Ordinära differentialekvationer 6

Läraren går igenom avsnitt 2.1 och Exempel 1 och 2. Sedan gör du övningarna med penna och papper och med MATLAB.

1.1 Allmän och speciell ekvation

Vi hållller på att studera några speciella typer av differentialekvationer som kan lösas analytiskt, dvs med en formel. Det är:

- separabel ekvation: $u'(x) = h(x)/g(u(x))$;
- linjär ekvation av första ordningen: $u'(x) + a(x)u(x) = f(x)$
- linjär ekvation av andra ordningen med konstanta koefficienter: $u''(x)+bu'(x)+cu(x) = f(x)$
- system av första ordningen med konstanta koefficienter: $u'(x) = Au(x)$ (i ALA-c).

Vi har redan gjort separabla ekvationer. Nu kommer linjära ekvationer av första och andra ordningen.

2.1 Linear differential equation—first order

$$u' + a(t)u = f(t). \tag{1}$$

Here $u = u(t)$ is an unknown function of an independent variable t. The equation is called *homogeneous* if $f(t) \equiv 0$ and *nonhomogeneous* otherwise. The differential operator $Lu = u'+a(t)u$ has *constant coefficient* if $a(t) = a$ is constant and it has *variable coefficient* otherwise. The equation is said to be a *linear equation*, because the operator L is a *linear operator*:

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv, \quad (\alpha, \beta \in \mathbf{R},\ u = u(t),\ v = v(t))$$

i.e., it preserves linear combinations of functions. Check this!

Solution method: multiply by the *integrating factor* $e^{A(t)}$ with $A(t) = \int_0^t a(s)\,ds$, and integrate.

This goes like this. We consider the initial-value problem:

$$\begin{aligned} &u' + a(t)u = f(t), \quad t > 0,\\ &u(0) = u_0. \end{aligned} \tag{2}$$

We multiply by the integrating factor $e^{A(t)}$:

$$e^{A(t)}u'(t) + e^{A(t)}a(t)u(t) = e^{A(t)}f(t)$$

Using the product rule and the chain rule ($De^{A(t)} = e^{A(t)}A'(t) = e^{A(t)}a(t)$) we see that the left side is an exact derivative:

$$\frac{d}{dt}\Big(e^{A(t)}u(t)\Big) = e^{A(t)}u'(t) + e^{A(t)}a(t)u(t) = e^{A(t)}f(t)$$

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Then we integrate $\int_0^t \cdots ds$ and use the initial values $u(0) = u_0$, $A(0) = 0$:

$$\begin{aligned}
\left[e^{A(s)}u(s)\right]_{s=0}^{t} &= \int_0^t e^{A(s)} f(s)\, ds \\
e^{A(t)}u(t) - e^{A(0)}u(0) &= \int_0^t e^{A(s)} f(s)\, ds \\
e^{A(t)}u(t) &= u_0 + \int_0^t e^{A(s)} f(s)\, ds \\
u(t) &= u_0 e^{-A(t)} + e^{-A(t)} \int_0^t e^{A(s)} f(s)\, ds \\
u(t) &= u_0 e^{-A(t)} + \int_0^t e^{A(s)-A(t)} f(s)\, ds
\end{aligned}$$

Problem 1.1. (constant coefficient, homogeneous) Solve the following. Sketch the graph of the solution.

(a)

$$\begin{aligned}
&u' + 2u = 0, \quad t > 0, \\
&u(0) = u_0.
\end{aligned}$$

(b)

$$\begin{aligned}
&u' - 2u = 0, \quad t > 0, \\
&u(0) = u_0.
\end{aligned}$$

Problem 1.2. (constant coefficient, nonhomogeneous) Solve the following.

(a)

$$\begin{aligned}
&u' + 2u = f(t), \quad t > 0, \\
&u(0) = u_0.
\end{aligned}$$

(b)

$$\begin{aligned}
&u' - 2u = f(t), \quad t > 0, \\
&u(0) = u_0.
\end{aligned}$$

(c)

$$\begin{aligned}
&u' + 2u = 1, \quad t > 0, \\
&u(0) = 5.
\end{aligned}$$

Also: solve the equation by MATLAB and `myode`.

