TMV035 Analysis and Linear Algebra A, 2006

THE NUMBER SYSTEMS

In this lecture we present a brief introduction to the mathematics courses. Then we introduce the number systems. See also Adams: P1.

1. Introduction

You will have four obligatory mathematics courses:

- ALA-A. Goal: solve algebraic equation $f(x) = 0$.

  Example.

  $$x^2 + 4x - 5 = 0$$
  $$\Rightarrow x = -1 \quad \text{or} \quad x = 5$$

- ALA-B. Goal: solve ordinary differential equation (ODE) $u'(x) - f(x, u(x)) = 0$.

  Example.

  $$u'(x) + u(x)^2 = 0$$
  $$\Rightarrow u(x) = \frac{1}{x+c}$$

- ALA-C and Applied Mathematics. Goal: solve partial differential equation (PDE) of the form $-\nabla \cdot (a\nabla u) = f$.

The equations in the examples above are *special* equations for which there are explicit solution formulas. For equations in the *general* forms, $f(x) = 0$, $u'(x) - f(x, u(x)) = 0$, $-\nabla \cdot (a\nabla u) = f$, there are no solution formulas. In our courses we emphasize solution methods for general equations, these methods construct solutions by means of algorithms that can also be implemented in computer programs. We shall spend a lot of time writing such programs in the Matlab environment. We shall also solve special equations with pencil and paper and solution formulas.

When we can solve general equations we can use them to model processes in chemical engineering. We shall do this together with the chemistry course.

In order to study algebraic equations, $f(x) = 0$, in ALA-A, we begin with the number systems.

2. The natural numbers

The natural numbers are

$$\mathbf{N} = \{1, 2, 3, \dots\}$$

These are the numbers that we use for counting how many elements that are contained in a set. We have two arithmetic operations ("räkneoperationer"): addition and multiplication. The sum $m + n$ is the number of elements of the set which is the union of a set with m elements and a set with n elements. The product $m \cdot n$ is repeated addition:

$$m \cdot n = n + n + \cdots + n \quad (m \text{ times})$$

It is easy to prove the following rules:

$$\begin{aligned} &m + n = n + m, && m \cdot n = n \cdot m, && \text{commutative laws} \\ &m + (n + p) = (m + n) + p, && m \cdot (n \cdot p) = (m \cdot n) \cdot p, && \text{associative laws} \\ &m \cdot (n + p) = m \cdot n + m \cdot p, && && \text{the distributive law} \end{aligned} \tag{1}$$

The associative laws mean that we may skip the parentheses and write $m + n + p$ and $m \cdot n \cdot p$. We usually skip the $\cdot$ and write mn instead of $m \cdot n$.

Date: August 26, 2006, Stig Larsson, Mathematical Sciences, Chalmers University of Technology.

We also define the power ("potens") by repeated multiplication:

$$n^m = n \cdot n \cdots n \quad (m \text{ times}) \tag{2}$$

It useful to represent the natural numbers by marking them on the number line.

There is also a natural *order relation* ("ordningsrelation") between the natural numbers: we know what it means to say that m is less than n, $m < n$. We may then introduce the related notation $m > n$, $m \leq n$, $m \geq n$.

There is a concept of *subtraction* for $m \geq n$, namely, $m - n$ is the number of elements that remain if we remove a subset of n elements from a set of m elements, with zero being the number of elements of the empty set $\emptyset$, i.e., $0 = m - m$.

Note the special roles played by the numbers 0 and 1:

$$m + 0 = m, \quad m \cdot 1 = m. \tag{3}$$

3. The integers

In order to solve equations of the form $m + x = n$ (with solution $x = n - m$) for arbitrary natural numbers m, n we need to introduce negative numbers.

The integers ("de hela talen") are

$$\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Here we have invented new numbers as follows: 0 (zero) and for each $n \in \mathbf{N}$ a negative number denoted $-n$.

We extend the addition and the multiplication to these new numbers as follows: (here $m, n \in \mathbf{N}$)

$$m + 0 = m, \quad 0 + 0 = 0, \quad m + (-n) = \begin{cases} m - n & \text{if } m \geq n, \\ -(n - m) & \text{if } m < n, \end{cases}$$

$$m \cdot 0 = 0, \quad 0 \cdot 0 = 0, \quad m \cdot (-n) = -(m \cdot n), \quad -(m) \cdot (-n) = m \cdot n.$$

Here we relate operations involving negative numbers and zero to the corresponding operations for positive numbers. In this way all the arithmetic rules in (1) hold also for the integers, i.e, for $m, n, p \in \mathbf{Z}$.

The order relation, $m < n$, is also extended to all integers $m, n \in \mathbf{Z}$ as follows:

$$-n < -m \quad \text{if} \quad m > n, \ m, n \in \mathbf{N}.$$

It useful to represent these numbers by marking them on the number line.

