Maximum likelihood estimation in a spatio-temporal marked point process

Ottmar Cronie[1]

[1]Mathematical Sciences
Chalmers University of Technology & University of Gothenburg
Sweden

August 19, 2010

Motivation: Dynamics of a Scots pines stand

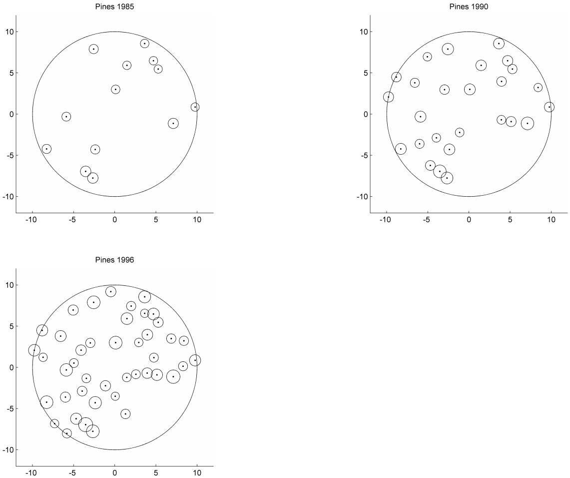


Figure: Swedish Scots pines with scaled radii (factor 10) recorded in 1985 (upper left), 1990 (right) and 1996 (lower left).

Our spatio-temporal marked point process

Employ a random process, $\{\mathbb{X}(t)\}_{t \geq 0}$, to model this phenomenon.

Necessary ingredients of $\{\mathbb{X}(t)\}_{t \geq 0}$:

- A stochastic process, $N(t)$, which controls the number of trees present at time t.
- A random structure, $\{[X_i, M_i(t)]\}$, which describes the locations and the sizes of the trees at time t.

Use the Renshaw-Särkkä growth-interaction model (RS-model): A spatio-temporal marked point process,
$\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$.

Our spatio-temporal marked point process

Employ a random process, $\{\mathbb{X}(t)\}_{t\geq 0}$, to model this phenomenon.
Necessary ingredients of $\{\mathbb{X}(t)\}_{t\geq 0}$:

- A stochastic process, $N(t)$, which controls the number of trees present at time t.
- A random structure, $\{[X_i, M_i(t)]\}$, which describes the locations and the sizes of the trees at time t.

Use the Renshaw-Särkkä growth-interaction model (RS-model):
A spatio-temporal marked point process,
$\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$.

Our spatio-temporal marked point process

Employ a random process, $\{\mathbb{X}(t)\}_{t\geq 0}$, to model this phenomenon.
Necessary ingredients of $\{\mathbb{X}(t)\}_{t\geq 0}$:

- A stochastic process, $N(t)$, which controls the number of trees present at time t.
- A random structure, $\{[X_i, M_i(t)]\}$, which describes the locations and the sizes of the trees at time t.

Use the Renshaw-Särkkä growth-interaction model (RS-model):
A spatio-temporal marked point process,
$\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$.

Our spatio-temporal marked point process

Employ a random process, $\{\mathbb{X}(t)\}_{t\geq 0}$, to model this phenomenon.
Necessary ingredients of $\{\mathbb{X}(t)\}_{t\geq 0}$:

- A stochastic process, $N(t)$, which controls the number of trees present at time t.
- A random structure, $\{[X_i, M_i(t)]\}$, which describes the locations and the sizes of the trees at time t.

Use the Renshaw-Särkkä growth-interaction model (RS-model):
A spatio-temporal marked point process,
$\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$.

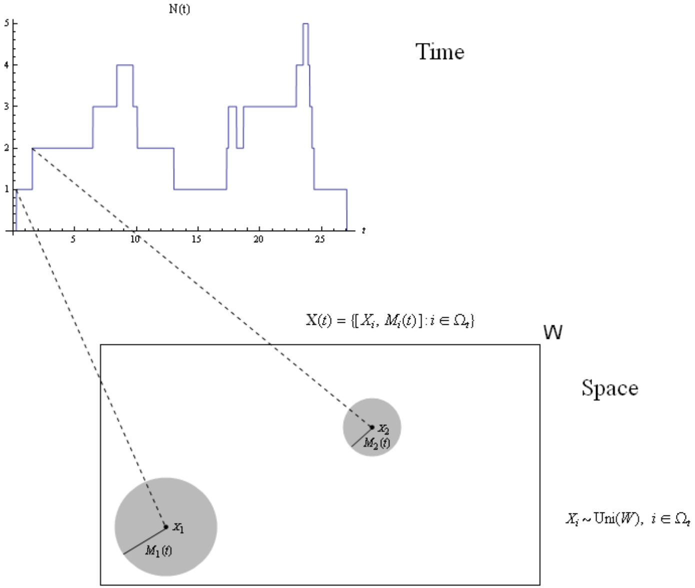
Our spatio-temporal marked point process

Employ a random process, $\{\mathbb{X}(t)\}_{t\geq 0}$, to model this phenomenon.
Necessary ingredients of $\{\mathbb{X}(t)\}_{t\geq 0}$:

- A stochastic process, $N(t)$, which controls the number of trees present at time t.
- A random structure, $\{[X_i, M_i(t)]\}$, which describes the locations and the sizes of the trees at time t.

Use the Renshaw-Särkkä growth-interaction model (RS-model):
A spatio-temporal marked point process,
$\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$.

The RS-model, $\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$, $t \in [0, T]$



The immigration-death process, $\{N(t)\}_{t\geq 0}$

- We use the immigration-death process for $\{N(t)\}_{t\geq 0}$.
- It is a Markov chain in continuous time.
- State space $E = \{0, 1, \ldots\}$.
  Parameters $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.
- A particular type of birth-death process.
  $N(t) = B(t; \alpha) - D(t; \mu) =$
  $\#\{$individuals alive at time t in a population$\}$
  where:
  1) New individuals arrive according to a Poisson process, $B(t)$, with intensity α.
  2) Individuals get iid $Exp(\mu)$-distributed life-times ($D(t)$ counts deaths).
- It can also be viewed as an $M/M/\infty$-queue.

The immigration-death process, $\{N(t)\}_{t\geq 0}$

- We use the immigration-death process for $\{N(t)\}_{t\geq 0}$.
- It is a Markov chain in continuous time.
- State space $E = \{0, 1, \dots\}$.
  Parameters $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.
- A particular type of birth-death process.
  $N(t) = B(t; \alpha) - D(t; \mu) =$
  $\#\{\text{individuals alive at time } t \text{ in a population}\}$
  where:
  1) New individuals arrive according to a Poisson process, $B(t)$, with intensity α.
  2) Individuals get iid $Exp(\mu)$-distributed life-times ($D(t)$ counts deaths).
- It can also be viewed as an $M/M/\infty$-queue.