(d)

$$\begin{aligned}
&u' + 2u = t, \quad t > 0, \\
&u(0) = 5.
\end{aligned}$$

Problem 1.3. (constant coefficient, nonhomogeneous) Solve the following.

$$\begin{aligned}
&u' + au = f(t), \quad t > 0, \\
&u(0) = u_0
\end{aligned}$$

Problem 1.4. (variable coefficient, nonhomogeneous) Solve the following.

$$\begin{aligned}
&u' + 2tu = f(t), \quad t > 0, \\
&u(0) = u_0.
\end{aligned}$$

2.2 Linear differential equation—second order—constant coefficients

$$u'' + a_1 u' + a_0 u = f(t). \tag{3}$$

The equation is called *homogeneous* if $f(t) \equiv 0$ and *nonhomogeneous* otherwise. We assume that the differential operator $Lu = u'' + a_1 u' + a_0 u$ has *constant coefficients* a_1 and a_0. Check that the operator L is linear!

Variable coefficients: Linear differential equations of second order with variable coefficients $u'' + a_1(t)u' + a_0(t)u = f(t)$, cannot be solved analytically, except in some special cases. We do not discuss this here.

Homogeneous equation

The homogeneous equation (3) may be written

$$D^2 u + a_1 D u + a_0 u = 0, \tag{4}$$

or

$$P(D)u = 0,$$

where

$$P(r) = r^2 + a_1 r + a_0$$

is the *characteristic polynomial* of the equation. The *characteristic equation* $P(r) = 0$ has two roots r_1 and r_2. Hence $P(r) = (r - r_1)(r - r_2)$. All solutions of equation (3) are obtained as linear combinations

$$\begin{aligned} u(t) &= c_1 e^{r_1 t} + c_2 e^{r_2 t}, && \text{if } r_1 \neq r_2, \\ u(t) &= (c_1 + c_2 t) e^{r_1 t}, && \text{if } r_1 = r_2, \end{aligned} \tag{5}$$

where c_2, c_2 are arbitrary coefficients. The coefficients may be determined from an initial condition of the form

$$u(0) = u_0,\ u'(0) = u_1.$$

The formula (5) is called the *general solution* of homogeneous linear equation (4).

Example 1. We solve

$$u'' + u' - 12u = 0; \quad u(0) = u_0,\ u'(0) = u_1.$$

The equation is written $(D^2 - D - 12)u = 0$ and the characteristic equation is $r^2 + r - 12 = 0$ with roots $r_1 = 3$, $r_2 = -4$. The general solution is

$$u(t) = c_1 e^{3t} + c_2 e^{-4t}$$

with the derivative

$$u'(t) = 3c_1 e^{3t} - 4c_2 e^{-4t}.$$

The initial condition gives

$$\begin{aligned} u_0 &= u(0) = c_1 + c_2 \\ u_1 &= u'(0) = 3c_1 - 4c_2 \end{aligned}$$

which implies $c_1 = (4u_0 + u_1)/7$, $c_2 = (3u_0 - u_1)/7$. The solution is

$$u(t) = \frac{4u_0 + u_1}{7} e^{3t} + \frac{3u_0 - u_1}{7} e^{-4t}.$$

Complex roots

If the characteristic polynomial $P(r)$ has real coefficients, then its roots are either real numbers or a pair of conjugate complex numbers. In the latter case we have $r_1 = \alpha + i\omega$ and $r_2 = \alpha - i\omega$ and the solution (5) becomes

$$\begin{aligned} u(t) &= c_1 e^{(\alpha+i\omega)t} + c_2 e^{(\alpha-i\omega)t} \\ &= e^{\alpha t}\Big(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\Big) \\ &= e^{\alpha t}\Big(c_1\big(\cos(\omega t) + i\sin(\omega t)\big) + c_2\big(\cos(\omega t) - i\sin(\omega t)\big)\Big) \\ &= e^{\alpha t}\Big(d_1\cos(\omega t) + d_2\sin(\omega t)\Big), \end{aligned}$$

with $d_1 = c_1 + c_2$, $d_2 = i(c_1 - c_2)$.

Nonhomogeneous equation

The solution of the nonhomogeneous equation $P(D)u = f(t)$ is given by

$$u(t) = u_h(t) + u_p(t), \tag{6}$$

where u_h is the general solution (5) of the corresponding homogeneous equation, i.e., $P(D)u_h = 0$, and u_p is a *particular solution* of the nonhomogeneous equation, i.e., $P(D)u_p = f(t)$.

Proof: If u is given by (6), then $Lu = L(u_h + u_p) = Lu_h + Lu_p = 0 + f = f$, so that u solves the nonhomogeneous equation. On the other hand: if u_p is a particular solution and u is any other solution of the nonhomogeneous equation, then $L(u - u_p) = Lu - Lu_p = f - f = 0$, i.e., $u - u_p$ solves the homogeneous equation. Thus $u - u_p = u_h$, which is (6).