We can now define subtraction for all integers:

$$m - n = m + (-n)$$

and we can solve the equation $m + x = n$ as follows:

$$m + x = n \Rightarrow m + x + (-m) = n + (-m) \Rightarrow x + m + (-m) = n + (-m)$$

$$\Rightarrow x + 0 = n + (-m) \Rightarrow x = n + (-m) = n - m.$$

4. The rational numbers

In order to solve equations of the form $m \cdot x = n$ (with solution $x = n/m$) for arbitrary integers m, n, $m \neq 0$, we need to introduce rational numbers. Since we have not yet defined the fraction p/q, we first define the rational numbers as the set of all pairs $x = (p, q)$ with $p, q \in \mathbf{Z}$, $q \neq 0$, where p and q are supposed to represent the numerator and denominator, respectively.

The rational numbers ("de rationella talen") are

$$\mathbf{Q} = \Big\{x = (p, q) : p, q \in \mathbf{Z}, \ q \neq 0\Big\}$$

Two rational numbers are considered to be equal if the numerator and denominator have a common factor:

$$(mp, mq) = (p, q), \quad m \in \mathbf{Z}.$$

The integers are identified with the rational numbers that have the denominator $= 1$:

$$p = (p, 1), \quad p \in \mathbf{Z}.$$

In particular, $(p, p) = (1, 1) = 1$.

We define addition and multiplication, for $x = (p, q)$, $y = (r, s)$, as follows:

$$x + y = (s \cdot p + r \cdot q, q \cdot s), \quad x \cdot y = (p \cdot r, q \cdot s)$$

which are suggested by the expected formulas

$$x + y = \frac{p}{q} + \frac{r}{s} = \frac{s \cdot p + r \cdot q}{q \cdot s}, \quad x \cdot y = \frac{p}{q} \cdot \frac{r}{s} = \frac{p \cdot r}{q \cdot s}$$

In this way all the arithmetic rules in (1) hold also for the rational numbers.

We also define the inverse of x:

$$x^{-1} = (p, q)^{-1} = (q, p) \quad \text{for } x \neq 0$$

Note that

$$x^{-1} \cdot x = (p, q)^{-1} \cdot (p, q) = (qp, qp) = (1, 1) = 1.$$

We can now define division:

$$\frac{y}{x} = y \cdot x^{-1} = (r \cdot q, s \cdot p) \quad \text{for } x \neq 0$$

and we write the rational numbers in fractional form:

$$x = (p, q) = \frac{p}{q}$$

We can now solve the equation $a \cdot x = b$ for $a, b \in \mathbf{Z}$, $a \neq 0$:

$$a \cdot x = b \Rightarrow a^{-1} \cdot a \cdot x = a^{-1} \cdot b \Rightarrow 1 \cdot x = a^{-1} \cdot b \Rightarrow x = a^{-1} \cdot b = \frac{b}{a}$$

The order relation, $x < y$, can also be extended to rational numbers. We note (without proof) the important implication (where $a, b, c \in \mathbf{Z}$)

(4) $$a < b \Rightarrow \begin{cases} ca < cb & \text{if } c > 0 \\ ca > cb & \text{if } c < 0 \end{cases}$$

We define intervals of rational numbers:

(5) $$\begin{aligned} (m, n) &= \{x \in \mathbf{Z} : m < x < n\} \\ [m, n] &= \{x \in \mathbf{Z} : m \leq x \leq n\} \\ (m, \infty) &= \{x \in \mathbf{Z} : m < x\} \\ (-\infty, n) &= \{x \in \mathbf{Z} : x < n\} \end{aligned}$$

Note that $\{x \in \mathbf{Z} : m < x < n\}$ reads "the set of all x that belong to $\mathbf{Z}$ such that x is between m and n".

In order to measure the size of a rational number, irrespective of its sign, we define *absolute value* ("absolutbelopp")

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Note that $|x|$ is the distance of x from zero, and $|x - y|$ is the distance from x to y measured along the number line.

Note the following:

(6) $$|-x| = |x|$$

(7) $$|xy| = |x||y|$$

(8) $$|x|^2 = x^2$$

(9) $$x \leq |x|$$

Prove them!

The following inequality is very important.

Theorem. *(The triangle inequality)*

$$|a+b| \le |a| + |b|, \quad a, b \in \mathbf{Z}. \tag{10}$$

Proof. It is easier to compute with the square instead of the absolute value, so we use (8) and then (7) and (9) to get

$$|a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2 \le a^2 + |2ab| + b^2 = |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2$$

It follows that $|a+b| \le |a| + |b|$ if we take the square root of both sides and use the next theorem with $x = |a+b|$ and $y = |a| + |b|$. □

Theorem. *If $x, y > 0$ then*

$$x^2 \le y^2 \Rightarrow x \le y. \tag{11}$$

Proof. Let $x, y > 0$ and $x^2 \le y^2$. Assume that the conlusion is false, i.e., assume that $y < x$. Then multiply this inequality by the positive numbers y and x and use (4) to get

$$y^2 < yx \quad \text{and} \quad xy < x^2.$$

It follows that $y^2 < x^2$, which is a contradiction ("motsägelse") to our assumption that $x^2 \le y^2$. Hence the assumption $y < x$ leads to a contradiction and it must be false. We conclude that $x \le y$. This kind of proof is called "proof by contradiction" ("motsägelsebevis"). □

So far we have discussed the basic properties of the integers and rational numbers. This should be familiar to you: you already know very well how to compute with these numbers.