The immigration-death process, $\{N(t)\}_{t\geq 0}$

- We use the immigration-death process for $\{N(t)\}_{t\geq 0}$.
- It is a Markov chain in continuous time.
- State space $E = \{0, 1, \ldots\}$.
  Parameters $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.
- A particular type of birth-death process.
  $N(t) = B(t; \alpha) - D(t; \mu) =$
  $\#\{\text{individuals alive at time } t \text{ in a population}\}$
  where:
  1) New individuals arrive according to a Poisson process, $B(t)$, with intensity α.
  2) Individuals get iid $Exp(\mu)$-distributed life-times ($D(t)$ counts deaths).
- It can also be viewed as an $M/M/\infty$-queue.

The immigration-death process, $\{N(t)\}_{t\geq 0}$

- We use the immigration-death process for $\{N(t)\}_{t\geq 0}$.
- It is a Markov chain in continuous time.
- State space $E = \{0, 1, \ldots\}$.
  Parameters $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.
- A particular type of birth-death process.
  $N(t) = B(t;\alpha) - D(t;\mu) =$
  $\#\{\text{individuals alive at time } t \text{ in a population}\}$
  where:
  1) New individuals arrive according to a Poisson process, $B(t)$, with intensity α.
  2) Individuals get iid $Exp(\mu)$-distributed life-times ($D(t)$ counts deaths).
- It can also be viewed as an $M/M/\infty$-queue.

The immigration-death process, $\{N(t)\}_{t\geq 0}$

- We use the immigration-death process for $\{N(t)\}_{t\geq 0}$.
- It is a Markov chain in continuous time.
- State space $E = \{0, 1, \ldots\}$.
  Parameters $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.
- A particular type of birth-death process.
  $N(t) = B(t; \alpha) - D(t; \mu) =$
  $\#\{\text{individuals alive at time } t \text{ in a population}\}$
  where:
  1) New individuals arrive according to a Poisson process, $B(t)$, with intensity α.
  2) Individuals get iid $Exp(\mu)$-distributed life-times ($D(t)$ counts deaths).
- It can also be viewed as an $M/M/\infty$-queue.

The immigration-death process, $\{N(t)\}_{t \geq 0}$

- We use the immigration-death process for $\{N(t)\}_{t \geq 0}$.
- It is a Markov chain in continuous time.
- State space $E = \{0, 1, \ldots\}$.
  Parameters $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.
- A particular type of birth-death process.
  $N(t) = B(t; \alpha) - D(t; \mu) =$
  $\#\{\text{individuals alive at time } t \text{ in a population}\}$
  where:
  1) New individuals arrive according to a Poisson process, $B(t)$, with intensity α.
  2) Individuals get iid $Exp(\mu)$-distributed life-times ($D(t)$ counts deaths).
- It can also be viewed as an $M/M/\infty$-queue.

The immigration-death process, $\{N(t)\}_{t\geq 0}$

- We use the immigration-death process for $\{N(t)\}_{t\geq 0}$.
- It is a Markov chain in continuous time.
- State space $E = \{0, 1, \ldots\}$.
  Parameters $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.
- A particular type of birth-death process.
  $N(t) = B(t; \alpha) - D(t; \mu) =$
  $\#\{\text{individuals alive at time } t \text{ in a population}\}$
  where:
  1) New individuals arrive according to a Poisson process, $B(t)$, with intensity α.
  2) Individuals get iid $Exp(\mu)$-distributed life-times ($D(t)$ counts deaths).
- It can also be viewed as an $M/M/\infty$-queue.

The immigration-death process, $\{N(t)\}_{t\geq 0}$

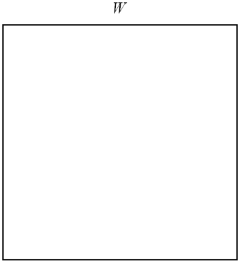
- We use the immigration-death process for $\{N(t)\}_{t\geq 0}$.
- It is a Markov chain in continuous time.
- State space $E = \{0, 1, \ldots\}$.
  Parameters $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.
- A particular type of birth-death process.
  $N(t) = B(t; \alpha) - D(t; \mu) =$
  $\#\{\text{individuals alive at time } t \text{ in a population}\}$
  where:
  1) New individuals arrive according to a Poisson process, $B(t)$, with intensity α.
  2) Individuals get iid $Exp(\mu)$-distributed life-times ($D(t)$ counts deaths).
- It can also be viewed as an $M/M/\infty$-queue.

The immigration-death process, $\{N(t)\}_{t\geq 0}$

- We use the immigration-death process for $\{N(t)\}_{t\geq 0}$.
- It is a Markov chain in continuous time.
- State space $E = \{0, 1, \ldots\}$.
  Parameters $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.
- A particular type of birth-death process.
  $N(t) = B(t; \alpha) - D(t; \mu) =$
  $\#\{\text{individuals alive at time } t \text{ in a population}\}$
  where:
  1) New individuals arrive according to a Poisson process, $B(t)$, with intensity α.
  2) Individuals get iid $Exp(\mu)$-distributed life-times ($D(t)$ counts deaths).
- It can also be viewed as an $M/M/\infty$-queue.

The RS-model, $\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}, t \in [0, T]$

- Wrap $W \subseteq \mathbb{R}^2$ onto a torus.
- $N(t) = B(t; \alpha) - D(t; \mu)$ controls $\Omega_t = \{\text{individuals present in } W \text{ at time } t\}$.
- Arrival of individual i at time t_i^0:
  - Location: $X_i \sim Uni(W)$
  - Initial mark: $M_i(t_i^0) = M_i^0 > 0$ (radius of the closed disk $B_{X_i}[M_i(t)]$)
- $\mathbb{X}(t)$: (thinned) spatial marked Poisson process.
- The $N = B(T; \alpha) \sim Poi(\alpha T)$ marks change size:



$$dM_1(t) = \Big[f(M_1(t); \psi) + \sum_{j \in \Omega_t} h(M_1(t), M_j(t), X_1, X_j; \psi)\Big] dt + \sigma dW_1(t)$$

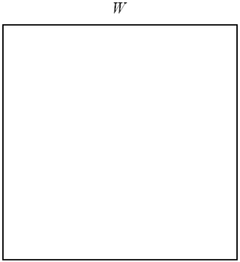
$$\vdots$$

$$dM_N(t) = \Big[f(M_N(t); \psi) + \sum_{j \in \Omega_t} h(M_N(t), M_j(t), X_N, X_j; \psi)\Big] dt + \sigma dW_N(t)$$

$W_1(t), \ldots, W_N(t)$ Brownian motions.

