4 - 15i)$. Then solve $z^2 = w_i$, $i = 1, 2$.

22.7 (a) $\{(0, y) : y \in\}$ (To see why, rewrite as $|z - (-i)| = |z - i|$)
(b) $\{(x, y) : xy = 1\}$ (Because $z^2 = (x^2 - y^2, 2xy)$)
(c) $\{(x, y) : |y| \leq x\}$.

22.8 If $z = r(\cos(\theta), \sin(\theta))$ and $\zeta = \rho(\cos(\phi), \sin(\phi))$, then $z/\zeta = (r/\rho)(\cos(\theta - \phi), \sin(\theta - \phi))$.

22.10 (a) The complex plane is first rotated around the origin the angle Arg a and streched by the factor $|a|$, through the mulitiplication of z by a, then translated by the addition of b.
(b) The complex number $z = r(\cos(\theta), \sin(\theta))$ is mapped onto the complex number $r^2(\cos(2\theta), \sin(2\theta))$, that is the argument is doubled and the modulus squared.

Chapter 23

23.1 Write $x^3 = (\bar{x} + x - \bar{x})^3 = \bar{x}^3 + 3\bar{x}^2(x - \bar{x}) + 3\bar{x}(x - \bar{x})^2 + (x - \bar{x})^3$. This leads to the identity $x^3 = \bar{x}^3 + 3\bar{x}^2(x - \bar{x}) + E_f(x, \bar{x})$, with the error term $E_f(x, \bar{x}) = 3\bar{x}(x - \bar{x})^2 + (x - \bar{x})^3$. Note that $|E_f(x, \bar{x}| = |2\bar{x} + x|(x - \bar{x})^2$, and thus the derivative of x^3 is $3x^2$. The proof for x^4 is similar.

23.2 The error term is $E_f(x,\bar{x}) = \sqrt{x} - \sqrt{\bar{x}} - (x-\bar{x})/2\sqrt{\bar{x}} = (1/(\sqrt{x} + \sqrt{\bar{x}}) - 1/2\sqrt{\bar{x}})(x-\bar{x})$. Furthermore $1/(\sqrt{x}+\sqrt{\bar{x}}) - 1/2\sqrt{\bar{x}} = \sqrt{\bar{x}} - \sqrt{x})/(\sqrt{\bar{x}}+\sqrt{x})2\sqrt{\bar{x}} = (\bar{x}-x)/(\sqrt{\bar{x}}+\sqrt{x})^2 2\sqrt{\bar{x}}$. Collecting the results we get $E_f(x,\bar{x}) \leq K_f(x,\bar{x})(x-\bar{x})^2$ with $K_f(x,\bar{x}) = 1/|(\sqrt{\bar{x}}+\sqrt{x})^2 2\sqrt{\bar{x}}| \approx 1/8\bar{x}^{3/2}$, for x close to $\bar{x}$.

23.3 We calculate the derivative of $\sqrt{x}$ at $\bar{x} = 0.5$ using the difference quotient $f'(\bar{x}) \approx f_h'(\bar{x}) = (f(\bar{x}+h) - f(\bar{x}))/h$ for $h_j = 2^{-j}$ for $j = 0, 1, \ldots, 40$ using matlab. Then we calculate the error in the numerical approximation $e_h(\bar{x}) = |f'(\bar{x}) - f_h'(\bar{x})|$. Using formula (23.27) we get $h_{optimal} = \sqrt{eps/K_f}$, where eps is the smallest number in Matlab and $K_f(x,\bar{x}) \approx 1/(8\bar{x}^{3/2})$. See the figure.

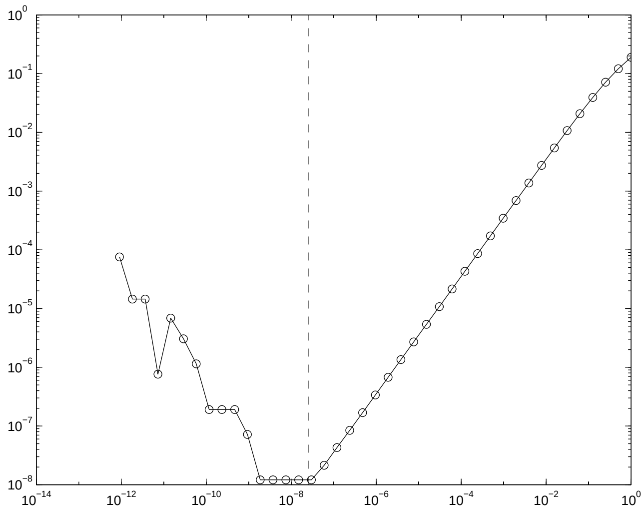


Figure 16: The error e_h in the numerical derivative as a function of h and the predicted optimal choice of h marked by a vertical dashed line. Note that this is a log-log plot! (Problem 23.3)

23.4 Using Taylors formula and proceeding in the same way as in Chapter 23.13 we get the following formula for the optimal choice of h: $h_{optimal} = (eps/K_f)^{1/3}$, with $K_f = f'''(\bar{x})/6$. For $f(x) = \sqrt{x}$ we show the error in the difference quotient as a function of h as well as the the predicted optimal h (vertical dashed line).

