92 Analytical computation of integrals

92.1. You must be familiar with the following elementary primitive functions. Prove them or find them in the book. Here $F'(x) = f(x)$.

	$f(x)$	$F(x)$
(92.1)	x^a	$\frac{x^{a+1}}{a+1} \quad (a \neq -1)$
(92.2)	x^{-1}	$\log(\lvert x\rvert)$
(92.3)	$\frac{h'(x)}{h(x)}$	$\log(\lvert h(x)\rvert)$
(92.4)	e^x	e^x
(92.5)	a^x	$\frac{a^x}{\log(a)} \quad (a \neq 1)$
(92.6)	$\sin(x)$	$-\cos(x)$
(92.7)	$\cos(x)$	$\sin(x)$
(92.8)	$\frac{1}{\cos^2(x)}$	$\tan(x)$
(92.9)	$\frac{1}{\sin^2(x)}$	$-\cot(x)$
(92.10)	$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (a > 0)$
(92.11)	$\frac{1}{\sqrt{x^2 + a}}$	$\log(\lvert x + \sqrt{x^2 + a}\rvert)$
(92.12)	$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \arctan\left(\frac{x}{a}\right)$

92.2. Compute (hint: integrate by parts)

$$\text{(a)} \int_1^x \log(y)\, dy \qquad \text{(b)} \int_1^x y \log(y)\, dy \qquad \text{(c)} \int_1^x y^2 \log(y)\, dy$$

92.3. Compute (hint: integrate by parts)

$$\text{(a)} \int_0^x t \sin t\, dt \qquad \text{(b)} \int_0^x t^2 \cos t\, dt \qquad \text{(c)} \int_0^x t^2 e^t\, dt \qquad \text{(d)} \int_0^x e^t \cos t\, dt$$

92.4. Compute $F(x) = \int_0^x f(t)\, dt$ with $f(t) = t$ for $t \in [0, 1]$, $f(t) = -1$ for $t \in [1, 2]$. Draw the graphs of f and F. Compute and plot F with MATLAB.

92.5. Compute the following. For which values of t are the integrals defined? (hint for (c): partial fractions)

$$\text{(a)} \int_0^t \frac{1}{x+3}\, dx \qquad \text{(b)} \int_0^t \frac{1}{x-5}\, dx \qquad \text{(c)} \int_0^t \frac{1}{x^2 - 2x - 15}\, dx$$

92.6. Compute (hint: change of variable)

(a) $$\int_0^2 xe^{-x^2}\,dx$$

(b) $$\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx \quad \text{(hint: } t=\sin x\text{)}$$

(c) $$\int_1^2 x\sqrt{x-1}\,dx \quad \text{(hint: } t=\sqrt{x-1}\text{)}$$

(d) $$\int_0^y \cos^3 x\,dx \quad \text{(hint: } t=\sin x\text{)}$$

92.7. A function $f:[-a,a]\to \mathbf{R}$ is called *even* if $f(-x)=f(x)$ for all $x\in[-a,a]$, and *odd* if $f(-x)=-f(x)$ for all $x\in[-a,a]$. (a) Find examples of even and odd functions. Sketch their graphs. (b) Show that

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{if } f \text{ is even,} \qquad \int_{-a}^a f(x)\,dx = 0 \quad \text{if } f \text{ is odd.}$$

92.8. Use the result in Problem 92.7 to compute

(a) $\int_{-\pi}^{\pi} |x|\cos x\,dx$ (b) $\int_{-\pi}^{\pi} \sin^2 x\,dx$

(c) $\int_{-\pi}^{\pi} x\sin^2 x\,dx$ (d) $\int_{-\pi}^{\pi} \arctan(x+3x^3)\,dx$

92.9. (Physical chemistry.) When a gas expands from volume V_1 to volume V_2 it does work on its surrounding according to the formula $w=\int_{V_1}^{V_2} p\,dV$. Compute the work during an isothermal expansion in the following cases. See: P. Atkins and L. Jones, *Chemical Principles. The Quest for Insight.* Freeman, New York, second edition, 2002, pp. 209–210.

(a) $$pV = nRT \quad \text{(ideal gas)}$$

(b) $$(p + a\frac{n^2}{V^2})(V-nb) = nRT \quad \text{(van der Waals gas)}$$

92.10. (Advanced, not obligatory.) (Hyperbolic functions, AMBS 32.5) Show that

(a) $$\operatorname{arcsinh}(y) = \log\left(y+\sqrt{y^2+1}\right) \quad \text{for } -\infty<y<\infty$$

(b) $$\operatorname{arccosh}(y) = \log\left(y+\sqrt{y^2-1}\right) \quad \text{for } y\ge 1$$

(c) $$\cosh^2(x)-\sinh^2(x)=1$$

(d) $$D\operatorname{arcsinh}(y) = \frac{1}{\sqrt{y^2+1}}$$

(e) $$D\operatorname{arccosh}(y) = \frac{1}{\sqrt{y^2-1}} \quad \text{for } y>1$$

(f) $$\int_0^x \frac{dy}{\sqrt{y^2+1}} = \log\left(x+\sqrt{x^2+1}\right)$$

(g) $$\int_1^x \frac{dy}{\sqrt{y^2-1}} = \log\left(x+\sqrt{x^2-1}\right) \quad \text{for } x>1$$

Answers and solutions

92.2.