The RS-model, $\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$, $t \in [0, T]$

- Wrap $W \subseteq \mathbb{R}^2$ onto a torus.
- $N(t) = B(t;\alpha) - D(t;\mu)$ controls $\Omega_t = \{\text{individuals present in } W \text{ at time } t\}$.
- Arrival of individual i at time t_i^0:
  - Location: $X_i \sim Uni(W)$
  - Initial mark: $M_i(t_i^0) = M_i^0 > 0$ (radius of the closed disk $B_{X_i}[M_i(t)]$)
- $\mathbb{X}(t)$: (thinned) spatial marked Poisson process.
- The $N = B(T;\alpha) \sim Poi(\alpha T)$ marks change size:



$$dM_1(t) = \Big[f(M_1(t);\psi) + \sum_{j\in\Omega_t} h(M_1(t), M_j(t), X_1, X_j;\psi)\Big]dt + \sigma dW_1(t)$$

$$\vdots$$

$$dM_N(t) = \Big[f(M_N(t);\psi) + \sum_{j\in\Omega_t} h(M_N(t), M_j(t), X_N, X_j;\psi)\Big]dt + \sigma dW_N(t)$$

$W_1(t), \ldots, W_N(t)$ Brownian motions.

The RS-model, $\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$, $t \in [0, T]$

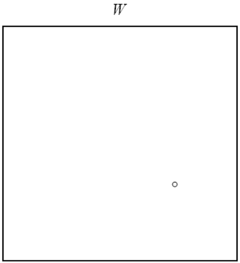
- Wrap $W \subseteq \mathbb{R}^2$ onto a torus.
- $N(t) = B(t;\alpha) - D(t;\mu)$ controls $\Omega_t = \{\text{individuals present in } W \text{ at time } t\}$.
- Arrival of individual i at time t_i^0:
  - Location: $X_i \sim Uni(W)$
  - Initial mark: $M_i(t_i^0) = M_i^0 > 0$ (radius of the closed disk $B_{X_i}[M_i(t)]$)
- $\mathbb{X}(t)$: (thinned) spatial marked Poisson process.
- The $N = B(T;\alpha) \sim Poi(\alpha T)$ marks change size:

$$dM_1(t) = \Big[f(M_1(t);\psi) + \sum_{j\in\Omega_t} h(M_1(t), M_j(t), X_1, X_j;\psi)\Big]dt + \sigma dW_1(t)$$

$$\vdots$$

$$dM_N(t) = \Big[f(M_N(t);\psi) + \sum_{j\in\Omega_t} h(M_N(t), M_j(t), X_N, X_j;\psi)\Big]dt + \sigma dW_N(t)$$

$W_1(t), \ldots, W_N(t)$ Brownian motions.



The RS-model, $\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$, $t \in [0, T]$

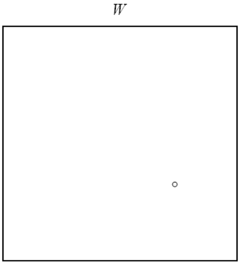
- Wrap $W \subseteq \mathbb{R}^2$ onto a torus.
- $N(t) = B(t; \alpha) - D(t; \mu)$ controls $\Omega_t = \{$individuals present in W at time $t\}$.
- Arrival of individual i at time t_i^0:
  - Location: $X_i \sim Uni(W)$
  - Initial mark: $M_i(t_i^0) = M_i^0 > 0$ (radius of the closed disk $B_{X_i}[M_i(t)]$)
- $\mathbb{X}(t)$: (thinned) spatial marked Poisson process.
- The $N = B(T; \alpha) \sim Poi(\alpha T)$ marks change size:

$$dM_1(t) = \left[f(M_1(t); \psi) + \sum_{j \in \Omega_t} h(M_1(t), M_j(t), X_1, X_j; \psi)\right] dt + \sigma dW_1(t)$$

$$\vdots$$

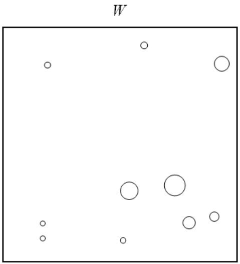
$$dM_N(t) = \left[f(M_N(t); \psi) + \sum_{j \in \Omega_t} h(M_N(t), M_j(t), X_N, X_j; \psi)\right] dt + \sigma dW_N(t)$$

$W_1(t), \ldots, W_N(t)$ Brownian motions.



The RS-model, $\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$, $t \in [0, T]$

- Wrap $W \subseteq \mathbb{R}^2$ onto a torus.
- $N(t) = B(t;\alpha) - D(t;\mu)$ controls $\Omega_t = \{\text{individuals present in } W \text{ at time } t\}$.
- Arrival of individual i at time t_i^0:
  - Location: $X_i \sim Uni(W)$
  - Initial mark: $M_i(t_i^0) = M_i^0 > 0$ (radius of the closed disk $B_{X_i}[M_i(t)]$)
- $\mathbb{X}(t)$: (thinned) spatial marked Poisson process.
- The $N = B(T;\alpha) \sim Poi(\alpha T)$ marks change size:



$$dM_1(t) = \left[f(M_1(t);\psi) + \sum_{j\in\Omega_t} h(M_1(t), M_j(t), X_1, X_j;\psi)\right] dt + \sigma dW_1(t)$$

$$\vdots$$

$$dM_N(t) = \left[f(M_N(t);\psi) + \sum_{j\in\Omega_t} h(M_N(t), M_j(t), X_N, X_j;\psi)\right] dt + \sigma dW_N(t)$$

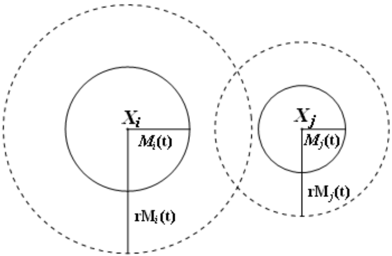
$W_1(t), \ldots, W_N(t)$ Brownian motions.

Influence zones and deaths

Definition

The influence zone of individual i at time t is the closed disk, $B_{X_i}[rM_i(t)]$, centred at X_i with radius $rM_i(t)$, $r > 0$.

Two individuals interact at time t if their influence zones overlap.



Possible death scenarios for individual i:

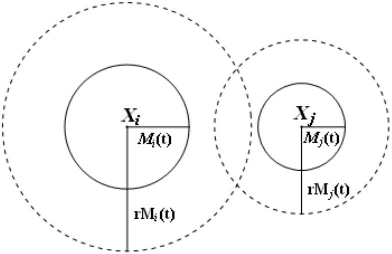
- Natural death: Its $Exp(\mu)$-distributed lifetime expires.
- Competitive death: $M_i(t) \leq 0$.