23.5 For simplicity, compute the derivative at $\overline{x} = 1$. Then the relative error for a specific choice of $h = x - \overline{x}$ is $e_n(h) = \frac{\frac{(1+h)^n - 1}{h} - n}{n}$. The relative errors for a few different choices of n are plotted as function of h in the figure below. For $n = 1$ one should choose a large value of h since the linearization error

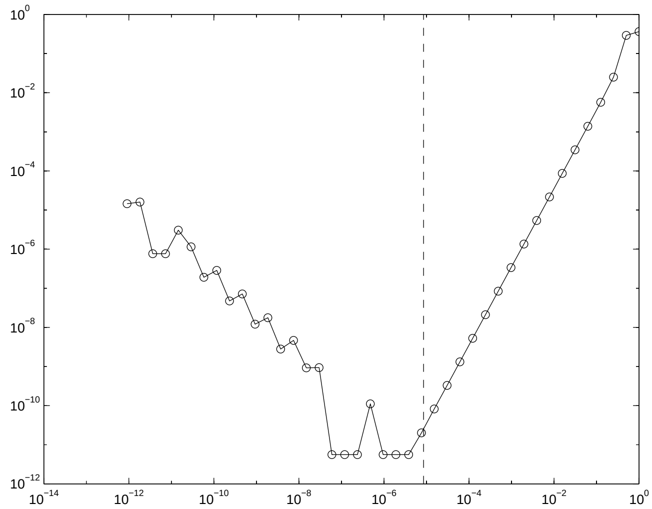


Figure 17: The error e_h in the numerical derivative as a function of h and the predicted optimal choice of h marked by a vertical dashed line. Note that this is a log-log plot! (Problem 23.4)

is zero and we only need to worry about round-off/computational error. For larger n there is an optimal value of h.

23.6 Perhaps the correct answer to this question is no, since we have not yet defined $\sin(x)$ and $\cos(x)$, but we still may find the correct answer.

Two alternatives: (i) Realize geometrically that $\sin(x) \approx x$ for small $|x|$. Then use the relations $\sin(x) - \sin(\overline{x}) = 2\sin\left(\frac{x-\overline{x}}{2}\right)\cos\left(\frac{x+\overline{x}}{2}\right)$ and $\cos(x) - \cos(\overline{x}) = -2\sin\left(\frac{x-\overline{x}}{2}\right)\sin\left(\frac{x+\overline{x}}{2}\right)$. (ii) The second alternative is to realize directly geometrically that the derivative of $\sin(x)$ is $\cos(x)$ and the derivative of $\cos(x)$ is $-\sin(x)$.

23.7 Use Theorem 23.1 to get a lower bound for L and then show that the function is really Lipschitz continuous with this L.

23.8 Use the fact that $f(x_i) = f(0) + (x-0)f'(0) + E_f(x_i, 0)$ and $g(x_i) = g(0) + (x-0)g'(0) + E_g(x_i, 0)$, which gives $f(x_i) = xf'(0) + E_f(x_i, 0)$ for f and the same for g. Divide by x and realize that the limit is $f'(0)/g'(0)$.

23.9 This problem should perhaps be in the next chapter?

Generalize l'Hopital's rule to $x_i \to \overline{x}$ and compute the derivatives at $\overline{x} = 1$. The limits are $1/2$ and r.

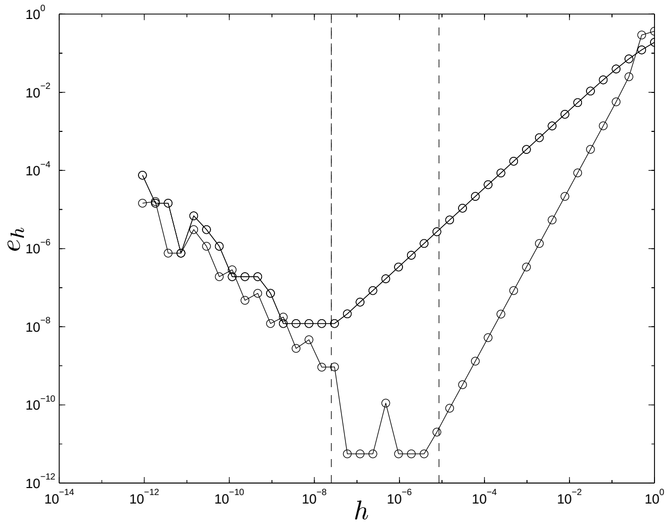


Figure 18: Comparision between the results in Problem 23.3 and 23.4. Note the improvement using the symmetric difference formula.(Problem 23.4)