A particular solution can sometimes be found by guess-work: we make an Ansatz for u_p of the same form as f.

Example 2. $u'' + u' - 12u = t$. Here $f(t) = t$ is a polynomial of degree 1 and we make the Ansatz $u_p(t) = At + B$, i.e., a polynomial of degree 1. Substitution into the equation gives $A - 12(At + B) = t$. Identification of coefficients gives $A = -\frac{1}{12}$, $B = -\frac{1}{144}$, so that $u_p(t) = -\frac{1}{12}t - \frac{1}{144}$. The general solution of the homogeneous equation is $u_h(t) = c_1 e^{3t} + c_2 e^{-4t}$, see Example 1. Hence we get

$$u(t) = u_h(t) + u_p(t) = c_1 e^{3t} + c_2 e^{-4t} - \frac{1}{12}t - \frac{1}{144}.$$

Rewriting as a system of first order equations

By setting $w_1 = u$, $w_2 = u'$, $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, we can rewrite (3) as a system of first order equations

$$w'(t) = Aw(t) + F(t); \quad w(0) = w_0,$$

where

$$w_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

To see this we compute

$$w' = \begin{bmatrix} u' \\ u'' \end{bmatrix} = \begin{bmatrix} u' \\ -a_0 u - a_1 u' + f(t) \end{bmatrix} = \begin{bmatrix} w_2 \\ -a_0 w_1 - a_1 w_2 + f(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

It is necessary to do this rewriting before we can use our MATLAB programs to solve (3).

2.3 System of linear differential equations of first order

Constant coefficients—homogeneous equations

We finally mention

$$\begin{aligned} &u' + Au = 0, \quad t > 0, \\ &u(0) = u_0, \end{aligned} \tag{7}$$

where $u(t), u_0 \in \mathbf{R}^d$, and $A \in \mathbf{R}^{d\times d}$ is a constant matrix of coefficients. This kind of system will studied by means of eigenvalues and eigenvectors in the following course ALA-C.

Problems

Problem 1.5. Write the following equations as $P(D)u = 0$ and solve the initial value problem.

(a) $u'' - u' - 2u = 0; \quad u(0) = u_0, \ u'(0) = u_1.$

(b) $u'' - k^2 u = 0; \quad u(0) = u_0, \ u'(0) = u_1.$

(c) $u'' + 4u' + 4u = 0; \quad u(0) = u_0, \ u'(0) = u_1.$

Problem 1.6. Solve the *boundary value problem*

$$\begin{aligned} &u''(x) - k^2 u(x) = 0, \quad 0 < x < L, \\ &u(0) = 0, \ u(L) = u_L. \end{aligned}$$

Problem 1.7. Write the equation as $P(D)u = 0$ and solve the initial value problem.

(a) $u'' + 4u' + 13u = 0; \quad u(0) = u_0, \ u'(0) = u_1.$

(b) $u'' + \omega^2 u = 0; \quad u(0) = u_0, \ u'(0) = u_1.$

Problem 1.8. Solve the following.

(a) $u'' - u' - 2u = e^t$ Ansatz: $u_p(t) = Ae^t$

(b) $u'' - u' - 2u = \cos(t)$ Ansatz: $u_p(t) = A\cos(t) + B\sin(t)$

(c) $u'' - u' - 2u = t^3$ Ansatz: $u_p(t) = At^3 + Bt^2 + Ct + D$

(d) $u'' - u' - 2u = e^{-t}$ Ansatz: $u_p(t) = Ate^{-t}$

Problem 1.9. Write the equation in Problem 1.8(a) as a system of first order. Choose initial values and solve the problem with your MATLAB program `myode.m`.

Problem 1.10. Prove the solution formula (5) by writing the equation as

$$P(D)u = (D - r_1)(D - r_2)u = 0$$

and by solving two first order equations $(D - r_1)v = 0$ and $(D - r_2)u = v$ as in Problems 1.1 and 1.2.

Answers and solutions

1.1.

(a) $u(t) = e^{-2t}u_0$

(b) $u(t) = e^{2t}u_0$

1.2.