Influence zones and deaths

Definition

The influence zone of individual i at time t is the closed disk, $B_{X_i}[rM_i(t)]$, centred at X_i with radius $rM_i(t)$, $r > 0$.

Two individuals interact at time t if their influence zones overlap.



Possible death scenarios for individual i:

- Natural death: Its $Exp(\mu)$-distributed lifetime expires.
- Competitive death: $M_i(t) \leq 0$.

The RS-model: Individual growth- and interaction-function

Here $\psi = (\lambda, K, c, r)$.

- The linear (individual) growth function:

$$f(M_i(t); \psi) = \lambda \left(1 - \frac{M_i(t)}{K}\right)$$

Growth rate: $\lambda > 0$
Carrying capacity (upper bound): $K > 0$

- Area interaction:

$$h(M_i(t), M_j(t), X_i, X_j; \psi) = c \frac{\nu\left(B_{X_i}[rM_i(t)] \cap B_{X_j}[rM_j(t)]\right)}{\nu\left(B_{X_i}[rM_i(t)]\right)}$$

The range of interaction: $r > 0$
The force of interaction: $c > 0$
Note: Large individuals affect small individuals more than the other way around.

The RS-model: Individual growth- and interaction-function

Here $\psi = (\lambda, K, c, r)$.

- The linear (individual) growth function:

$$f(M_i(t);\psi) = \lambda\left(1 - \frac{M_i(t)}{K}\right)$$

Growth rate: $\lambda > 0$
Carrying capacity (upper bound): $K > 0$

- Area interaction:

$$h(M_i(t), M_j(t), X_i, X_j; \psi) = c\frac{\nu\left(B_{X_i}[rM_i(t)] \cap B_{X_j}[rM_j(t)]\right)}{\nu\left(B_{X_i}[rM_i(t)]\right)}$$

The range of interaction: $r > 0$
The force of interaction: $c > 0$
Note: Large individuals affect small individuals more than the other way around.

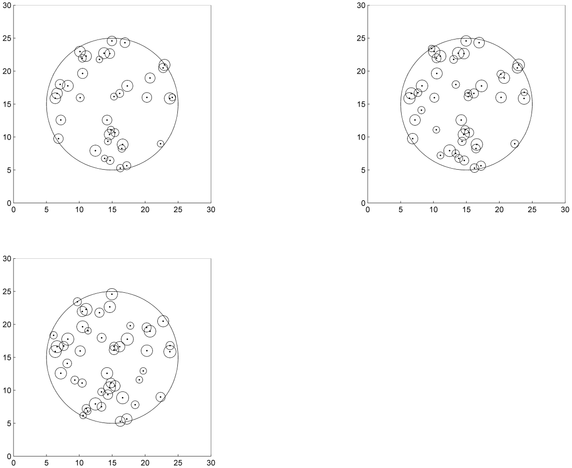


Figure: Simulated on $W = [0, 30] \times [0, 30]$ (radii scaled by a factor of 10), sampled at $T_1 = 22$ (top left), $T_2 = 27$ (right), $T_3 = 33$ (bottom left) in the circular region.

What is $\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$, $t \in [0, T]$?

- A marked spatial Poisson process Φ with intensity $T\alpha\nu(W)$, living on the torus W.
  Φ has points $X_1, \ldots, X_N$ and associated (dependent) marks $M_1(t), \ldots, M_N(t)$ which are diffusions.
  (The arrival and death times control the supports of the sample paths $M_1(t;\omega), \ldots, M_N(t;\omega)$).
- A multivariate diffusion, $\mathbb{M}(t, \Phi) = (M_1(t, \Phi), \ldots, M_N(t, \Phi))$, parametrised by a spatial Poisson process $\Phi = [X_1, \ldots, X_N]$ with intensity $T\alpha\nu(W)$, living on the torus W.
  (The arrival and death times control the supports of the sample paths $M_1(t, \Phi; \omega), \ldots, M_N(t, \Phi; \omega)$).
- Note that since $\nu(W) < \infty$ and $T < \infty$ we have that $N = B(T; \alpha\nu(W)) < \infty$ a.s.

What is $\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$, $t \in [0, T]$?

- A marked spatial Poisson process Φ with intensity $T\alpha\nu(W)$, living on the torus W.
  Φ has points $X_1, \dots, X_N$ and associated (dependent) marks $M_1(t), \dots, M_N(t)$ which are diffusions.
  (The arrival and death times control the supports of the sample paths $M_1(t; \omega), \dots, M_N(t; \omega)$).
- A multivariate diffusion, $\mathbb{M}(t, \Phi) = (M_1(t, \Phi), \dots, M_N(t, \Phi))$, parametrised by a spatial Poisson process $\Phi = [X_1, \dots, X_N]$ with intensity $T\alpha\nu(W)$, living on the torus W.
  (The arrival and death times control the supports of the sample paths $M_1(t, \Phi; \omega), \dots, M_N(t, \Phi; \omega)$).
- Note that since $\nu(W) < \infty$ and $T < \infty$ we have that $N = B(T; \alpha\nu(W)) < \infty$ a.s.

What is $\mathbb{X}(t) = \{[X_i, M_i(t)] : i \in \Omega_t\}$, $t \in [0, T]$?

- A marked spatial Poisson process Φ with intensity $T\alpha\nu(W)$, living on the torus W.
  Φ has points $X_1, \dots, X_N$ and associated (dependent) marks $M_1(t), \dots, M_N(t)$ which are diffusions.
  (The arrival and death times control the supports of the sample paths $M_1(t;\omega), \dots, M_N(t;\omega)$).
- A multivariate diffusion, $\mathbb{M}(t, \Phi) = (M_1(t, \Phi), \dots, M_N(t, \Phi))$, parametrised by a spatial Poisson process $\Phi = [X_1, \dots, X_N]$ with intensity $T\alpha\nu(W)$, living on the torus W.
  (The arrival and death times control the supports of the sample paths $M_1(t, \Phi;\omega), \dots, M_N(t, \Phi;\omega)$).
- Note that since $\nu(W) < \infty$ and $T < \infty$ we have that $N = B(T; \alpha\nu(W)) < \infty$ a.s.

