PDE Project Course

3. FEniCS: FIAT, FFC, DOLFIN

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Lecture plan

- Component structure of FEniCS
- FIAT
- FFC
- DOLFIN
- Solving a PDE with FEniCS

FEniCS

Goal of FEniCS

- One goal: automatic solution of PDE (variational formulation)
- Why?
  - Efficiency
    - Execution time
    - Development time
  - Safety - reduce chance of human error (bugs)

Component structure of FEniCS

- Automatic generation of finite elements (FIAT)

$$e = (K, P, \mathcal{N})$$

- Automatic evaluation of variational forms (FFC)

$$a(v, u) = \int_\Omega \nabla v \cdot \nabla u dx$$

- Automatic assembly of discrete systems (DOLFIN)

```
$A = 0$
for all elements $K \in \mathcal{T}_\Omega$
    $A$ += $A^K$
```

FIAT, the finite element tabulator

FIAT: FInite element Automatic Tabula

- Automates the generation of finite element basis functions
- Simplifies the specification of new elements
- Continuous and discontinuous Lagrange elements of arbitrary order
- Crouziex-Raviart (CR) elements
- Nedelec elements
- Raviart-Thomas (RT) elements
- Brezzi-Douglas-Marini (BDM) elements
- ...

FIAT: Implementation

- Use orthonormal basis on triangles and tets (Dubiner)
- Express basis functions in terms of the orthonormal basis
- Translate conditions of function space into linear algebraic relations
- Implemented in Python (as a Python module)

FIAT: Example

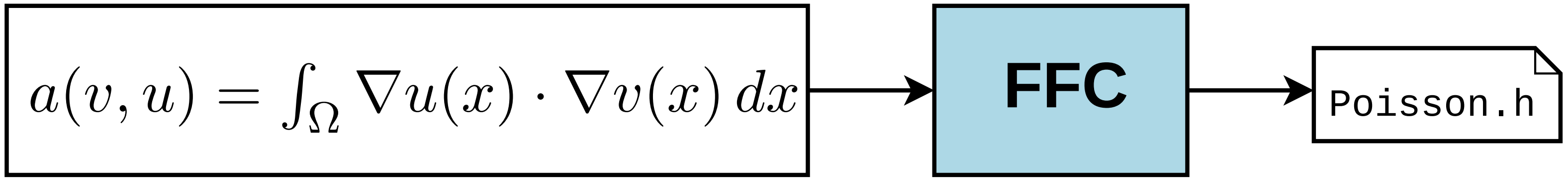
```
>>> from FIAT.Lagrange import *
>>> from FIAT.shapes import *
>>> element = Lagrange(TRIANGLE, 2)
>>> basis = element.function_space()
>>> v = basis[0]
>>> v([-1.0, -1.0])
1.0
```

(note reference triangle starts at (-1, -1) in FIAT)

FFC, the form compiler

FFC: the FEniCS Form Compiler

- Automates a key step in the implementation of finite element methods for partial differential equations
- Input: a variational form and a finite element
- Output: optimal C/C++

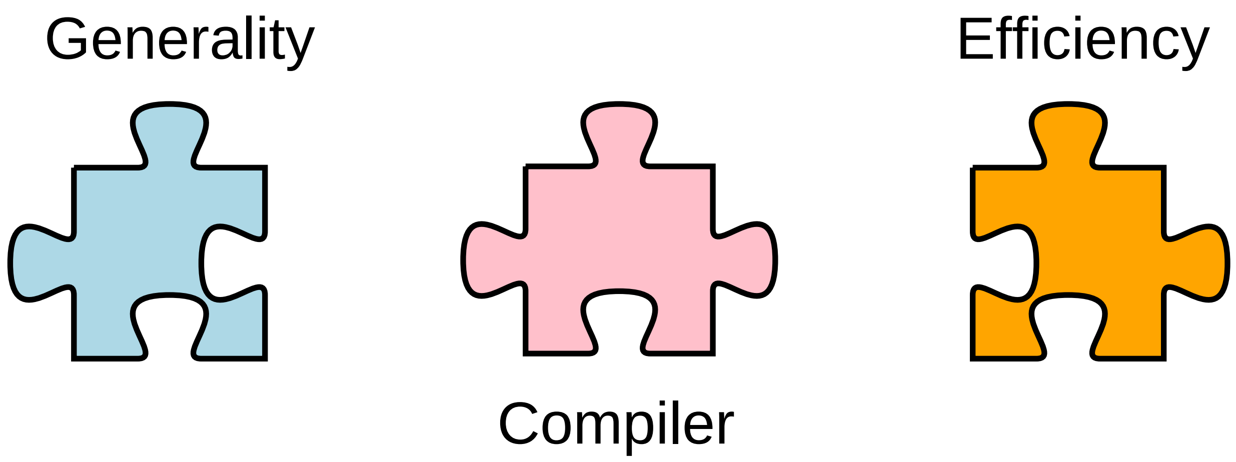


```
>> ffc [-l language] poisson.form
```