Chapter 24

24.1 The rules for differentiating x^r, the quotient rule, and the chain rule gives:

$$D\left(\sqrt{x^{11}+\sqrt{\frac{x^{111}}{x^{-1.1}+x^{1.1}}}}\right)=\frac{1}{2\sqrt{x^{11}+\sqrt{\frac{x^{111}}{x^{-1.1}+x^{1.1}}}}}\cdot\Bigg(11x^{10}+$$

$$+\frac{1}{2\sqrt{\frac{x^{111}}{x^{-1.1}+x^{1.1}}}}\cdot\frac{111x^{110}(x^{-1.1}+x^{1.1})-x^{111}(-1.1x^{-2.1}+1.1x^{0.1})}{(x^{-1.1}+x^{1.1})^2}\Bigg)$$

24.2 $\frac{\partial f}{\partial x_1}=\frac{\partial}{\partial x_1}(x_1^2+x_2^4)\partial x_2(x_1^2+x_2^4)=4x_2^3$

24.3 We plot q_n for $n=2^1,2^2,...,2^{15}$: By increasing n, we find that q_n converges to $0.6931...=\ln 2$. Now, in the Chapter A Very Short Course in Calculus, we saw that $\lim_{\to\infty}(1+\frac{1}{n})^n=e$. We see the connection to $\lim_{n\to\infty}(1+\frac{q_n}{n})^n=2$, by noting that $\lim_{n\to\infty}q_n=D2^x(0)$ and $De^x(0)=1$

24.4 Let $f(x)=2^x$ and suppose that we know $f'(0)$ (see Problem 24.3) We have $2^x=2^{x-\bar{x}}2^{\bar{x}}$, i.e., $f(x)=f(x-\bar{x})f(\bar{x})$. The chain rule gives $f'(x)=f'(x-\bar{x})f(\bar{x})$, so that $f'(\bar{x})=f'(0)f(\bar{x})$.

24.5 (i) $a+b=\frac{1}{2}$ (match the two pieces at $x=1$)

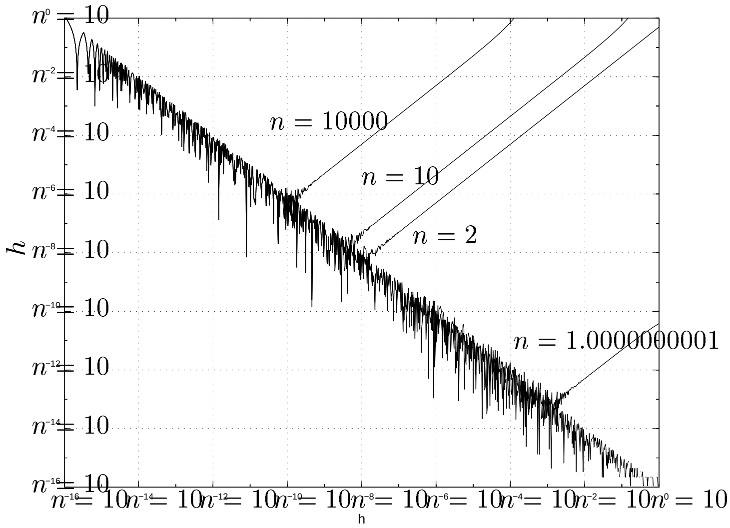


Figure 19: Relative errors for the numerical derivative as function of h. (Problem 23.5)

(ii) $a = -2$, $b = \frac{5}{2}$ (match also the left- and right-handderivatives at $x = 1$)

Chapter 25

25.4 $\bar{x} = \{-3.3027, -1.6180, 0.3027, 0.6180, 1\}$, *Note that the answers have not been rounded off.*

25.5 $\bar{x} = 3.0608$

25.7 Probably an error in the assignment. $x_0 = 1/\sqrt{3}$ is more interesting.

25.8 (a) E.g. $x_i - \bar{x} \approx \frac{1}{1-g'(x_i)}(x_i - g(x_i))$

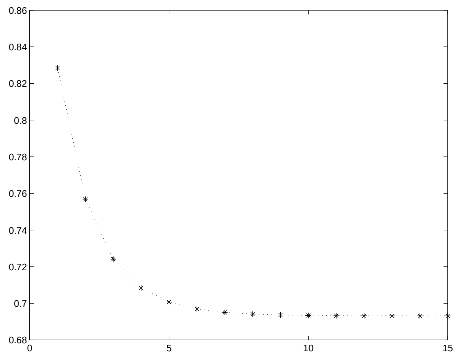


Figure 20: Problem 24.3.