ML-estimation

- $\mathbb{X}(t)$ is observed at $0 < T_1 < \ldots < T_n$ as $\{\mathbf{X}_j\}_{j=1}^n$.
- Estimate the parameters $\Lambda = (\alpha, \mu, \lambda, K, c, r, \sigma)$:
  - Immigration-death parameters: α, μ
  - Spatial growth and interaction parameters: $\psi = (\lambda, K, c, r)$
  - Diffusion parameter: $\sigma > 0$
- Aim: Simultaneous ML-estimation of Λ
- $\mathbb{X}(t)$ is a Markov process:
  - $(M_1(t), \ldots, M_N(t))$ is a Markov process (diffusion)
  - $N(t)$ is a continuous time Markov chain
- Find the transition densities, $P_{\mathbb{X}(T_j)|\mathbb{X}(T_{j-1})}(\mathbf{X}_j|\mathbf{X}_{j-1}; \Lambda)$:

$$L(\Lambda) = \prod_{i=1}^{n} P_{\mathbb{X}(T_j)|\mathbb{X}(T_{j-1})}(\mathbf{X}_j|\mathbf{X}_{j-1}; \Lambda)$$

Estimation of the mark parameters

Estimation of $(\lambda, K, c, r, \sigma)$ from $\{(M_1(T_j), \dots, M_N(T_j))\}_{j=1}^{n}$.

- When $\sigma = 0$ (Särkkä, 2006): Deterministic mark growth;

$$dM_i(t) = \left[f(M_i(t); \psi) + \sum_{j \in \Omega_t} h(M_i(t), M_j(t), X_i, X_j; \psi) \right] dt.$$

  $\Rightarrow$ Least squares estimation of ψ, ML-estimation of α, μ.

- When $c = 0$: No spatial interaction, i.e. independent SDEs;

$$dM_i(t) = f(M_i(t); \psi)dt + \sigma dW_i(t)$$

  Transition densities (linear growth): $(M_i(t) | M_i(s) = m_s) \sim N\left(K + (m_s + K)e^{-\lambda t/K}, \frac{\sigma^2(1-e^{-2\lambda t/K})}{2\lambda/K}\right)$

- When $c, \sigma \neq 0$: $(M_1(t), \dots, M_N(t))$ multivariate SDE (components interact):

$$dM_i(t) = \left[f(M_i(t); \psi) + \sum_{j \in \Omega_t} h(M_i(t), M_j(t), X_i, X_j; \psi) \right] dt + \sigma dW_i(t)$$

  Find transition densities of $(M_1(t), \dots, M_N(t))$.

Estimation of the mark parameters

Estimation of $(\lambda, K, c, r, \sigma)$ from $\{(M_1(T_j), \ldots, M_N(T_j))\}_{j=1}^{n}$.

- When $\sigma = 0$ (Särkkä, 2006): Deterministic mark growth;

$$dM_i(t) = \left[f(M_i(t); \psi) + \sum_{j \in \Omega_t} h(M_i(t), M_j(t), X_i, X_j; \psi) \right] dt.$$

  $\Rightarrow$ Least squares estimation of ψ, ML-estimation of α, μ.

- When $c = 0$: No spatial interaction, i.e. independent SDEs;

$$dM_i(t) = f(M_i(t); \psi)dt + \sigma dW_i(t)$$

  Transition densities (linear growth): $(M_i(t)|M_i(s) = m_s) \sim N\left(K + (m_s + K)e^{-\lambda t/K}, \frac{\sigma^2(1-e^{-2\lambda t/K})}{2\lambda/K}\right)$

- When $c, \sigma \neq 0$: $(M_1(t), \ldots, M_N(t))$ multivariate SDE (components interact):

$$dM_i(t) = \left[f(M_i(t); \psi) + \sum_{j \in \Omega_t} h(M_i(t), M_j(t), X_i, X_j; \psi) \right] dt + \sigma dW_i(t)$$

  Find transition densities of $(M_1(t), \ldots, M_N(t))$.

Estimation of the mark parameters

Estimation of $(\lambda, K, c, r, \sigma)$ from $\{(M_1(T_j), \ldots, M_N(T_j))\}_{j=1}^{n}$.

- When $\sigma = 0$ (Särkkä, 2006): Deterministic mark growth;

$$dM_i(t) = \left[f(M_i(t);\psi) + \sum_{j\in\Omega_t} h(M_i(t), M_j(t), X_i, X_j;\psi)\right]dt.$$

  $\Rightarrow$ Least squares estimation of ψ, ML-estimation of α, μ.

- When $c = 0$: No spatial interaction, i.e. independent SDEs;

$$dM_i(t) = f(M_i(t);\psi)dt + \sigma dW_i(t)$$

  Transition densities (linear growth): $(M_i(t)|M_i(s) = m_s) \sim N\left(K + (m_s + K)e^{-\lambda t/K}, \frac{\sigma^2(1-e^{-2\lambda t/K})}{2\lambda/K}\right)$

- When $c, \sigma \neq 0$: $(M_1(t), \ldots, M_N(t))$ multivariate SDE (components interact):

$$dM_i(t) = \left[f(M_i(t);\psi) + \sum_{j\in\Omega_t} h(M_i(t), M_j(t), X_i, X_j;\psi)\right]dt + \sigma dW_i(t)$$

  Find transition densities of $(M_1(t), \ldots, M_N(t))$.

Estimation of the mark parameters

Estimation of $(\lambda, K, c, r, \sigma)$ from $\{(M_1(T_j), \ldots, M_N(T_j))\}_{j=1}^{n}$.

- When $\sigma = 0$ (Särkkä, 2006): Deterministic mark growth;
$$dM_i(t) = \left[f(M_i(t); \psi) + \sum_{j \in \Omega_t} h(M_i(t), M_j(t), X_i, X_j; \psi) \right] dt.$$
$\Rightarrow$ Least squares estimation of ψ, ML-estimation of α, μ.
- When $c = 0$: No spatial interaction, i.e. independent SDEs;
$$dM_i(t) = f(M_i(t); \psi)dt + \sigma dW_i(t)$$
Transition densities (linear growth): $(M_i(t)|M_i(s) = m_s) \sim N\left(K + (m_s + K)e^{-\lambda t/K}, \frac{\sigma^2(1-e^{-2\lambda t/K})}{2\lambda/K}\right)$
- When $c, \sigma \neq 0$: $(M_1(t), \ldots, M_N(t))$ multivariate SDE (components interact):
$$dM_i(t) = \left[f(M_i(t); \psi) + \sum_{j \in \Omega_t} h(M_i(t), M_j(t), X_i, X_j; \psi) \right] dt + \sigma dW_i(t)$$
Find transition densities of $(M_1(t), \ldots, M_N(t))$.