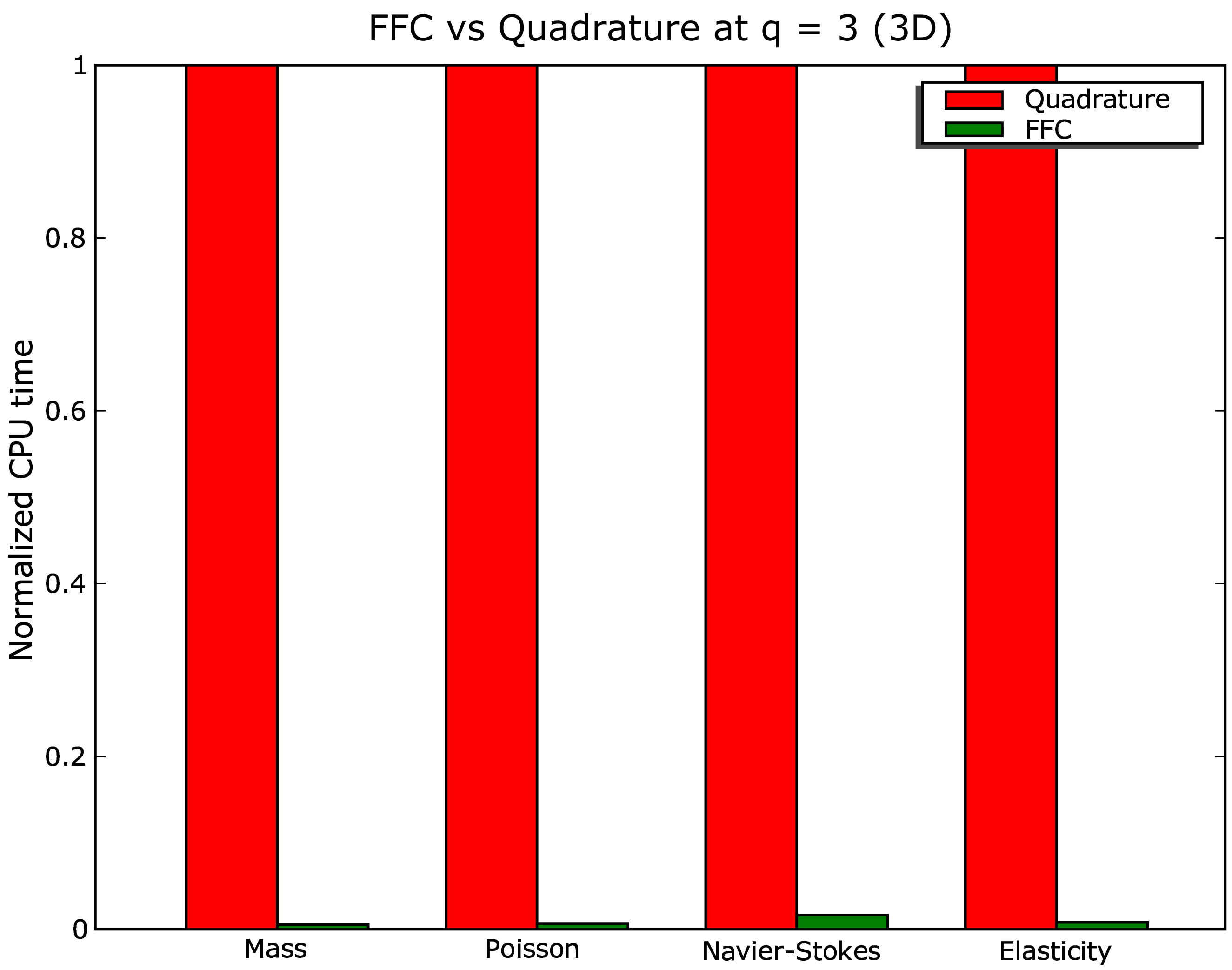
FFC: Design goals

- Any form
- Any element
- Maximum efficiency

Possible to combine generality with efficiency by using a compiler approach:



Impressive speedups



Basic example: Poisson's equation

- Strong form: Find $u \in \mathcal{C}^2(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$ such that

$$-\Delta u = f \quad \text{in } \Omega$$

- Weak form: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx \quad \text{for all } v \in H_0^1(\Omega)$$

- Standard notation: Find $u \in V$ such that

$$a(v, u) = L(v) \quad \text{for all } v \in \hat{V}$$

with $a : \hat{V} \times V \to \mathbb{R}$ a *bilinear form* and $L : \hat{V} \to \mathbb{R}$ a *linear form* (functional)

Obtaining the discrete system

Let V and $\hat{V}$ be discrete function spaces. Then

$$a(v, U) = L(v) \quad \text{for all } v \in \hat{V}$$

is a discrete linear system for the approximate solution $U \approx u$. With $V = \mathrm{span}\{\phi_i\}_{i=1}^M$ and $\hat{V} = \mathrm{span}\{\hat{\phi}_i\}_{i=1}^M$, we obtain the linear system

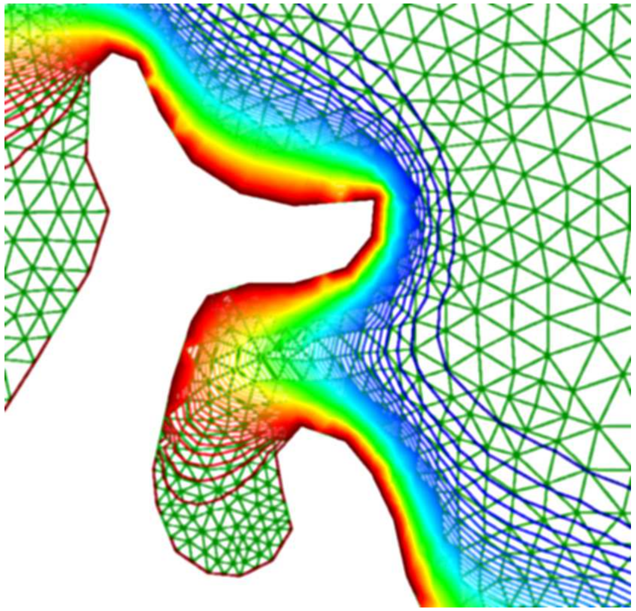
$$Ax = b$$

for the degrees of freedom $x = (x_i)$ of $U = \sum_{i=1}^M x_i \phi_i$, where

$$A_{ij} = a(\hat{\phi}_i, \phi_j)$$

$$b_i = L(\hat{\phi}_i)$$

Computing the linear system: assembly



Noting that $a(v,u) = \sum_{K\in\mathcal{T}} a_K(v,u)$, the matrix A can be assembled by

```
$A = 0$
for all elements $K \in \mathcal{T}$
    $A$ += $A^K$
```

The *element matrix* A^K is defined by

$$A_{ij}^K = a_K(\hat{\phi}_i, \phi_j)$$

for all local basis functions $\hat{\phi}_i$ and ϕ_j on K

Tensor representation

In general, the element tensor A^K can be represented as the product of a *reference tensor* A^0 and a *geometry tensor* G_K:

$$A_i^K = A_{i\alpha}^0 G_K^\alpha$$

- A^0: a tensor of rank $|i| + |\alpha| = r + |\alpha|$
- G_K: a tensor of rank $|\alpha|$

Basic idea:

- Precompute A^0 at compile-time
- Generate optimal code for run-time evaluation of G_K and the product $A_{i\alpha}^0 G_K^\alpha$