(a) $\displaystyle\int_1^x \log(y)\,dy = \int_1^x 1\log(y)\,dy = \Big[y\log(y)\Big]_1^x - \int_1^x y\frac{1}{y}\,dy = x\log(x) - x + 1.$

(b) $\displaystyle\frac{x^2}{2}\log(x) - \frac{x^2}{4} + \frac{1}{4}.$

(c) $\displaystyle\frac{x^3}{3}\log(x) - \frac{x^3}{9} + \frac{1}{9}.$

92.3.

(a) $-x\cos x + \sin x.$

(b) $(x^2-2)\sin x + 2x\cos x.$

(c) $(x^2-2x+2)e^x - 2.$

(d) $\frac{1}{2}(e^x\cos x + e^x\sin x - 1).$

92.4.

$$F(x) = \begin{cases} \int_0^x t\,dt = \Big[\frac{t^2}{2}\Big]_0^x = \frac{x^2}{2} & \text{for } x\in[0,1] \\ \int_0^1 t\,dt + \int_1^x(-1)\,dt = \Big[\frac{t^2}{2}\Big]_0^1 + \Big[-t\Big]_1^x = \frac{1}{2} + 1 - x & \text{for } x\in[1,2] \end{cases}$$

Matlab function file:

```
function y=funk1(x,u)
if  x<1
  y=x;
else
  y=-1;
end
```

Matlab command:

```
>> [x,U]=my_ode('funk1',[0 2],0,1e-2);  plot(x,U)
```

92.5.

(a)

$$\int_0^t \frac{1}{x+3}\,dx = \Big[\log(x+3)\Big]_0^t = \log(t+3) - \log(3) = \log\Big(\frac{t+3}{3}\Big), \quad \text{for } t > -3$$

(b)

$$\int_0^t \frac{1}{x-5}\,dx = -\int_0^t \frac{1}{5-x}\,dx = \left\{ \begin{array}{c} z = 5-x \\ dz = -dx \end{array} \right\} = \int_5^{5-t} \frac{1}{z}\,dz$$
$$= \Big[\log(z)\Big]_5^{5-t} = \log(5-t) - \log(5) = \log\Big(\frac{5-t}{5}\Big), \quad \text{for } t < 5$$

(c) We split the function into *partial fractions*:

$$\frac{1}{x^2-2x-15} = \frac{1}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3} = \frac{A(x+3)+B(x-5)}{(x-5)(x+3)}$$

Identification of the coefficients gives $A = 1/8$, $B = -1/8$, so that

$$\frac{1}{x^2-2x-15} = \frac{1}{(x-5)(x+3)} = \frac{1/8}{x-5} - \frac{1/8}{x+3}$$

Hence, using (a) and (b), we get

$$\int_0^t \frac{1}{x^2-2x-15}\,dx = \frac{1}{8}\int_0^t \frac{1}{x-5}\,dx - \frac{1}{8}\int_0^t \frac{1}{x+3}\,dx$$
$$= \frac{1}{8}\log\Big(\frac{5-t}{5}\Big) - \frac{1}{8}\log\Big(\frac{t+3}{3}\Big) = \frac{1}{8}\log\Big(\frac{3}{5}\cdot\frac{5-t}{t+3}\Big), \quad \text{for } -3<t<5$$

92.6.

(a) $\int_0^2 xe^{-x^2}\,dx = \Big\{t = x^2,\ dt = 2x\,dx\Big\} = \frac{1}{2}\int_0^4 e^{-t}\,dt = \frac{1}{2}(1-e^{-4})$.

(b) $\pi/4$.

(c) $\frac{2}{5}+\frac{2}{3}$.

(d) $\sin y - \frac{1}{3}\sin^3 y$.

92.7.

(a) even: 1, x^2, $\cos x$, odd: x, x^3, $\sin x$

92.8.

(a) $2\displaystyle\int_0^\pi x\cos x\,dx = -4$. (b) π. (c) 0. (d) 0.

92.10.

(a) $w = nRT\log(V_2/V_1)$

(b) $w = nRT\log\Big(\dfrac{V_2-nb}{V_1-nb}\Big) - an^2\Big(\dfrac{1}{V_1}-\dfrac{1}{V_2}\Big)$

92.10.

(a) Hint: to solve the equation $\frac{1}{2}(e^x-e^{-x}) = y$ we set $z = e^x$ and find the equation $z^2-2yz-1 = 0$. Solve for z and finally get $x = \log(z)$.

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