ML-estimation: Discretely sampled immigration-death

- Sample $\{N(t)\}_{t\geq 0}$ as $(N(0), N(T_1), \ldots, N(T_n)) = (0, N_1, \ldots, N_n)$.
- The log-likelihood function of $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$:

$$l_n(\theta) = \sum_{k=1}^{n} \log p_{N_{k-1}N_k}(T_k - T_{k-1}; \theta).$$

Proposition (Transition probabilities)

The transition probabilities $p_{ij}(t;\theta) := \mathbb{P}\left(N(h+t) = j | N(h) = i\right)$:

$$\begin{aligned} p_{ij}(t;\theta) &= \left(f_{Poi(\alpha(1-e^{-\mu t})/\mu)} * f_{Bin(i,e^{-\mu t})}\right)(j) \\ &= \sum_{k=0}^{j} f_{Poi(\alpha(1-e^{-\mu t})/\mu)}(k) f_{Bin(i,e^{-\mu t})}(j-k) \end{aligned}$$

where $i,j \in E = \mathbb{N}$, $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.

ML-estimation: Discretely sampled immigration-death

- Sample $\{N(t)\}_{t\geq 0}$ as $(N(0), N(T_1), \ldots, N(T_n)) = (0, N_1, \ldots, N_n)$.
- The log-likelihood function of $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$:

$$l_n(\theta) = \sum_{k=1}^{n} \log p_{N_{k-1}N_k}(T_k - T_{k-1}; \theta).$$

Proposition (Transition probabilities)

The transition probabilities $p_{ij}(t;\theta) := \mathbb{P}\left(N(h+t) = j | N(h) = i\right)$:

$$\begin{aligned} p_{ij}(t;\theta) &= \left(f_{Poi(\alpha(1-e^{-\mu t})/\mu)} * f_{Bin(i,e^{-\mu t})}\right)(j) \\ &= \sum_{k=0}^{j} f_{Poi(\alpha(1-e^{-\mu t})/\mu)}(k) f_{Bin(i,e^{-\mu t})}(j-k) \end{aligned}$$

where $i, j \in E = \mathbb{N}$, $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.

ML-estimation: Discretely sampled immigration-death

- Sample $\{N(t)\}_{t\geq 0}$ as $(N(0), N(T_1), \ldots, N(T_n)) = (0, N_1, \ldots, N_n)$.
- The log-likelihood function of $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$:

$$l_n(\theta) = \sum_{k=1}^{n} \log p_{N_{k-1}N_k}(T_k - T_{k-1}; \theta).$$

Proposition (Transition probabilities)

The transition probabilities $p_{ij}(t;\theta) := \mathbb{P}\left(N(h+t) = j | N(h) = i\right)$*:*

$$\begin{aligned} p_{ij}(t;\theta) &= \left(f_{Poi(\alpha(1-e^{-\mu t})/\mu)} * f_{Bin(i,e^{-\mu t})}\right)(j) \\ &= \sum_{k=0}^{j} f_{Poi(\alpha(1-e^{-\mu t})/\mu)}(k) f_{Bin(i,e^{-\mu t})}(j-k) \end{aligned}$$

where $i, j \in E = \mathbb{N}$, $\theta = (\alpha, \mu) \in \Theta \subseteq \mathbb{R}_+^2$.

The ML-estimation

- The ML-estimator of $\theta = (\alpha, \mu) \in \Theta$:

$$(\hat{\alpha}_n, \hat{\mu}_n) = \hat{\theta}_n(N(T_1), \dots, N(T_n)) = \arg\max_{\theta \in \Theta} l_n(\theta).$$

- No closed form expression for $(\hat{\alpha}_n, \hat{\mu}_n)$.
  Dimension reduction: $\hat{\alpha}_n = \hat{\alpha}_n(\mu)$ is a function of μ and the sample $\{N(T_1), \dots, N(T_n)\}$. Maximise $l_n(\hat{\alpha}_n(\mu), \mu)$ w.r.t. μ.
- Assume now that $T_k - T_{k-1} = t$, $k = 1, \dots, n$, so that

$$l_n(\theta) = \sum_{i,j \in E} N_n(i,j) \log p_{ij}(t; \theta),$$

where $N_n(i,j) = \sum_{k=1}^{n} \mathbf{1}\{(N_{k-1}, N_k) = (i,j)\}$.

The ML-estimation

- The ML-estimator of $\theta = (\alpha, \mu) \in \Theta$:

$$(\hat{\alpha}_n, \hat{\mu}_n) = \hat{\theta}_n(N(T_1), \dots, N(T_n)) = \arg\max_{\theta \in \Theta} l_n(\theta).$$

- No closed form expression for $(\hat{\alpha}_n, \hat{\mu}_n)$.
  Dimension reduction: $\hat{\alpha}_n = \hat{\alpha}_n(\mu)$ is a function of μ and the sample $\{N(T_1), \dots, N(T_n)\}$. Maximise $l_n(\hat{\alpha}_n(\mu), \mu)$ w.r.t. μ.
- Assume now that $T_k - T_{k-1} = t$, $k = 1, \dots, n$, so that

$$l_n(\theta) = \sum_{i,j \in E} N_n(i,j) \log p_{ij}(t; \theta),$$

where $N_n(i,j) = \sum_{k=1}^{n} \mathbf{1}\{(N_{k-1}, N_k) = (i,j)\}$.

The ML-estimation

- The ML-estimator of $\theta = (\alpha, \mu) \in \Theta$:

$$(\hat{\alpha}_n, \hat{\mu}_n) = \hat{\theta}_n(N(T_1), \dots, N(T_n)) = \arg\max_{\theta \in \Theta} l_n(\theta).$$

- No closed form expression for $(\hat{\alpha}_n, \hat{\mu}_n)$.
  Dimension reduction: $\hat{\alpha}_n = \hat{\alpha}_n(\mu)$ is a function of μ and the sample $\{N(T_1), \dots, N(T_n)\}$. Maximise $l_n(\hat{\alpha}_n(\mu), \mu)$ w.r.t. μ.
- Assume now that $T_k - T_{k-1} = t$, $k = 1, \dots, n$, so that

$$l_n(\theta) = \sum_{i,j \in E} N_n(i,j) \log p_{ij}(t; \theta),$$

where $N_n(i,j) = \sum_{k=1}^{n} \mathbf{1}\{(N_{k-1}, N_k) = (i,j)\}$.

Asymptotics

Denote by $\theta_0 = (\alpha_0, \mu_0) \in \Theta$ the true parameter pair.

Proposition (Strong consistency)

Let Θ be any compact subset of $\mathbb{R}_+^2$. Then, as $n \to \infty$, the maximum likelihood estimator for the immigration-death process satisfies

$$(\hat{\alpha}_n, \hat{\mu}_n) \xrightarrow{a.s.} (\alpha_0, \mu_0).$$

Proposition (Asymptotic normality)

Let Θ be any compact subset of $\mathbb{R}_+^2$. Furthermore, assume that $(\log(\alpha_0 + \mu_0) - \log(\alpha_0))/\mu_0 \geq 2t$. Then, as $n \to \infty$,

$$\sqrt{n}\left((\hat{\alpha}_n, \hat{\mu}_n) - (\alpha_0, \mu_0)\right) \xrightarrow{d} N\left(\mathbf{0}, I(\theta_0)^{-1}\right),$$

where $I(\theta_0)^{-1}$ is the inverse of the Fisher information matrix.