Example 1: the mass matrix

- Form:

$$a(v,u) = \int_\Omega v(x)u(x)\,dx$$

- Evaluation:

$$\begin{aligned} A_i^K &= \int_K \phi_{i_1}\phi_{i_2}\,dx \\ &= \det F_K' \int_{K_0} \Phi_{i_1}(X)\Phi_{i_2}(X)\,dX = A_i^0 G_K \end{aligned}$$

with $A_i^0 = \int_{K_0} \Phi_{i_1}(X)\Phi_{i_2}(X)\,dX$ and $G_K = \det F_K'$

Example 2: Poisson

- Form:

$$a(v,u) = \int_\Omega \nabla v(x) \cdot \nabla u(x)\, dx$$

- Evaluation:

$$\begin{aligned} A_i^K &= \int_K \nabla \phi_{i_1}(x) \cdot \nabla \phi_{i_2}(x)\, dx \\ &= \det F_K' \frac{\partial X_{\alpha_1}}{\partial x_\beta} \frac{\partial X_{\alpha_2}}{\partial x_\beta} \int_{K_0} \frac{\partial \Phi_{i_1}}{\partial X_{\alpha_1}} \frac{\partial \Phi_{i_2}}{\partial X_{\alpha_2}}\, dX = A^0_{i\alpha} G_K^\alpha \end{aligned}$$

with $A^0_{i\alpha} = \int_{K_0} \frac{\partial \Phi_{i_1}}{\partial X_{\alpha_1}} \frac{\partial \Phi_{i_2}}{\partial X_{\alpha_2}}\, dX$ and $G_K^\alpha = \det F_K' \frac{\partial X_{\alpha_1}}{\partial x_\beta} \frac{\partial X_{\alpha_2}}{\partial x_\beta}$

Example 3: Navier–Stokes

- Form:

$$a(v,u) = \int_\Omega v \cdot (w \cdot \nabla) u \, dx$$

- Evaluation:

$$\begin{aligned} A_i^K &= \int_K \phi_{i_1} \cdot (w \cdot \nabla) \phi_{i_2} \, dx \\ &= \det F_K' \frac{\partial X_{\alpha_3}}{\partial x_{\alpha_1}} w_{\alpha_2} \int_{K_0} \Phi_{i_1}[\beta] \Phi_{\alpha_2}[\alpha_1] \frac{\partial \Phi_{i_2}[\beta]}{\partial X_{\alpha_3}} \, dX = A_{i\alpha}^0 G_K^\alpha \end{aligned}$$

with $A_{i\alpha}^0 = \int_{K_0} \Phi_{i_1}[\beta] \Phi_{\alpha_2}[\alpha_1] \frac{\partial \Phi_{i_2}[\beta]}{\partial X_{\alpha_3}} \, dX$ and

$G_K^\alpha = \det F_K' \frac{\partial X_{\alpha_3}}{\partial x_{\alpha_1}} w_{\alpha_2}$

FFC: Example form file (Poisson)

```
# The bilinear form a(v, u) and linear form L(v) for
# Poisson’s equation, 2D version
#
# Compile this form with FFC: ffc Poisson2D.form

element = FiniteElement("Lagrange", "triangle", 1)

v = BasisFunction(element)
u = BasisFunction(element)
f = Function(element)

a = dot(grad(u), grad(v))*dx
L = f*v*dx
```

FFC: Example output (Poisson)

Note the factors 0.5 which are simply computed integrals of the integrand 1 on a triangle.

```
...

void eval(real block[], const AffineMap& map) const
{
  // Compute geometry tensors
  const real G0_0_0 = map.det*map.g00*map.g00 + map.det*map.g01*map.g01;
  const real G0_0_1 = map.det*map.g00*map.g10 + map.det*map.g01*map.g11;
  const real G0_1_0 = map.det*map.g10*map.g00 + map.det*map.g11*map.g01;
  const real G0_1_1 = map.det*map.g10*map.g10 + map.det*map.g11*map.g11;

  // Compute element tensor
  block[0] = 4.999999999999998e-01*G0_0_0 + 4.999999999999997e-01*G0_0_1 + ...
  block[1] = -4.999999999999998e-01*G0_0_0 - 4.999999999999997e-01*G0_0_1;
  block[2] = -4.999999999999997e-01*G0_1_0 - 4.999999999999996e-01*G0_1_1;
  ...
```

DOLFIN, C++ interface for solving PDEs

Assembly in DOLFIN

```
// Iterate over all cells in the mesh
for (CellIterator cell(mesh); !cell.end(); ++cell)
{
  // Update affine map
  map.update(*cell);

  // Update form
  a.update(map);

  // Compute maps from local to global degrees of freedom
  test_element.dofmap(test_dofs, *cell, mesh);
  trial_element.dofmap(trial_dofs, *cell, mesh);


  // Compute element matrix
  a.eval(block, map);

  // Add element matrix to global matrix
  A.add(block, test_dofs, m, trial_dofs, n);
}
```

Demonstration

- ...