You also know the real numbers. We need some preparations before we can introduce them. For example, we need decimal expansions.

5. Periodic decimal expansion of rational numbers

If we perform a long division ("liggande stolen") of a rational number, then two things can happen: (i) the division stops after a finite number of decimals have been generated; or (ii) the division does not stop but the decimals repeat themselves. Loosely speaking, this is because there are only finitely many possible numbers for the remainder that occurs in each step and so after a finite number of steps it we get the same remainder and the calculation repeats itself. Try this! For example:

$$\frac{3}{4} = 0.75$$
$$\frac{1}{3} = 0.33333333333\ldots$$
$$\frac{16}{7} = 2.\underbrace{285714}\underbrace{285714}\underbrace{285714}\underbrace{285714}\ldots$$

In the first case we have a finite ("ändlig") decimal expansion and the number can be expressed exactly in terms of powers of 10, e.g., $\frac{3}{4} = 7 \cdot 10^{-1} + 5 \cdot 10^{-2}$. In the other cases we have an infinite, periodic, decimal expansion and the number cannot be expressed exactly with powers of 10.

(Note, by the way, that also a finite decimal expansion can be considered to be periodic with trailing zeros repeated: $\frac{3}{4} = 0.75000\ldots$)

Suppose on the other hand that we have a periodic decimal expansion. Does it represent a rational number? If so: which number is it? Take, for example,

$$0.1818181818181818\ldots$$

Let $p_m = 0.1818\dots 18_m$ be the number that we get if we truncate it after m periods:

$$\begin{aligned} p_m &= 0.181818\dots 18_m \quad (m \text{ times}) = 18\cdot 10^{-2} + 18\cdot 10^{-4} + 18\cdot 10^{-6} + \cdots + 18\cdot 10^{-2m} \\ &= 18\cdot 10^{-2}(1 + 10^{-2} + 10^{-4} + \cdots + 10^{-2m+2}) \\ &= 18\cdot 10^{-2}(1 + 10^{-2} + (10^{-2})^2 + \cdots + (10^{-2})^{m-1}) \\ &= 18\cdot 10^{-2}\frac{1-(10^{-2})^m}{1-10^{-2}} = \frac{18}{10^2-1}\big(1-(10^{-2})^m\big) = \frac{18}{99}\big(1-(10^{-2})^m\big) = \frac{2}{11}\big(1-(10^{-2})^m\big). \end{aligned}$$

Here we used the formula for a geometric sum:

$$1 + a + a^2 + \cdots + a^{m-1} = \frac{1-a^m}{1-a}, \quad a \neq 1,$$

with $a = 10^{-2}$. We find that

$$\left|\frac{2}{11} - p_m\right| = \frac{2}{11}\cdot 10^{-2m} < 10^{-2m}.$$

This means that the distance between the rational numbers p_m and $2/11$ is less than 10^{-2m}. In other words: p_m is an approximation of $2/11$ with $2m$ correct decimals. By taking m large enough we can compute a decimal approximation of $2/11$ which is correct to any number of decimals. This is what we mean when we write

$$\frac{2}{11} = 0.181818\dots$$

More generally, let

$$0.\underbrace{q_1q_2\dots q_n}\underbrace{q_1q_2\dots q_n}\underbrace{q_1q_2\dots q_n}\dots$$

be a periodic decimal expansion and let p_m be the number that we get if we truncate it after m periods. A similar calculation gives

$$\left|\frac{q_1q_2\cdots q_n}{10^n-1} - p_m\right| < 10^{-nm}$$

and we conclude that p_m approximates the rational number

$$p = \frac{q_1q_2\cdots q_n}{10^n-1}$$

to nm correct decimals. We write

$$\frac{q_1q_2\cdots q_n}{10^n-1} = 0.\underbrace{q_1q_2\dots q_n}\underbrace{q_1q_2\dots q_n}\underbrace{q_1q_2\dots q_n}\dots$$

6. The real numbers

We now define the set of *real numbers* $\mathbf{R}$ as *the set of all decimal expansions*, finite, periodic, or non-periodic. This set includes the integers and the rational numbers but also many new numbers. For example,

$$\pi = 3.141592\dots$$
$$\sqrt{2} = 1.41421356\dots$$

It is known that these decimal expansions are not periodic, and hence that these numbers are not rational, therefore they are called irrational numbers. By the way, the word rational refers to ratio ("kvot, bråk").

When we do numerical computations on the computer, we actually compute decimal expansions. A typical numerical algorithm can compute a certain number to any desired accuracy, in other words, it can produce as many decimals from the decimal expansion as we wish. For example, the task may be to compute a certain number to a certain accuracy, for example, six correct decimals. Another time we may need ten decimals. Then we have run the algorithm again. However, unless the decimal expansion is periodic (rational number) we can not compute the whole expansion.

We will discuss the real numbers more later in the course.