(a) $u(t) = e^{-2t}u_0 + \int_0^t e^{-2(t-s)} f(s)\, ds$

(b) $u(t) = e^{2t}u_0 + \int_0^t e^{2(t-s)} f(s)\, ds$

(c) $u(t) = 5e^{-2t} + e^{-2t} \int_0^t e^{2s}\, ds = 5e^{-2t} + e^{-2t}\frac{1}{2}(e^{2t} - 1) = \frac{9}{2}e^{-2t} + \frac{1}{2}$

Function file:

```
function y=funk(t,u)
y=-2*u+1;
```

Command line:

```
>> [t,U]=myode(@funk,[0,3],5,1e-3);
>> plot(t,U)
```

(d) Partial integration gives: $u(t) = 5e^{-2t} + e^{-2t} \int_0^t e^{2s} s\, ds = 5e^{-2t} + e^{-2t}(\big[\frac{1}{2}e^{2s}s\big]_0^t - \frac{1}{2}\int_0^t e^{2s}\, ds) = \frac{19}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}$

1.3. $u(t) = e^{-at}u_0 + \int_0^t e^{-a(t-s)} f(s)\, ds$

1.4. Integrating factor: e^{t^2}. Solution $u(t) = e^{-t^2}u_0 + \int_0^t e^{-(t^2-s^2)} f(s)\, ds$.

1.5.

(a) $u(t) = \frac{1}{3}(2u_0 - u_1)e^{-t} + \frac{1}{3}(u_0 + u_1)e^{2t}$.

(b) $u(t) = c_1 e^{kt} + c_2 e^{-kt} = d_1 \cosh(kt) + d_2 \sinh(kt)$, $d_1 = c_1 + c_2$, $d_2 = c_1 - c_2$. The initial condition gives $u(t) = \frac{1}{2}(u_0 + u_1/k)e^{kt} + \frac{1}{2}(u_0 - u_1/k)e^{-kt}$ or alternatively $u(t) = u_0 \cosh(kt) + (u_1/k)\sinh(kt)$.

(c) $u(t) = (u_0 + (2u_0 + u_1)t)e^{-2t}$.

1.6. $u(x) = u_L \sinh(kx)/\sinh(kL)$.

1.7.

(a) $u(t) = e^{-2t}\Big(u_0 \cos(3t) + \frac{1}{3}(2u_0 + u_1)\sin(3t)\Big)$.

(b) $u(t) = u_0 \cos(\omega t) + (u_1/\omega)\sin(\omega t)$. Compare to Problem 1.5 (b).

1.8.

(a) $u(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{1}{2}e^t$.

(b) $u(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{3}{10}\cos(t) - \frac{1}{10}\sin(t)$.

(c) $u(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{1}{2}t^3 + \frac{3}{4}t^2 - \frac{9}{4}t + \frac{15}{8}$.

(d) $u(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{1}{3}te^{-t}$. Note: the Ansatz $u_p(t) = Ae^{-t}$ does not work, because e^{-t} is a solution of the homogeneous equation, $P(D)e^{-t} = 0$, i.e., e^{-t} is contained in u_h.

1.9. $\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}$.

Function file:

```
function y=funk(t,w)
y(1,1)= w(2);
y(2,1)=2*w(1)+w(2)+exp(t);
```

Command line:

```
>> [t,W]=myode(@funk,[0,3],[1;0],1e-3)
>> plot(t,W)
```

1.10. The equation for v is $(D - r_1)v = 0$, or $v' - r_1 v = 0$, $v(0) = v_0$, with unique solution $v(t) = v_0 e^{r_1 t}$. The equation for u is $(D - r_2)u = v$, or $u' - r_2 u = v_0 e^{r_1 t}$, $u(0) = u_0$, with unique solution, see Problem 1.3,

$$\begin{aligned} u(t) &= u_0 e^{r_2 t} + e^{r_2 t} \int_0^t e^{-r_2 s} v_0 e^{r_1 s}\, ds \\ &= u_0 e^{r_2 t} + v_0 e^{r_2 t} \int_0^t e^{(r_1 - r_2)s}\, ds \\ &= u_0 e^{r_2 t} + v_0 e^{r_2 t} \Big[\frac{e^{(r_1 - r_2)s}}{r_1 - r_2} \Big]_{s=0}^{t} \\ &= u_0 e^{r_2 t} + \frac{v_0}{r_1 - r_2} \big(e^{r_1 t} - e^{r_2 t} \big) \\ &= \frac{v_0}{r_1 - r_2} e^{r_1 t} + \Big(u_0 - \frac{v_0}{r_1 - r_2} \Big) e^{r_2 t} \\ &= c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad \text{if } r_1 \neq r_2. \end{aligned}$$

If $r_1 = r_2$, then we have instead

$$u(t) = u_0 e^{r_2 t} + e^{r_2 t} \int_0^t e^{-r_2 s} v_0 e^{r_1 s}\, ds = u_0 e^{r_1 t} + v_0 e^{r_1 t} \int_0^t ds = u_0 e^{r_1 t} + v_0 t e^{r_1 t} = (c_1 + c_2 t) e^{r_1 t}.$$