Asymptotics

Denote by $\theta_0 = (\alpha_0, \mu_0) \in \Theta$ the true parameter pair.

Proposition (Strong consistency)

Let Θ be any compact subset of $\mathbb{R}_+^2$. Then, as $n \to \infty$, the maximum likelihood estimator for the immigration-death process satisfies

$$(\hat{\alpha}_n, \hat{\mu}_n) \xrightarrow{a.s.} (\alpha_0, \mu_0).$$

Proposition (Asymptotic normality)

Let Θ be any compact subset of $\mathbb{R}_+^2$. Furthermore, assume that $(\log(\alpha_0 + \mu_0) - \log(\alpha_0))/\mu_0 \geq 2t$. Then, as $n \to \infty$,

$$\sqrt{n}\left((\hat{\alpha}_n, \hat{\mu}_n) - (\alpha_0, \mu_0)\right) \xrightarrow{d} N\left(\mathbf{0}, I(\theta_0)^{-1}\right),$$

where $I(\theta_0)^{-1}$ is the inverse of the Fisher information matrix.

Asymptotics

Denote by $\theta_0 = (\alpha_0, \mu_0) \in \Theta$ the true parameter pair.

Proposition (Strong consistency)

Let Θ be any compact subset of $\mathbb{R}^2_+$. Then, as $n \to \infty$, the maximum likelihood estimator for the immigration-death process satisfies

$$(\hat{\alpha}_n, \hat{\mu}_n) \xrightarrow{a.s.} (\alpha_0, \mu_0).$$

Proposition (Asymptotic normality)

Let Θ be any compact subset of $\mathbb{R}^2_+$. Furthermore, assume that $(\log(\alpha_0 + \mu_0) - \log(\alpha_0))/\mu_0 \geq 2t$. Then, as $n \to \infty$,

$$\sqrt{n}\left((\hat{\alpha}_n, \hat{\mu}_n) - (\alpha_0, \mu_0)\right) \xrightarrow{d} N\left(\mathbf{0}, I(\theta_0)^{-1}\right),$$

where $I(\theta_0)^{-1}$ is the inverse of the Fisher information matrix.

Numerical evaluation

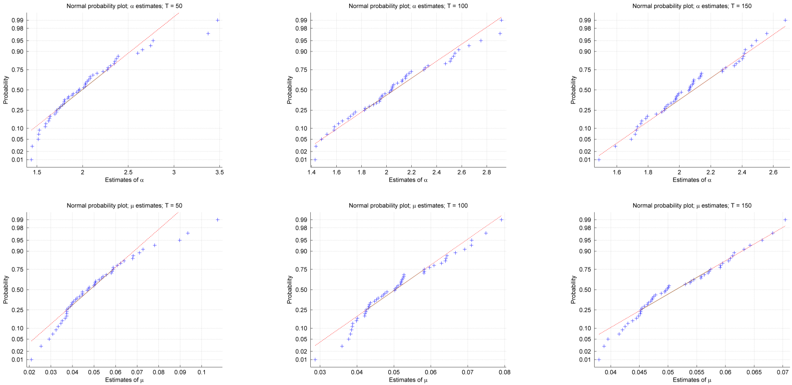
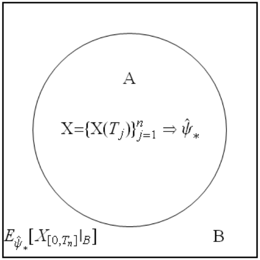


Figure: Normal probability plots of the estimates of $(\alpha_0, \mu_0) = (0.4, 0.01)$ based on 50 sample paths sampled at times $T_k = kt$, $t = 1$, $k = 1, \ldots, T$. Upper row: The estimates of α_0 at final times $T = 50$ (left), $T = 100$ (middle) and $T = 150$ (right). Lower row: The estimates of μ_0 at final times $T = 50$ (left), $T = 100$ (middle) and $T = 150$ (right).

Thank you for listening!

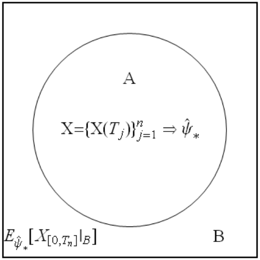
Idea used in the spatio-temporal edge correction

- Data, $\mathbb{X} = \{\mathbb{X}(T_j)\}_{j=1}^n$, is sampled in region A.
- No information about individuals in $B = W \setminus A$ who interact with $\mathbb{X}$ - Edge effects! $\Rightarrow$ Biased estimates!
- Existing edge corrections are not easily altered to the spatio-temporal case or remove data.
- Idea behind the new approach:
  1. From $\{\mathbb{X}(T_j)\}_{j=1}^n$: non-edge corrected estimates $\hat{\psi}_*$.
  2. In region B: The "expected process", $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$, under the regime of $\hat{\psi}_*$.
  3. Re-estimate ψ while letting $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$ interact with $\{\mathbb{X}(T_j)\}_{j=1}^n$.
  4. New estimates $\hat{\psi}$ have been influenced by the "expected interaction" between $\mathbb{X}$ and $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$.



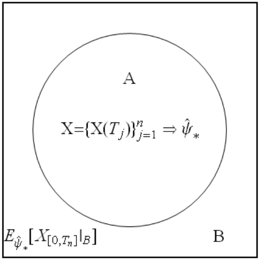
Idea used in the spatio-temporal edge correction

- Data, $\mathbb{X} = \{\mathbb{X}(T_j)\}_{j=1}^{n}$, is sampled in region A.
- No information about individuals in $B = W \setminus A$ who interact with $\mathbb{X}$ - Edge effects! $\Rightarrow$ Biased estimates!
- Existing edge corrections are not easily altered to the spatio-temporal case or remove data.
- Idea behind the new approach:
  1. From $\{\mathbb{X}(T_j)\}_{j=1}^{n}$: non-edge corrected estimates $\hat{\psi}_*$.
  2. In region B: The "expected process", $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|B\right]$, under the regime of $\hat{\psi}_*$.
  3. Re-estimate ψ while letting $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|B\right]$ interact with $\{\mathbb{X}(T_j)\}_{j=1}^{n}$.
  4. New estimates $\hat{\psi}$ have been influenced by the "expected interaction" between $\mathbb{X}$ and $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|B\right]$.



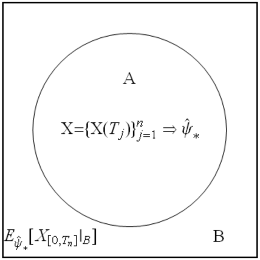
Idea used in the spatio-temporal edge correction

- Data, $\mathbb{X} = \{\mathbb{X}(T_j)\}_{j=1}^{n}$, is sampled in region A.
- No information about individuals in $B = W \setminus A$ who interact with $\mathbb{X}$ - Edge effects! $\Rightarrow$ Biased estimates!
- Existing edge corrections are not easily altered to the spatio-temporal case or remove data.
- Idea behind the new approach:
  1. From $\{\mathbb{X}(T_j)\}_{j=1}^{n}$: non-edge corrected estimates $\hat{\psi}_*$.
  2. In region B: The "expected process", $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$, under the regime of $\hat{\psi}_*$.
  3. Re-estimate ψ while letting $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$ interact with $\{\mathbb{X}(T_j)\}_{j=1}^{n}$.
  4. New estimates $\hat{\psi}$ have been influenced by the "expected interaction" between $\mathbb{X}$ and $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$.



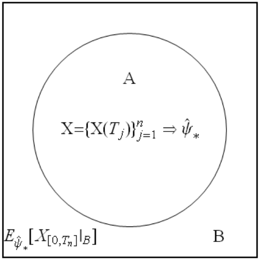
Idea used in the spatio-temporal edge correction

- Data, $\mathbb{X} = \{\mathbb{X}(T_j)\}_{j=1}^{n}$, is sampled in region A.
- No information about individuals in $B = W \setminus A$ who interact with $\mathbb{X}$ - Edge effects! $\Rightarrow$ Biased estimates!
- Existing edge corrections are not easily altered to the spatio-temporal case or remove data.
- Idea behind the new approach:
  1. From $\{\mathbb{X}(T_j)\}_{j=1}^{n}$: non-edge corrected estimates $\hat{\psi}_*$.
  2. In region B: The "expected process", $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|B\right]$, under the regime of $\hat{\psi}_*$.
  3. Re-estimate ψ while letting $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|B\right]$ interact with $\{\mathbb{X}(T_j)\}_{j=1}^{n}$.
  4. New estimates $\hat{\psi}$ have been influenced by the "expected interaction" between $\mathbb{X}$ and $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|B\right]$.



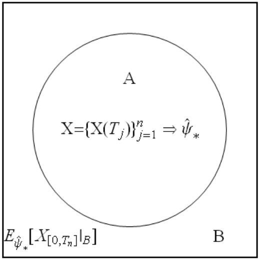
Idea used in the spatio-temporal edge correction

- Data, $\mathbb{X} = \{\mathbb{X}(T_j)\}_{j=1}^{n}$, is sampled in region A.
- No information about individuals in $B = W \setminus A$ who interact with $\mathbb{X}$ - Edge effects! $\Rightarrow$ Biased estimates!
- Existing edge corrections are not easily altered to the spatio-temporal case or remove data.
- Idea behind the new approach:
  1. From $\{\mathbb{X}(T_j)\}_{j=1}^{n}$: non-edge corrected estimates $\hat{\psi}_*$.
  2. In region B: The "expected process", $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$, under the regime of $\hat{\psi}_*$.
  3. Re-estimate ψ while letting $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$ interact with $\{\mathbb{X}(T_j)\}_{j=1}^{n}$.
  4. New estimates $\hat{\psi}$ have been influenced by the "expected interaction" between $\mathbb{X}$ and $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$.



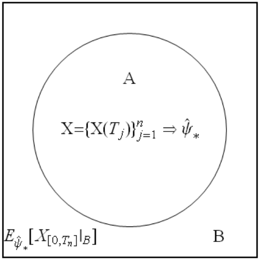
Idea used in the spatio-temporal edge correction

- Data, $\mathbb{X} = \{\mathbb{X}(T_j)\}_{j=1}^{n}$, is sampled in region A.
- No information about individuals in $B = W \setminus A$ who interact with $\mathbb{X}$ - Edge effects! $\Rightarrow$ Biased estimates!
- Existing edge corrections are not easily altered to the spatio-temporal case or remove data.
- Idea behind the new approach:
  1. From $\{\mathbb{X}(T_j)\}_{j=1}^{n}$: non-edge corrected estimates $\hat{\psi}_*$.
  2. In region B: The "expected process", $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$, under the regime of $\hat{\psi}_*$.
  3. Re-estimate ψ while letting $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$ interact with $\{\mathbb{X}(T_j)\}_{j=1}^{n}$.
  4. New estimates $\hat{\psi}$ have been influenced by the "expected interaction" between $\mathbb{X}$ and $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$.



Idea used in the spatio-temporal edge correction

- Data, $\mathbb{X} = \{\mathbb{X}(T_j)\}_{j=1}^{n}$, is sampled in region A.
- No information about individuals in $B = W \setminus A$ who interact with $\mathbb{X}$ - Edge effects! $\Rightarrow$ Biased estimates!
- Existing edge corrections are not easily altered to the spatio-temporal case or remove data.
- Idea behind the new approach:
  1. From $\{\mathbb{X}(T_j)\}_{j=1}^{n}$: non-edge corrected estimates $\hat{\psi}_*$.
  2. In region B: The "expected process", $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$, under the regime of $\hat{\psi}_*$.
  3. Re-estimate ψ while letting $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$ interact with $\{\mathbb{X}(T_j)\}_{j=1}^{n}$.
  4. New estimates $\hat{\psi}$ have been influenced by the "expected interaction" between $\mathbb{X}$ and $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$.



Idea used in the spatio-temporal edge correction

- Data, $\mathbb{X} = \{\mathbb{X}(T_j)\}_{j=1}^n$, is sampled in region A.
- No information about individuals in $B = W \setminus A$ who interact with $\mathbb{X}$ - Edge effects! $\Rightarrow$ Biased estimates!
- Existing edge corrections are not easily altered to the spatio-temporal case or remove data.
- Idea behind the new approach:
  1. From $\{\mathbb{X}(T_j)\}_{j=1}^n$: non-edge corrected estimates $\hat{\psi}_*$.
  2. In region B: The "expected process", $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$, under the regime of $\hat{\psi}_*$.
  3. Re-estimate ψ while letting $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$ interact with $\{\mathbb{X}(T_j)\}_{j=1}^n$.
  4. New estimates $\hat{\psi}$ have been influenced by the "expected interaction" between $\mathbb{X}$ and $\mathbb{E}_{\hat{\psi}_*}\left[\mathbb{X}_{[0,T_n]}|_B\right]$.